Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 916243, 7 pages
http://dx.doi.org/10.1155/2014/916243

Research Article

Convex Combination Inequalities of the Line and Plane

Zlatko Pavić,[1] Shanhe Wu,[2] and Štefanija Klarić[1]

[1] *Mechanical Engineering Faculty in Slavonski Brod, University of Osijek, Trg Ivane Brlić-Mažuranić 2, 35000 Slavonski Brod, Croatia*
[2] *Department of Mathematics and Computer Science, Longyan University, Longyan, Fujian 364012, China*

Correspondence should be addressed to Shanhe Wu; shanhewu@gmail.com

Received 19 July 2014; Revised 27 October 2014; Accepted 27 October 2014; Published 16 November 2014

Academic Editor: Irena Lasiecka

The paper deals with convex combinations, convex functions, and Jensen's functionals. The main idea of this work is to present the given convex combination by using two other convex combinations with minimal number of points. For example, as regards the presentation of the planar combination, we use two trinomial combinations. Generalizations to higher dimensions are also considered.

1. Introduction

Let $\mathscr{X}$ be a real vector space. A set $\mathscr{A} \subseteq \mathscr{X}$ is affine if it contains the lines passing through all pairs of its points (all binomial affine combinations in $\mathscr{A}$, i.e., the combinations $p_1P_1 + p_2P_2$ of points $P_1, P_2 \in \mathscr{A}$ and coefficients $p_1, p_2 \in \mathbb{R}$ of the sum $p_1 + p_2 = 1$). A function $f : \mathscr{A} \to \mathbb{R}$ is affine if it satisfies the equality $f(p_1P_1 + p_2P_2) = p_1f(P_1) + p_2f(P_2)$ for all binomial affine combinations in $\mathscr{A}$.

A set $\mathscr{C} \subseteq \mathscr{X}$ is convex if it contains the line segments connecting all pairs of its points (all binomial convex combinations in $\mathscr{C}$, i.e., the combinations $p_1P_1 + p_2P_2$ of points $P_1, P_2 \in \mathscr{C}$ and nonnegative coefficients $p_1, p_2 \in \mathbb{R}$ of the sum $p_1 + p_2 = 1$). A function $f : \mathscr{C} \to \mathbb{R}$ is convex if it satisfies the inequality $f(p_1P_1 + p_2P_2) \leq p_1f(P_1) + p_2f(P_2)$ for all binomial convex combinations in $\mathscr{C}$.

Using the mathematical induction, it can be proved that every affine function $f : \mathscr{A} \to \mathbb{R}$ satisfies the equality

$$f\left(\sum_{i=1}^{n} p_iP_i\right) = \sum_{i=1}^{n} p_if\left(P_i\right) \tag{1}$$

for all affine combinations in $\mathscr{A}$ and that every convex function $f : \mathscr{C} \to \mathbb{R}$ satisfies the Jensen inequality

$$f\left(\sum_{i=1}^{n} p_iP_i\right) \leq \sum_{i=1}^{n} p_if\left(P_i\right) \tag{2}$$

for all convex combinations in $\mathscr{C}$.

For an affine or a convex combination $P = \sum_{i=1}^{n} p_iP_i$ the point P itself is called the combination center, and it is important to mathematical inequalities. Recognizing the importance of the combination center, the authors (see [1]) have recently considered inequalities on simplexes and their cones.

A general overview of convex sets, convex functions, and its applications can be found in [2]. In working with means and their inequalities, we can rely on the book in [3]. Many details of the branch of mathematical inequalities are written in [4].

2. Convex Combinations of the Line

The section shows the importance of convex combination centers in deriving inequalities. The main result is Theorem 2.

If $a, b \in \mathbb{R}$ are different numbers, say $a < b$, then every number $x \in \mathbb{R}$ can be uniquely presented as the affine combination

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b. \tag{3}$$

The above binomial combination is convex if and only if the number x belongs to the interval $[a, b]$. Given the function $f : \mathbb{R} \to \mathbb{R}$, let $f^{\text{line}}_{\{a,b\}} : \mathbb{R} \to \mathbb{R}$ be the function of the line

passing through the points $(a, f(a))$ and $(b, f(b))$ of the graph of f. Using the affinity of $f^{\text{line}}_{\{a,b\}}$, we get the equation

$$f^{\text{line}}_{\{a,b\}}(x) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b). \tag{4}$$

If the function f is convex, then, using the definition of convexity, we obtain the inequality

$$f(x) \le f^{\text{line}}_{\{a,b\}}(x), \quad \text{if } x \in [a,b], \tag{5}$$

and the reverse inequality

$$f(x) \ge f^{\text{line}}_{\{a,b\}}(x), \quad \text{if } x \notin (a,b). \tag{6}$$

By the end of this section we will use an interval $\mathcal{I} \subseteq \mathbb{R}$ with the nonempty interior $\mathcal{I}^0$.

The following lemma represents a systematised version of [5, Proposition 2] and deals with two convex combinations having the same center. Assigning the convex function to such convex combinations, we obtain the following Jensen type inequality.

Lemma 1. *Let $a, b \in \mathcal{I} \subseteq \mathbb{R}$ be points such that $a \le b$. Let $\sum_{i=1}^{n} p_i x_i$ be a convex combination with points $x_i \in [a,b]$, and let $\sum_{j=1}^{m} q_j y_j$ be a convex combination with points $y_j \in \mathcal{I} \setminus (a,b)$.*

If the above convex combinations have the same center

$$\sum_{i=1}^{n} p_i x_i = \sum_{j=1}^{m} q_j y_j, \tag{7}$$

then every convex function $f : \mathcal{I} \to \mathbb{R}$ satisfies the inequality

$$\sum_{i=1}^{n} p_i f(x_i) \le \sum_{j=1}^{m} q_j f(y_j). \tag{8}$$

If f is concave, then the reverse inequality is valid in (8).

Proof. Assume f is convex. If $a < b$, the right-hand side follows from the series of inequalities

$$\begin{aligned} \sum_{i=1}^{n} p_i f(x_i) &\le \sum_{i=1}^{n} p_i f^{\text{line}}_{\{a,b\}}(x_i) = f^{\text{line}}_{\{a,b\}}\left(\sum_{i=1}^{n} p_i x_i\right) \\ &= f^{\text{line}}_{\{a,b\}}\left(\sum_{j=1}^{m} q_j y_j\right) = \sum_{j=1}^{m} q_j f^{\text{line}}_{\{a,b\}}(y_j) \\ &\le \sum_{j=1}^{m} q_j f(y_j) \end{aligned} \tag{9}$$

derived applying the inequality in (5) to x_i and the inequality in (6) to y_j. If $a = b$, we use any support line $f^{\text{line}}_{\{a\}}$ instead of the chord line $f^{\text{line}}_{\{a,b\}}$. □

Lemma 1 is the generalization of Jensen's inequality: applying this lemma to the convex combination center equality

$$1c = \sum_{i=1}^{n} p_i x_i, \tag{10}$$

with the assumption $a = b = c$, we come to the Jensen inequality

$$f\left(\sum_{i=1}^{n} p_i x_i\right) = 1 f(c) \le \sum_{i=1}^{n} p_i f(x_i). \tag{11}$$

So, the discrete form of the famous Jensen inequality (discrete form in [6] and integral form in [7]) can be derived applying the convexity definition and the affinity of the chord or support line. The different forms of Jensen's inequality can be seen in [8].

Theorem 2. *Let $a, b, a_1, b_1 \in \mathcal{I} \subseteq \mathbb{R}$ be points such that $a_1 < a < b < b_1$. Let $c = \sum_{i=1}^{n} p_i x_i$ be a convex combination of the points $x_i \in [a_1, b_1] \setminus (a,b)$ with the center $c \in (a,b)$.*

Then there exist two binomial convex combinations $\alpha a + \beta b$ and $\alpha_1 a_1 + \beta_1 b_1$ so that

$$\alpha a + \beta b = \sum_{i=1}^{n} p_i x_i = \alpha_1 a_1 + \beta_1 b_1, \tag{12}$$

and consequently, every convex function $f : \mathcal{I} \to \mathbb{R}$ satisfies the inequality

$$\alpha f(a) + \beta f(b) \le \sum_{i=1}^{n} p_i f(x_i) \le \alpha_1 f(a_1) + \beta_1 f(b_1). \tag{13}$$

If f is concave, then the reverse inequality is valid in (13).

Proof. We use the formula in (3) to calculate the coefficients $\alpha = (b-c)/(b-a)$ and $\beta = (c-a)(b-a)$ that satisfy $c = \alpha a + \beta b$ and also for α_1 and β_1. Now we need to apply Lemma 1 to both sides of the obtained equality in (12). □

The graphical representation of the equality in (12) and the inequality in (13) is shown in Figure 1.

Binomial convex combinations are included into the definition of convexity. The following corollary demonstrates how the binomial combinations may be assigned to each convex combination.

Corollary 3. *Let $\sum_{i=1}^{n} p_i x_i$ be a convex combination in $\mathcal{I} \subseteq \mathbb{R}$. Then there exist two binomial convex combinations $p_0 x_0 + q_0 y_0$ and $px + qy$ with points x_0, y_0, x, y from the set $\{x_1, \ldots, x_n\}$ so that*

$$p_0 x_0 + q_0 y_0 = \sum_{i=1}^{n} p_i x_i = px + qy, \tag{14}$$

and consequently, every convex function $f : \mathcal{I} \to \mathbb{R}$ satisfies the inequality

$$p_0 f(x_0) + q_0 f(y_0) \le \sum_{i=1}^{n} p_i f(x_i) \le p f(x) + q f(y). \tag{15}$$

Proof. Put $c = \sum_{i=1}^{n} p_i x_i$ and suppose $x_1 \le \cdots \le x_n$. Take $x = x_1$ and $y = x_n$. If $x = y$, we take $p = q = 1/2$. If $x < y$, we calculate the coefficients p and q by the formula in (3) to get $c = px + qy$.

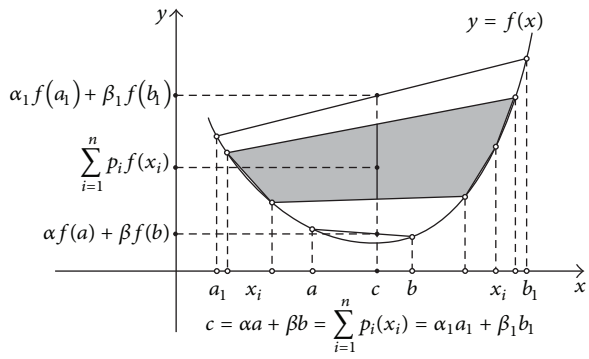

Figure 1: Line convex combinations with the same center c.

If c is equal to some x_{i_0}, then we take $x_0 = y_0 = x_{i_0}$ and $p_0 = q_0 = 1/2$. Otherwise, it must be $c \in (x_i, x_{i+1})$ for some pair, in which case we take $x_0 = x_i$ and $y_0 = x_{i+1}$. Calculating p_0 and q_0 by (3), we also get $c = p_0x_0 + q_0y_0$.

It remains to apply Lemma 1 to both sides of the obtained equality in (14). □

Respecting the Jensen inequality, the formula in (15) can be expressed in the extended form:

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le p_0 f(x_0) + q_0 f(y_0) \le \sum_{i=1}^{n} p_i f(x_i) \le pf(x) + qf(y). \tag{16}$$

Let us show how the intermediate member in (13) can be transformed into integral. Let $\mathscr{L} \subseteq \mathscr{I}$ be a bounded set with the length $|\mathscr{L}| > 0$, and let f be an integrable (in the sense of Riemann or Lebesgue) function on $\mathscr{L}$. Given the positive integer n, we employ a partition

$$\mathscr{L} = \bigcup_{i=1}^{n} \mathscr{L}_{ni}, \tag{17}$$

where each of pairwise disjoint subsets $\mathscr{L}_{ni}$ contracts to the point as n approaches infinity. Take one point $x_{ni} \in \mathscr{L}_{ni}$ for every $i = 1, \ldots, n$ and then compose the convex combination c_n of the points $f(x_{ni})$ with the coefficients $p_{ni} = |\mathscr{L}_{ni}|/|\mathscr{L}|$; that is,

$$c_n = \frac{1}{|\mathscr{L}|} \sum_{i=1}^{n} |\mathscr{L}_{ni}| f(x_{ni}). \tag{18}$$

Letting n to infinity, the sequence $(c_n)_n$ approaches the point

$$c = \frac{1}{|\mathscr{L}|} \int_{\mathscr{L}} f(x)\, dx. \tag{19}$$

Using the integral method with convex combinations, we obtain the mixed discrete-integral form of Theorem 2 as follows.

Corollary 4. *Let $a, b, a_1, b_1 \in \mathscr{I} \subseteq \mathbb{R}$ be points such that $a_1 < a < b < b_1$. Let the barycenter c of the set $[a_1, b_1] \setminus (a, b)$ belongs to (a, b); that is,*

$$c = \frac{\int_{[a_1,b_1]\setminus(a,b)} x\, dx}{b_1 - a_1 - b + a} \in (a, b). \tag{20}$$

Then there exist two binomial convex combinations $\alpha a + \beta b$ and $\alpha_1 a_1 + \beta_1 b_1$ so that

$$\alpha a + \beta b = c = \alpha_1 a_1 + \beta_1 b_1, \tag{21}$$

and consequently, every convex function $f : \mathscr{I} \to \mathbb{R}$ satisfies the inequality

$$\alpha f(a) + \beta f(b) \le \frac{\int_{[a_1,b_1]\setminus(a,b)} f(x)\, dx}{b_1 - a_1 - b + a} \le \alpha_1 f(a_1) + \beta_1 f(b_1). \tag{22}$$

If f is concave, then the reverse inequality is valid in (22).

The summarizing Jensen's functional of a function $f : \mathscr{I} \to \mathbb{R}$ for the given convex combination $\sum_{i=1}^{n} p_i x_i$ in $\mathscr{I}$ is defined with

$$J_{p_1x_1+\cdots+p_nx_n}(f) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right). \tag{23}$$

If the conditions of Theorem 2 hold, we get the functional inequality

$$J_{\alpha a+\beta b}(f) \le J_{p_1x_1+\cdots+p_nx_n}(f) \le J_{\alpha_1a_1+\beta_1b_1}(f). \tag{24}$$

The integrating Jensen's functional of an integrable function $f : \mathscr{I} \to \mathbb{R}$ for the given subset $\mathscr{L} \subseteq \mathscr{I}$ with the length $|\mathscr{L}| > 0$ is defined with

$$J_{\mathscr{L}}(f) = \frac{1}{|\mathscr{L}|} \int_{\mathscr{L}} f(x)\, dx - f\left(\frac{1}{|\mathscr{L}|} \int_{\mathscr{L}} x\, dx\right). \tag{25}$$

If the conditions of Corollary 4 are satisfied, we get

$$J_{\alpha a+\beta b}(f) \le J_{[a_1,b_1]\setminus(a,b)}(f) \le J_{\alpha_1a_1+\beta_1b_1}(f). \tag{26}$$

The inequalities in (24) and (26) offer the local bounds of Jensen's functionals. The global bounds of Jensen's summarizing functional were investigated in [9]. Some new Jensen type inequalities were obtained in [10].

3. Convex Combinations of the Plane

This section contains the main results, Theorem 6, and its consequences.

We assume that $\mathbb{R}^2$ is the real vector space treating its points as the vectors with the standard coordinate addition $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and the scalar multiplication $\alpha(x, y) = (\alpha x, \alpha y)$.

If $A(x_A, y_A)$, $B(x_B, y_B)$, and $C(x_C, y_C)$ are the planar points that do not belong to one line, respectively, the convex

hull conv$\{A, B, C\}$ is a real triangle, and then every point $P(x, y) \in \mathbb{R}^2$ can be presented by the unique affine combination

$$P = \alpha A + \beta B + \gamma C, \tag{27}$$

where

$$\alpha = \frac{\begin{vmatrix} x & y & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}}{\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}},$$

$$\beta = -\frac{\begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_C & y_C & 1 \end{vmatrix}}{\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}}, \tag{28}$$

$$\gamma = \frac{\begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix}}{\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}}.$$

The above trinomial combination is convex if and only if the point P belongs to the triangle conv$\{A, B, C\}$.

Given the triangle with vertices A, B, and C, the convex cone $\mathscr{C}_A$ with the vertex at A is the set spanned by the vectors $A - B$ and $A - C$ (similarly $\mathscr{C}_B$ and $\mathscr{C}_C$; all three cones can be viewed in Figure 2); that is,

$$\mathscr{C}_A = \{A + p(A - B) + q(A - C) : p, q \in \mathbb{R}, p, q \geq 0\}. \tag{29}$$

Given the function $f : \mathbb{R}^2 \to \mathbb{R}$, let $f^{\text{plane}}_{\{A,B,C\}} : \mathbb{R}^2 \to \mathbb{R}$ be the function of the plane passing through the points $(A, f(A))$, $(B, f(B))$, and $(C, f(C))$ of the graph of f. Due to the affinity of $f^{\text{plane}}_{\{A,B,C\}}$, it follows

$$f^{\text{plane}}_{\{A,B,C\}}(P) = \alpha f(A) + \beta f(B) + \gamma f(C). \tag{30}$$

For a convex function f, using the convexity definition, we get the inequality

$$f(P) \leq f^{\text{plane}}_{\{A,B,C\}}(P), \quad \text{if } P \in \text{conv}\{A, B, C\}, \tag{31}$$

and using the affinity of $f^{\text{plane}}_{\{A,B,C\}}$, the reverse inequality

$$f(P) \geq f^{\text{plane}}_{\{A,B,C\}}(P), \quad \text{if } P \in \mathscr{C}_A \cup \mathscr{C}_B \cup \mathscr{C}_C. \tag{32}$$

By the end of the section we will use a planar convex set $\mathscr{C} \subseteq \mathbb{R}^2$ with the nonempty interior $\mathscr{C}^0$. The area of a planar set $\mathscr{A}$ will be denoted with $\text{ar}(\mathscr{A})$.

Lemma 5. *Let $A, B, C \in \mathscr{C} \subseteq \mathbb{R}^2$ be points, $\Delta = \text{conv}\{A, B, C\}$, and $\mathscr{C}_\Delta = \mathscr{C}_A \cup \mathscr{C}_B \cup \mathscr{C}_C$ be the cone union. Let $\sum_{i=1}^n \alpha_i A_i$ be a convex combination of the points $A_i \in \Delta$, and let $\sum_{j=1}^m \beta_j B_j$ be a convex combination of the points $B_j \in \mathscr{C}_\Delta \cap \mathscr{C}$.*

If the above convex combinations have the same center

$$\sum_{i=1}^n \alpha_i A_i = \sum_{j=1}^m \beta_j B_j, \tag{33}$$

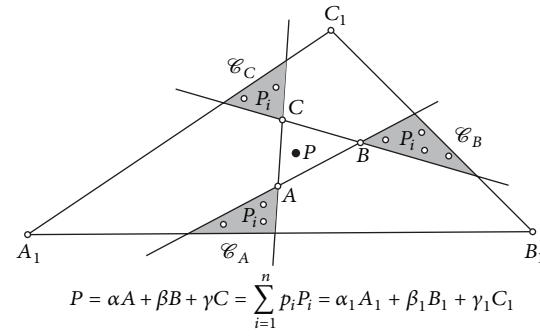

Figure 2: Plane convex combinations with the same center P.

then every convex function $f : \mathscr{C} \to \mathbb{R}$ satisfies the inequality

$$\sum_{i=1}^n \alpha_i f(A_i) \leq \sum_{j=1}^m \beta_j f(B_j). \tag{34}$$

Proof. If the set conv$\{A, B, C\}$ is a real triangle, we can apply the proof of Lemma 1 using $f^{\text{plane}}_{\{A,B,C\}}$ instead of $f^{\text{line}}_{\{a,b\}}$ respecting the plane inequalities in (31)-(32). If conv$\{A, B, C\}$ = conv$\{A, B\}$, then we rely on the proof of Lemma 1 with the chord line $f^{\text{line}}_{\{A,B\}}$. If conv$\{A, B, C\} = \{A\}$, we use any support line $f^{\text{line}}_{\{A\}}$ at the point A. □

Theorem 6. *Let $A, B, C; A_1, B_1, C_1 \in \mathscr{C} \subseteq \mathbb{R}^2$ be points such that*

$$\Delta = \text{conv}\{A, B, C\} \subset \text{conv}\{A_1, B_1, C_1\} = \Delta_1 \tag{35}$$

with $\Delta^0 \neq \emptyset$, and let $\mathscr{C}_\Delta = \mathscr{C}_A \cup \mathscr{C}_B \cup \mathscr{C}_C$ be the cone union. Let $P = \sum_{i=1}^n p_i P_i$ be a convex combination of the points $P_i \in \mathscr{C}_\Delta \cap \Delta_1$ with the center $P \in \Delta^0$.

Then there exist two trinomial convex combinations $\alpha A + \beta B + \gamma C$ and $\alpha_1 A_1 + \beta_1 B_1 + \gamma_1 C_1$ so that

$$\alpha A + \beta B + \gamma C = \sum_{i=1}^n p_i P_i = \alpha_1 A_1 + \beta_1 B_1 + \gamma_1 C_1, \tag{36}$$

and consequently, every convex function $f : \mathscr{C} \to \mathbb{R}$ satisfies the inequality

$$\alpha f(A) + \beta f(B) + \gamma f(C) \leq \sum_{i=1}^n p_i f(P_i) \leq \alpha_1 f(A_1) + \beta_1 f(B_1) + \gamma_1 f(C_1). \tag{37}$$

Proof. First we calculate the coefficients by the formula in (28) to get the equality in (36) and then apply Lemma 5 to its both sides. □

The graphical representation of the equality in (36) can be seen in Figure 2.

Applying the integral method with convex combinations, we obtain the form as follows.

Corollary 7. *Let $A, B, C; A_1, B_1, C_1 \in \mathscr{C} \subseteq \mathbb{R}^2$ be points such that*

$$\Delta = \operatorname{conv}\{A, B, C\} \subset \operatorname{conv}\{A_1, B_1, C_1\} = \Delta_1 \tag{38}$$

with $\Delta^0 \neq \emptyset$, and let $\mathscr{C}_\Delta = \mathscr{C}_A \cup \mathscr{C}_B \cup \mathscr{C}_C$ be the cone union. Let the barycenter P of the set $\mathscr{C}_\Delta \cap \Delta_1$ belong to Δ^0; that is,

$$P\left(\frac{\int_{\mathscr{C}_\Delta \cap \Delta_1} x\,dx\,dy}{\operatorname{ar}(\mathscr{C}_\Delta \cap \Delta_1)}, \frac{\int_{\mathscr{C}_\Delta \cap \Delta_1} y\,dx\,dy}{\operatorname{ar}(\mathscr{C}_\Delta \cap \Delta_1)}\right) \in \Delta^0. \tag{39}$$

Then there exist two trinomial convex combinations $\alpha A + \beta B + \gamma C$ and $\alpha_1 A_1 + \beta_1 B_1 + \gamma_1 C_1$ so that

$$\alpha A + \beta B + \gamma C = P = \alpha_1 A_1 + \beta_1 B_1 + \gamma_1 C_1, \tag{40}$$

and consequently, every convex function $f : \mathscr{C} \to \mathbb{R}$ satisfies the inequality

$$\begin{aligned} &\alpha f(A) + \beta f(B) + \gamma f(C) \\ &\quad \le \frac{\int_{\mathscr{C}_\Delta \cap \Delta_1} f(x, y)\,dx\,dy}{ar(\mathscr{C}_\Delta \cap \Delta_1)} \\ &\quad \le \alpha_1 f(A_1) + \beta_1 f(B_1) + \gamma_1 f(C_1). \end{aligned} \tag{41}$$

If f is concave, then the reverse inequality is valid in (41).

If the conditions of Theorem 6 are valid, then using the summarizing Jensen functional of f for the given convex combination $\sum_{i=1}^n p_i P_i$,

$$J_{p_1P_1+\cdots+p_nP_n}(f) = \sum_{i=1}^n p_i f(P_i) - f\left(\sum_{i=1}^n p_i P_i\right), \tag{42}$$

we get the functional inequality

$$\begin{aligned} J_{\alpha A+\beta B+\gamma C}(f) &\le J_{p_1P_1+\cdots+p_nP_n}(f) \\ &\le J_{\alpha_1A_1+\beta_1B_1+\gamma_1C_1}(f). \end{aligned} \tag{43}$$

If the conditions of Corollary 7 are satisfied, then applying another integrating Jensen functional of f for the given subset $\mathscr{A} \subseteq \mathscr{C}$ with the area $\operatorname{ar}(\mathscr{A}) > 0$,

$$\begin{aligned} J_{\mathscr{A}}(f) = &\frac{\int_{\mathscr{A}} f(x, y)\,dx\,dy}{\operatorname{ar}(\mathscr{A})} \\ &- f\left(\frac{\int_{\mathscr{A}} x\,dx\,dy}{\operatorname{ar}(\mathscr{A})}, \frac{\int_{\mathscr{A}} y\,dx\,dy}{\operatorname{ar}(\mathscr{A})}\right), \end{aligned} \tag{44}$$

we have the inequality

$$J_{\alpha A+\beta B+\gamma C}(f) \le J_{\mathscr{C}_\Delta \cap \Delta_1}(f) \le J_{\alpha_1A_1+\beta_1B_1+\gamma_1C_1}(f). \tag{45}$$

4. Generalization

The aim of the section is to generalize Theorem 6 and Corollary 7 to more dimensions.

If $A_1, \ldots, A_{m+1} \in \mathbb{R}^m$ are points such that the vectors $A_1 - A_2, \ldots, A_1 - A_{m+1}$ are linearly independent, then the convex hull

$$\Delta = \operatorname{conv}\{A_1, \ldots, A_{m+1}\} \tag{46}$$

is called the m-simplex with the vertices $A_1, \ldots, A_{m+1}$. All the simplex vertices can not belong to the same hyperplane in $\mathbb{R}^m$. Any point $P \in \mathbb{R}^m$ can be presented by the unique affine combination

$$P = \sum_{j=1}^{m+1} \alpha_j A_j. \tag{47}$$

If we use the point coordinates, then the coefficients α_j can be determined by the generalized coefficient formula in (28). The combination in (47) is convex if and only if the point P belongs to the m-simplex Δ.

Given the m-simplex with vertices $A_1, \ldots, A_{m+1}$, let $\mathscr{C}_{A_1}$ be the convex cone with the vertex at A_1 spanned by the vectors $A_1 - A_2, \ldots, A_1 - A_{m+1}$ (similarly $\mathscr{C}_{A_2}, \ldots, \mathscr{C}_{A_{m+1}}$); that is,

$$\mathscr{C}_{A_1} = \left\{A_1 + \sum_{k=2}^{m+1} p_k (A_1 - A_k) : p_k \in \mathbb{R}, p_k \ge 0\right\}. \tag{48}$$

Given the function $f : \mathbb{R}^m \to \mathbb{R}$, let $f_\Delta^{\mathrm{hp}} : \mathbb{R}^m \to \mathbb{R}$ be the function of the hyperplane (in $\mathbb{R}^{m+1}$) passing through the points $(A_j, f(A_j))$ of the graph of f. Applying the affinity of f_Δ^{hp} to the combination in (47), it follows

$$f_\Delta^{\mathrm{hp}}(P) = \sum_{j=1}^{m+1} \alpha_j f(A_j). \tag{49}$$

If we use the convex function f, then we get the inequality

$$f(P) \le f_\Delta^{\mathrm{hp}}(P), \quad \text{if } P \in \Delta, \tag{50}$$

and the reverse inequality

$$f(P) \ge f_\Delta^{\mathrm{hp}}(P), \quad \text{if } P \in \bigcup_{j=1}^{m+1} \mathscr{C}_{A_j} \tag{51}$$

which can be proved in the same way as the inequality in (32).

By the end of the section we will use a convex set $\mathscr{C} \subseteq \mathbb{R}^m$ with the nonempty interior $\mathscr{C}^0$. The volume of a set $\mathscr{V} \subset \mathbb{R}^m$ will be denoted with $\operatorname{vol}(\mathscr{V})$.

The generalization of Theorem 6 applied to m-simplexes is as follows.

Theorem 8. *Let $A_1, \ldots, A_{m+1}; B_1, \ldots, B_{m+1} \in \mathscr{C} \subseteq \mathbb{R}^m$ be points such that*

$$\Delta = \operatorname{conv}\{A_1, \ldots, A_{m+1}\} \subset \operatorname{conv}\{B_1, \ldots, B_{m+1}\} = \Delta_1 \tag{52}$$

with $\Delta^0 \neq \emptyset$, and let $\mathscr{C}_\Delta = \cup_{j=1}^{m+1} \mathscr{C}_{A_j}$ be the cone union. Let $P = \sum_{i=1}^{n} p_i P_i$ be a convex combination of the points $P_i \in \mathscr{C}_\Delta \cap \Delta_1$ with the center $P \in \Delta^0$.

Then there exist two $(m+1)$-membered convex combinations $\sum_{j=1}^{m+1} \alpha_j A_j$ and $\sum_{j=1}^{m+1} \beta_j B_j$ so that

$$\sum_{j=1}^{m+1} \alpha_j A_j = \sum_{i=1}^{n} p_i P_i = \sum_{j=1}^{m+1} \beta_j B_j, \tag{53}$$

and consequently every convex function $f : \mathscr{C} \to \mathbb{R}$ satisfies the inequality

$$\sum_{j=1}^{m+1} \alpha_j f\left(A_j\right) \leq \sum_{i=1}^{n} p_i f\left(P_i\right) \leq \sum_{j=1}^{m+1} \beta_j f\left(B_j\right). \tag{54}$$

Using the integrals, we get the form as follows.

Corollary 9. *Let $A_1, \ldots, A_{m+1}; B_1, \ldots, B_{m+1} \in \mathscr{C} \subseteq \mathbb{R}^m$ be points such that*

$$\Delta = \operatorname{conv}\{A_1, \ldots, A_{m+1}\} \subset \operatorname{conv}\{B_1, \ldots, B_{m+1}\} = \Delta_1 \tag{55}$$

with $\Delta^0 \neq \emptyset$, and let $\mathscr{C}_\Delta = \cup_{j=1}^{m+1} \mathscr{C}_{A_j}$ be the cone union. Let the barycenter P of the set $\mathscr{C}_\Delta \cap \Delta_1$ belong to Δ^0; that is,

$$P\left(\frac{\int_{\mathscr{C}_\Delta \cap \Delta_1} x_1 \, dx_1, \ldots, dx_m}{\operatorname{vol}\left(\mathscr{C}_\Delta \cap \Delta_1\right)}, \ldots, \frac{\int_{\mathscr{C}_\Delta \cap \Delta_1} x_m \, dx_1, \ldots, dx_m}{\operatorname{vol}\left(\mathscr{C}_\Delta \cap \Delta_1\right)}\right) \in \Delta^0. \tag{56}$$

Then there exist two $(m+1)$-membered convex combinations $\sum_{j=1}^{m+1} \alpha_j f(A_j)$ and $\sum_{j=1}^{m+1} \beta_j f(B_j)$ so that

$$\sum_{j=1}^{m+1} \alpha_j A_j = P = \sum_{j=1}^{m+1} \beta_j B_j, \tag{57}$$

and consequently every convex function $f : \mathscr{C} \to \mathbb{R}$ satisfies the inequality

$$\sum_{j=1}^{m+1} \alpha_j f\left(A_j\right) \leq \frac{\int_{\mathscr{C}_\Delta \cap \Delta_1} f\left(x_1, \ldots, x_m\right) dx_1, \ldots, dx_m}{\operatorname{vol}\left(\mathscr{C}_\Delta \cap \Delta_1\right)} \leq \sum_{j=1}^{m+1} \beta_j f\left(B_j\right). \tag{58}$$

If f is concave, then the reverse inequality is valid in (58).

If the conditions of Theorem 8 are valid, then using the summarizing Jensen functional of f for the given convex combination $\sum_{i=1}^{n} p_i P_i$, we get the functional inequality

$$J_{\alpha_1 A_1 + \cdots + \alpha_{m+1} A_{m+1}}(f) \leq J_{p_1 P_1 + \cdots + p_n P_n}(f) \leq J_{\beta_1 B_1 + \cdots + \beta_{m+1} B_{m+1}}(f). \tag{59}$$

If the conditions of Corollary 9 are satisfied, then applying another integrating Jensen functional of f for the given subset $\mathscr{V} \subseteq \mathscr{C}$ with the volume $\operatorname{vol}(\mathscr{V}) > 0$,

$$J_{\mathscr{V}}(f) = \frac{\int_{\mathscr{V}} f \, dx_1, \ldots, dx_m}{\operatorname{vol}(\mathscr{V})} - f\left(\frac{\int_{\mathscr{V}} x_1 \, dx_1, \ldots, dx_m}{\operatorname{vol}(\mathscr{V})}, \ldots, \frac{\int_{\mathscr{V}} x_m \, dx_1, \ldots, dx_m}{\operatorname{vol}(\mathscr{V})}\right), \tag{60}$$

we have the inequality

$$J_{\alpha_1 A_1 + \cdots + \alpha_{m+1} A_{m+1}}(f) \leq J_{\mathscr{C}_\Delta \cap \Delta_1}(f) \leq J_{\beta_1 B_1 + \cdots + \beta_{m+1} B_{m+1}}(f). \tag{61}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The work of the first and third authors has been fully supported by Mechanical Engineering Faculty in Slavonski Brod. The work of the second author has been supported by the Natural Science Foundation of Fujian province of China under Grant no. 2012J01014. The authors are grateful to Velimir Pavić (graphic designer at Školska knjiga Zagreb) who has graphically edited Figures 1 and 2.

References

[1] Z. Pavić and S. Wu, "Inequalities for convex functions on simplexes and their cones," *Abstract and Applied Analysis*, vol. 2014, Article ID 690803, 6 pages, 2014.

[2] C. P. Niculescu and L. E. Persson, *Convex Functions and Their Applications*, CMS Books in Mathematics, Springer, New York, NY, USA, 2006.

[3] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and Their Inequalities*, vol. 31 of *Mathematics and Its Applications (East European Series)*, D. Reidel, Dordrecht, The Netherlands, 1988.

[4] J. E. Peajcariaac, F. Proschan, and Y. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, New York, NY, USA, 1992.

[5] Z. Pavić, J. Pečarić, and I. Perić, "Integral, discrete and functional variants of Jensen's inequality," *Journal of Mathematical Inequalities*, vol. 5, no. 2, pp. 253–264, 2011.

[6] J. L. W. V. Jensen, "Om konvekse Funktioner og Uligheder mellem Middelværdier," *Nyt tidsskrift for matematik. B*, vol. 16, pp. 49–68, 1905.

[7] J. L. Jensen, "Sur les fonctions convexes et les inégalités entre les valeurs moyennes," *Acta Mathematica*, vol. 30, no. 1, pp. 175–193, 1906.

[8] J. Mićić, Z. Pavić, and J. Pečarić, "The inequalities for quasiarithmetic means," *Abstract and Applied Analysis*, vol. 2012, Article ID 203145, 25 pages, 2012.

[9] J. Sándor, "On global bounds for generalized Jensen's inequality," *Annals of Functional Analysis*, vol. 4, no. 1, pp. 18–24, 2013.

[10] C. P. Niculescu and C. I. Spiridon, "New Jensen-type inequalities," http://arxiv.org/abs/1207.6877.