

Research Article

Observation of a Class of Disturbance in Time Series Expansion for Fractional Order Systems

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This paper is concerned with the problem of designing disturbance observer for fractional order systems, of which the disturbance is in time series expansion. The stability of a special observer with the selected nonlinear weighted function and transient dynamics function is rigorously analyzed for slowly varying disturbance. In addition, the result is also extended to estimate slope forms disturbance and higher order disturbance of fractional order systems. The efficacy of the proposed method is validated through numerical examples.

1. Introduction

In recent years, fractional order systems (FOSs) have attracted considerable attention from control community, since many engineering plants and processes cannot be described concisely and precisely without the introduction of fractional order calculus [1–4]. Due to the tremendous efforts devoted by researchers, a number of valuable results on stability analysis [5–7] and controller synthesis [8–10] of FOSs have been reported in the literature.

Like the integer order system, the disturbance always exists in the FOSs and usually it is not possible or practical to obtain the exact model of the FOSs, so disturbance or uncertainty observation has been one of the major issues in the control field. A rich body of results about disturbance observation have been reported in the literature with different methods. For example, the so-called Q-filter method estimates the disturbance depending on the inversion of the transfer function. Many applications based on Q-filter have been reported in process control fields [11–13]. The second class mainly refers to the active disturbance rejection control technique proposed by Han [14]. Under this framework, the extended state observer can estimate the disturbance

effectively, which has been demonstrated in many fields [15–17]. The third class, called recursive filtering methods, is from a statistical point of view [18–20]. Some other methods about disturbance observation have been developed in [21–25].

A special kind of disturbance in time series expansion is considered to solve the friction compensation problem in [26]. The algorithms are brief and effective. Additionally, the time series expansion form is the basic form of the nonstochastic disturbance in engineering. Intuitively, such useful methods need to be generalized to FOSs. Maybe the stability is hard to be prove; such disturbance observation for FOSs has not been introduced so far in the literature to the best of the authors' knowledge. The proposed approaches in [26] reveal several important aspects: (1) the linear observer converges in the form of exponent, which means that it will take the observance error infinity time to converge to zero; (2) the observance value may bring forth big overshoot when the disturbance is discontinuous; meanwhile, the possible conflict between the overshoot and rapidity always exists; and (3) the disturbance is not general enough, since its degrees are restricted to integers. Motivated by the reasons stated above, in this paper we focus on the observation of a class of disturbance in time series expansion for FOSs.

The following section is devoted to some basic background materials and the main problems. Discussing the more general disturbance using a special nonlinear function, the main results are presented in Section 3. In Section 4, some numerical examples are provided to illustrate the effectiveness and advantages of the proposed approach. Conclusions are given in Section 5.

2. Problem Formulation and Preliminaries

Consider the following fractional order system:

$$\mathcal{D}^\alpha x = f(x, u; t) + Fd(t), \quad (1)$$

where the system order $0 < \alpha < 1$. $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $d \in \mathbb{R}^r$ are the system state, the control input, and the unknown disturbance, respectively. $f(\cdot)$ and the matrix F with $\text{rank}(F) = r$ are known. It is assumed that the state variable x is measured and the initial state condition $x(0)$ is known.

The reduced-order system can be expressed as

$$F^+ \mathcal{D}^\alpha x = F^+ f(x, u; t) + d(t), \quad (2)$$

where $F^+ = (F^T F)^{-1} F^T$ is the Moore-Penrose pseudoinverse of F .

Consider the disturbance has the following form:

$$d(t) = \sum_{i=0}^k d_i t^{n_i}, \quad (3)$$

where d_i ($i \in \mathcal{X} \triangleq \{0, 1, \dots, k\}$) is constant but unknown and $n_i < n_{i+1}$ ($i \in \mathcal{X} - \{k\}$) holds. Due to the magnitude of n_k , the disturbance $d(t)$ can be divided into the following three categories:

- (i) slowly varying disturbance $n_k < \alpha$;
- (ii) slope forms disturbance $\alpha \leq n_k < 2\alpha$;
- (iii) higher order disturbance $n_k \geq 2\alpha$.

The following Caputo definition [1] is adopted for fractional derivatives of order α for function $f(t)$:

$$\mathcal{D}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad (4)$$

where the fractional order $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, and the Gamma function $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$.

The so-called fractional order integral is just the dual operation of the fractional order differential. If $v(t)$ is the α th order differential of $y(t)$, then

$$\begin{aligned} v(t) &= \mathcal{D}^\alpha (y(t)), \\ y(t) &= \mathcal{F}_\alpha (v(t)). \end{aligned} \quad (5)$$

In order to describe briefly later, we introduce the following definition.

Definition 1. Consider polynomials $\varphi(s)$ with commensurate order α

$$\varphi(s) = s^{n\alpha} + a_0 s^{(n-1)\alpha} + \dots + a_{n-2} s^\alpha + a_{n-1}, \quad (6)$$

and define the corresponding characteristic matrix \mathcal{A} as

$$\mathcal{A} = \begin{bmatrix} -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \quad (7)$$

$\varphi(s)$ in (6) is said to be stable if all the eigenvalues of \mathcal{A} are in the region $\mathcal{D}(\alpha) \triangleq \{s \in \mathbb{C} : |\arg(s)| > \alpha\pi/2\}$.

3. Main Results

3.1. Slowly Varying Disturbance Observer

Theorem 2 (FDOB0). Given a constant matrix $\Lambda_0 = \text{diag}(\lambda_{01}, \lambda_{02}, \dots, \lambda_{0r})$, where $\lambda_{0i} > 0$, $i \in \mathcal{R} \triangleq \{1, \dots, r\}$, the disturbance observer given by

$$\hat{d} = \Lambda_0 g_0(e), \quad (8)$$

$$\mathcal{D}^\alpha z = F^+ f(x, u; t) + \frac{\hat{d}}{ts(t)}$$

is asymptotically convergent to the slowly varying disturbance when the nonlinear weighted function

$$g_0(e) = [g_{01}(e_1) g_{02}(e_2) \dots g_{0r}(e_r)]^T \quad (9)$$

is selected as

$$g_{0i}(e_i) = \begin{cases} \left| \frac{e_i}{w} \right|^\sigma \text{sgn}\left(\frac{e_i}{w}\right), & |e_i| \geq w, \\ \frac{e_i}{w}, & |e_i| < w, \end{cases} \quad (10)$$

where $e \triangleq F^+ x - z = [e_1 e_2 \dots e_r]^T$, $z \in \mathbb{R}^r$, $z(0) = F^+ x(0)$, $\sigma > 0$, $w > 0$ for any $i \in \mathcal{R}$, and the transient dynamics function $ts(t)$ is given by

$$ts(t) = \begin{cases} 0.51 + 0.49 \sin\left(\frac{\pi t}{T_0} - \frac{\pi}{2}\right), & t \leq T_0, \\ 1, & t > T_0. \end{cases} \quad (11)$$

Proof. Defining the observation error $\varepsilon = d - \hat{d}/ts(t)$ yields

$$\mathcal{D}^\alpha \varepsilon = \mathcal{D}^\alpha d - \mathcal{D}^\alpha \left(\frac{\hat{d}}{ts(t)} \right). \quad (12)$$

Based on the Leibniz rule for fractional differentiation, one has

$$\begin{aligned} \mathcal{D}^\alpha \left(\frac{\hat{d}}{ts(t)} \right) &= \sum_{i=0}^{\infty} \binom{\alpha}{i} \left(\frac{1}{ts(t)} \right)^{(i)} \mathcal{D}^{\alpha-i} \hat{d} \\ &= \frac{1}{ts(t)} \mathcal{D}^\alpha \hat{d} + \sum_{i=1}^{\infty} \binom{\alpha}{i} \left(\frac{1}{ts(t)} \right)^{(i)} \mathcal{D}^{\alpha-i} \hat{d} \\ &= \frac{1}{ts(t)} \mathcal{D}^\alpha \hat{d} + h(t), \end{aligned} \tag{13}$$

where the Newtons binomial generalized to noninteger orders as $\binom{\alpha}{i} = \Gamma(\alpha + 1)/\Gamma(\alpha - i + 1)\Gamma(i + 1)$.

Substituting (13) into (12) and linearizing the system at \bar{e} , one can get

$$\begin{aligned} \mathcal{D}^\alpha \varepsilon &= \mathcal{D}^\alpha d - \frac{1}{ts(t)} \Lambda_0 \mathcal{D}^\alpha g_0(e) - h(t) \\ &= \mathcal{D}^\alpha d - \frac{1}{ts(t)} \Lambda_0 J|_{e=\bar{e}} \mathcal{D}^\alpha g_0(e) - h(t) \\ &= \mathcal{D}^\alpha d - \frac{1}{ts(t)} \Lambda_0 J|_{e=\bar{e}} \varepsilon - h(t), \end{aligned} \tag{14}$$

where J is the pseudo-Jacobi matrix of $g_0(e)$ with

$$J = \begin{bmatrix} \frac{\partial g_{01}}{\partial e_1} & \dots & \frac{\partial g_{01}}{\partial e_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{0r}}{\partial e_1} & \dots & \frac{\partial g_{0r}}{\partial e_r} \end{bmatrix} = \begin{bmatrix} \frac{dg_{01}}{de_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{dg_{0r}}{de_r} \end{bmatrix}. \tag{15}$$

Define a new variable $\mu \triangleq 1/ts(t)$; then one has $\mu \in [1, 50]$ and the time-varying system in (14) can be regarded as a time-invariant system (16) with the interval uncertain parameter μ :

$$\mathcal{D}^\alpha \varepsilon = \mathcal{D}^\alpha d - h(t) - \mu \Lambda_0 J|_{e=\bar{e}} \varepsilon. \tag{16}$$

From (10), for all $i \in \mathcal{R}$, one can easily get

$$\frac{dg_{0i}(e_i)}{de_i} = \begin{cases} \frac{\sigma}{w} \left| \frac{e_i}{w} \right|^{\sigma-1}, & |e_i| \geq w, \\ \frac{1}{w}, & |e_i| < w. \end{cases} \tag{17}$$

Note that $\sigma > 0$ and $w > 0$, so the following result establishes:

$$\frac{dg_{0i}(e_i)}{de_i} \in \begin{cases} \left(0, \frac{1}{w} \right], & 0 < \sigma \leq 1, \\ \left[\frac{1}{w}, \infty \right), & \sigma > 1. \end{cases} \tag{18}$$

Thus, the matrix inequality $J|_{e=\bar{e}} > 0$ always holds.

From the Caputo definition of fractional derivatives, one has

$$\mathcal{D}^\alpha d = \sum_{i=0}^k \frac{d_i \Gamma(n_i + 1)}{\Gamma(n_i + 1 - \alpha)} t^{n_i - \alpha}. \tag{19}$$

Due to (8) and $n_k < \alpha$, one obtains that the first term of the right-hand side of (16) vanishes as $t \rightarrow \infty$ and the second term is equal to 0 when $t > T_0$. No matter $\mu = 1$ or $\mu = 50$, the error dynamic system (16) is stable, so (16) is stable for all the admissible uncertainties and the final value of observation error exists.

The Laplace transform of (16) is

$$\begin{aligned} s^\alpha E(s) - s^{\alpha-1} \varepsilon(0) &= s^\alpha D(s) - s^{\alpha-1} d(0) - \Lambda_0 J|_{e=\bar{e}} E(s) - H(s), \end{aligned} \tag{20}$$

where $E(s)$, $D(s)$, and $H(s)$ are the Laplace transforms of $\varepsilon(t)$, $d(t)$, and $h(t)$, respectively.

According to the initial condition, one has that

$$\hat{d}(0) = \Lambda_0 g_0(F^+ x(0) - z(0)) = \Lambda_0 g_0(0) = 0. \tag{21}$$

Consequently, the following equation holds:

$$\varepsilon(0) = d(0) - \hat{d}(0) = d(0). \tag{22}$$

Define $k_i = \mu(dg_{0i}(e_i)/de_i)|_{e_i=\bar{e}_i}$, $i \in \mathcal{R}$, and consider Λ_0 and $J|_{e=\bar{e}}$ are all diagonal; one gets the decoupled error dynamics from (19):

$$s^\alpha E_i(s) + \lambda_{0i} k_i E_i(s) = s^\alpha D_i(s) - H_i(s), \quad i \in \mathcal{R}. \tag{23}$$

For the slowly varying disturbance, one has $\alpha - n_k > 0$ and finite positive constant $\lambda_{0i} k_i$. Then using the Final-Value Theorem for any $i \in \mathcal{R}$, one obtains

$$\begin{aligned} e_i(\infty) &= \lim_{s \rightarrow 0} s E_i(s) \\ &= \lim_{s \rightarrow 0} s \frac{s^\alpha D_i(s) - H_i(s)}{s^\alpha + \lambda_{0i} k_i} \\ &= \lim_{s \rightarrow 0} s \frac{s^\alpha \sum_{i=0}^k (d_i \Gamma(n_i + 1) / s^{n_i+1}) - H_i(s)}{s^\alpha + \lambda_{0i} k_i} = 0. \end{aligned} \tag{24}$$

Therefore, the observation error is asymptotically stable and the initial error depends on the time derivative of the disturbance for any \bar{e} . In other words, the disturbance observer would exactly estimate the disturbance in the steady state. The proof is completed. \square

Remark 3. Regarding stability, any of the matrices whose eigenvalues are all in the region $\mathcal{D}(\alpha)$ can be adopted for Λ_0 . In order to have the decoupled error dynamics, we consider a class of diagonal matrices for Λ_0 and the nonlinear weighted function $g_0(e)$ consists of $g_{0i}(e_i)$ rather than $g_{0i}(e)$, $i \in \mathcal{R}$.

Remark 4. The functions $g_0(e)$ and $ts(t)$ in (8) are not unique. Considering only the stability, any nonlinear or linear function with positive derivative can be adopted for $g_0(e)$, and any nonlinear or linear function with positive value capable of arranging the transition dynamic can be adopted for $ts(t)$. However, without loss of generality, two classes of nonlinear functions with the forms (10) and (11) are considered.

Remark 5. Considering $\sigma = 1, w = 1, T_0 = 1$, and $g_0(e) = F^+x - z$, then FDOB0 will reduce to linear disturbance observer which are right a set of low-pass filters $\lambda_{0i}/(s^\alpha + \lambda_{0i})$, $i \in \mathcal{R}$. Actually, DOB0 in [26] can be viewed as a specific case of such linear FDOB0 with $\alpha = 1$.

3.2. Slope Forms Disturbance Observer. Consider the slope forms disturbance; Theorem 2 is extended to Theorem 6 as follows.

Theorem 6 (FDOB1). *Given two constant matrices $\Lambda_0 = \text{diag}(\lambda_{01}, \lambda_{02}, \dots, \lambda_{0r})$ and $\Lambda_1 = \text{diag}(\lambda_{11}, \lambda_{12}, \dots, \lambda_{1r})$, suppose that λ_{ij} ($i \in \{0, 1\}, j \in \mathcal{R}$) is chosen such that the polynomial $\varphi_j(s) = s^{2\alpha} + \lambda_{0j}s^\alpha + \lambda_{1j}$ is stable; then the disturbance observer given by*

$$\begin{aligned} \hat{d} &= \Lambda_0 g_0(e) + \Lambda_1 \mathcal{F}_\alpha(g_0(e)), \\ \mathcal{D}^\alpha z &= F^+ f(x, u; t) + \frac{\hat{d}}{ts(t)} \end{aligned} \quad (25)$$

is asymptotically convergent to the slope forms disturbance when the initial nonlinear weighted function $g_0(e)$ has the same form as Theorem 2 and the coefficients σ, w , and T_0 are properly selected according to Λ_0 and Λ_1 .

Proof. Consider the same observation error ε with Theorem 2 and linearize the system at \bar{e} ; one can get

$$\begin{aligned} \mathcal{D}^{2\alpha} \varepsilon &= \mathcal{D}^{2\alpha} d - \mathcal{D}^{2\alpha} \hat{d} \\ &= \mathcal{D}^{2\alpha} d - \mu \Lambda_0 \mathcal{D}^{2\alpha} g_0(e) - \mu \Lambda_1 \mathcal{D}^\alpha g_0(e) - h(t) \\ &= \mathcal{D}^{2\alpha} d - \mu \Lambda_0 J|_{e=\bar{e}} \mathcal{D}^{2\alpha} e - \mu \Lambda_1 J|_{e=\bar{e}} \mathcal{D}^\alpha e - h(t) \\ &= \mathcal{D}^{2\alpha} d - \mu \Lambda_0 J|_{e=\bar{e}} \mathcal{D}^\alpha \varepsilon - \mu \Lambda_1 J|_{e=\bar{e}} \varepsilon - h(t), \end{aligned} \quad (26)$$

where J is defined the same with Theorem 2 and $h(t)$ is equal to 0 when $t > T_0$, which is given by

$$h(t) = \sum_{i=1}^{\infty} \binom{2\alpha}{i} \mu^{(i)} \mathcal{D}^{2\alpha-i} \hat{d}. \quad (27)$$

Because Λ_0 and Λ_1 are diagonal matrices and $J|_{e=\bar{e}}$ is a positive definite matrix, one has the decoupled error dynamics. Therefore, the problem becomes that given stable polynomial $\varphi_i(s) = s^{2\alpha} + \lambda_{0i}s^\alpha + \lambda_{1i}$, $i \in \mathcal{R}$, properly select σ, w and T_0 to keep the polynomial $\bar{\varphi}_i(s) = s^{2\alpha} + \lambda_{0i}k_i s^\alpha + \lambda_{1i}k_i$ is always stable for all admissible uncertainties.

Set $\gamma = s^\alpha$; then the characteristic equations $\varphi_i(s) = 0$ and $\bar{\varphi}_i(s) = 0$ can be rewritten as (28) and (29), respectively:

$$\gamma^2 + \lambda_{0i}\gamma + \lambda_{1i} = 0, \quad (28)$$

$$\gamma^2 + \lambda_{0i}k_i\gamma + \lambda_{1i}k_i = 0. \quad (29)$$

Suppose γ_1 and γ_2 are the roots of (28); then the following discussion will be divided into two cases.

(A) γ_1 and γ_2 Are Nonconjugated Roots. Without loss of generality, we set $\gamma_1 = a$ and $\gamma_2 = b$ with $a \leq b < 0$. Then the characteristic equation of (29) can be described as

$$\gamma^2 - (a+b)k_i\gamma + abk_i = 0. \quad (30)$$

Calculating the roots of the quadratic equation yields

$$\gamma_3 = \frac{(a+b)k_i + \sqrt{\Delta}}{2}, \quad \gamma_4 = \frac{(a+b)k_i - \sqrt{\Delta}}{2}, \quad (31)$$

where the discriminant $\Delta = (a+b)^2 k_i^2 - 4abk_i$.

When $k_i \geq 4ab/(a+b)^2$, then $\Delta \geq 0$ and moreover

$$\gamma_4 \leq \gamma_3 < 0. \quad (32)$$

Thus $\bar{\varphi}_i(s)$ is stable.

When $k_i < 4ab/(a+b)^2$, then $\Delta < 0$ and moreover

$$|\arg(\gamma_3)| = |\arg(\gamma_4)| > \frac{\pi}{2} \geq \frac{\alpha\pi}{2}. \quad (33)$$

Thus $\bar{\varphi}_i(s)$ is stable.

Overall, when $\varphi_i(s)$ has two nonconjugated roots, as long as σ and w are positive, $\bar{\varphi}_i(s)$ is stable ($i \in \mathcal{R}$).

(B) γ_1 and γ_2 Are Conjugated Roots. Let $\gamma_1 = c + di$ and $\gamma_2 = c - di$ with $c, d \in \mathbb{R}$, $|\arg(\gamma_1)| > \alpha\pi/2$. Then the characteristic equation of (25) can be described as

$$\gamma^2 - 2ck_i\gamma + (c^2 + d^2)k_i = 0. \quad (34)$$

Calculating the roots of the quadratic equation yields

$$\gamma_3 = \frac{2ck_i + \sqrt{\Delta}}{2}, \quad \gamma_4 = \frac{2ck_i - \sqrt{\Delta}}{2}, \quad (35)$$

where the discriminant $\Delta = 4k_i^2 c^2 - 4k_i(c^2 + d^2)$.

When $k_i \geq (c^2 + d^2)/c^2$, $\Delta \geq 0$, γ_3 and γ_4 are real roots. If $c < 0$, one gets

$$\gamma_4 \leq \gamma_3 < 0. \quad (36)$$

Thus $\bar{\varphi}_i(s)$ is stable.

If $c \geq 0$, one gets

$$0 < \gamma_4 \leq \gamma_3. \quad (37)$$

Thus $\bar{\varphi}_i(s)$ is unstable.

When $k_i < (c^2 + d^2)/c^2$, $\Delta < 0$, γ_3 and γ_4 are conjugated roots. If $c < 0$, one gets

$$|\arg(\gamma_3)| = |\arg(\gamma_4)| > \frac{\pi}{2} \geq \frac{\alpha\pi}{2}. \quad (38)$$

Thus $\bar{\varphi}_i(s)$ is stable.

If $c \geq 0$, $|\arg(\gamma_3)| = |\arg(\gamma_4)| > \alpha\pi/2$ is equivalent to the following:

$$\tan(|\arg(\gamma_3)|) = \tan(|\arg(\gamma_4)|) = \left| \frac{\sqrt{-\Delta}}{2k_i c} \right| > \tan\left(\frac{\alpha\pi}{2}\right). \quad (39)$$

Thus one obtains that if $k_i < (c^2 + d^2)/c^2 [1 + \tan^2(\alpha\pi/2)]$, $\bar{\varphi}_i(s)$ is stable.

To sum up the above arguments, if $\varphi_i(s)$ has two conjugated roots, one gets the following results.

- (i) When $c < 0$, $\bar{\varphi}_i(s)$ ($i \in \mathcal{R}$) is always stable with $\sigma > 0$, $w > 0$.
- (ii) When $c \geq 0$, both $\sigma < 1$, $w < 50c^2[1 + \tan^2(\alpha\pi/2)]/(c^2 + d^2)^2$ and $\sigma \geq 1$, $w < c^2/(c^2 + d^2)^2$ can make sure $\bar{\varphi}_i(s)$ ($i \in \mathcal{R}$) is stable.

Consequently, selecting σ, w properly, all the error dynamic systems (26) are stable and the final value of observation errors exist.

When $0 < 2\alpha \leq 1$, using the Laplace transform to (26), one has

$$\begin{aligned} & s^{2\alpha} E(s) - s^{2\alpha-1} \varepsilon(0) \\ &= s^{2\alpha} D(s) - s^{2\alpha-1} d(0) - s^\alpha \mu \Lambda_0 J|_{e=\bar{e}} E(s) \\ & \quad + s^{\alpha-1} \varepsilon(0) - \mu \Lambda_1 J|_{e=\bar{e}} E(s) - H(s). \end{aligned} \tag{40}$$

According to Theorem 2, one knows $e(0) = d(0)$. Therefore, (40) can be simplified as

$$\begin{aligned} & s^{2\alpha} E(s) + s^\alpha \mu \Lambda_0 J|_{e=\bar{e}} E(s) + \mu \Lambda_1 J|_{e=\bar{e}} E(s) \\ &= s^{2\alpha} D(s) + s^{\alpha-1} \varepsilon(0) H(s). \end{aligned} \tag{41}$$

By using the Final-Value Theorem for any $i \in \mathcal{R}$, it follows that

$$\begin{aligned} \varepsilon_i(\infty) &= \lim_{s \rightarrow 0} s E_i(s) \\ &= \lim_{s \rightarrow 0} s \frac{s^{2\alpha} D_i(s) + s^{\alpha-1} \varepsilon_i(0)}{s^{2\alpha} + k_i \lambda_{0i} s^\alpha + k_i \lambda_{1i}} = 0. \end{aligned} \tag{42}$$

When $2\alpha > 1$, using the Laplace transform to (26), one has

$$\begin{aligned} & s^{2\alpha} E(s) - s^{2\alpha-1} \varepsilon(0) - s^{2\alpha-2} \dot{\varepsilon}(0) \\ &= s^{2\alpha} D(s) - s^{2\alpha-1} d(0) - s^{2\alpha-2} \dot{d}(0) \\ & \quad - s^\alpha \mu \Lambda_0 J|_{e=\bar{e}} E(s) + s^{\alpha-1} \varepsilon(0) \\ & \quad - \mu \Lambda_1 J|_{e=\bar{e}} E(s) - H(s). \end{aligned} \tag{43}$$

Proceeding forward, we have $e(0) = d(0)$. Based on the Final-Value Theorem for any $i \in \mathcal{R}$, one has

$$\begin{aligned} \varepsilon_i(\infty) &= \lim_{x \rightarrow 0} s E_i(s) \\ &= \lim_{x \rightarrow 0} s \frac{s^{2\alpha} D_i(s) + s^{\alpha-1} \varepsilon_i(0) + s^{2\alpha-2} \dot{\varepsilon}_i(0) - s^{2\alpha-2} \dot{d}_i(0)}{s^{2\alpha} + k_i \lambda_{0i} s^\alpha + k_i \lambda_{1i}} \\ &= 0. \end{aligned} \tag{44}$$

Therefore, for any \bar{e} the observation error is asymptotically stable and convergent to zero. To sum up, Theorem 6 has been completely proved. \square

Remark 7. Also considering that $\sigma = 1$, $w = 1$, $T_0 = 0$, and $g_0(e) = F^+ x - z$, then FODBI can reduce to linear disturbance observer, which actually consists of a set of low-pass filters $(\lambda_{0i} s^\alpha + \lambda_{1i}) / (s^{2\alpha} + \lambda_{0i} s^\alpha + \lambda_{1i})$, $i \in \mathcal{R}$. Set α further; FDOBI results in DOBI in [26].

3.3. Higher Order Disturbance Observer. According to the understanding of the aforementioned approaches, one generalizes Theorem 2 further to observe the higher order disturbance.

Theorem 8 (FDOBp). *Given constant matrices $\Lambda_i = \text{diag}(\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{ip})$ and nonlinear weighted functions $g_i(e) = \mathcal{F}_{i\alpha}(g_0(e))$, $i \in \mathcal{P} \triangleq \{0, 1, \dots, p\}$, where λ_{ij} ($j \in \mathcal{R}$) is chosen from the stable polynomial $\varphi_j(s) = s^{(p+1)\alpha} + \sum_{i=0}^p \lambda_{ij} s^{(p-i)\alpha}$, then the disturbance observer given by*

$$\hat{d} = \sum_{i=0}^p \Lambda_i g_i(e), \tag{45}$$

$$\mathcal{D}^\alpha z = F^+ f(x, u; t) + \frac{\hat{d}}{ts(t)}$$

is asymptotically convergent to the higher order disturbance with $p\alpha \leq n_k < (p+1)\alpha$ when $g_0(e)$ has the same form as Theorem 2 and the coefficients σ, w , and T_0 are properly selected according to $\Lambda_i, i \in \mathcal{P}$.

Remark 9. The theorem can be generalized by using the similar aforementioned approach; wherefore the proof is omitted here.

Remark 10. Set $\sigma = 1$, $w = 1$, $T_0 = 0$, and $g_0(e) = F^+ x - z$; then FDOBp becomes a linear disturbance observer, which is really a set of low-pass filters $(\lambda_{0i} s^{p\alpha} + \dots + \lambda_{(p-1)i} s + \lambda_{pi}) / (s^{(p+1)\alpha} + \dots + \lambda_{(p-1)i} s + \lambda_{pi})$, $i \in \mathcal{R}$. Furthermore, set $\alpha = 1$; then FODBp will reduce to the higher order in [26].

Remark 11. Under certain conditions, all the above mentioned theorems (Theorems 2, 6, and 8) can be extended to the case $1 < \alpha < 2$.

Remark 12. The proposed method has a great design freedom. With the appropriate parameters, the FDOBp is a fast nonovershooting disturbance observer for the FOSs with the disturbance ($p\alpha < (p+1)\alpha$). In addition, for those disturbances with small high order component, our method still works well. One application with the most potential is the realization of disturbance observance-based control (DOBC), which means that one can use the proposed observance method to estimate the external disturbance and the uncertainty and then achieve the desired response by an appropriate control.

4. Illustrative Examples

Example 1. Consider the system [27] as follows:

$$\mathcal{D}^{0.8} x(t) = f(x, u; t) + Fd(t), \quad x(0) = 4, \tag{46}$$

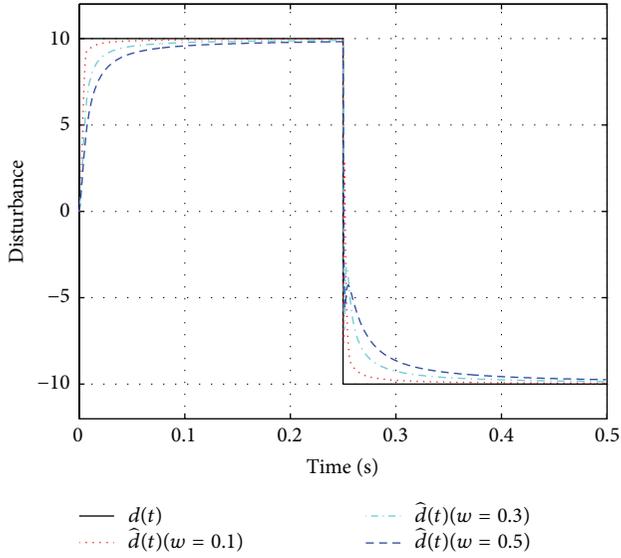


FIGURE 4: Results of Example 1 with different w .

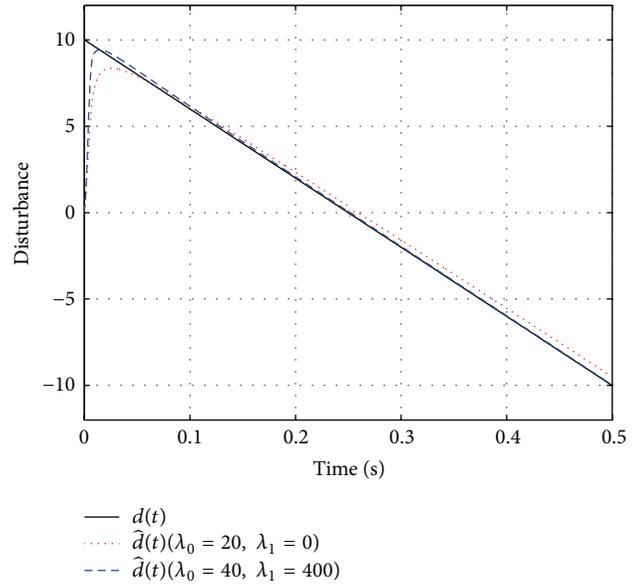


FIGURE 6: Results of Example 2 with FDOB0 and FDOB1.

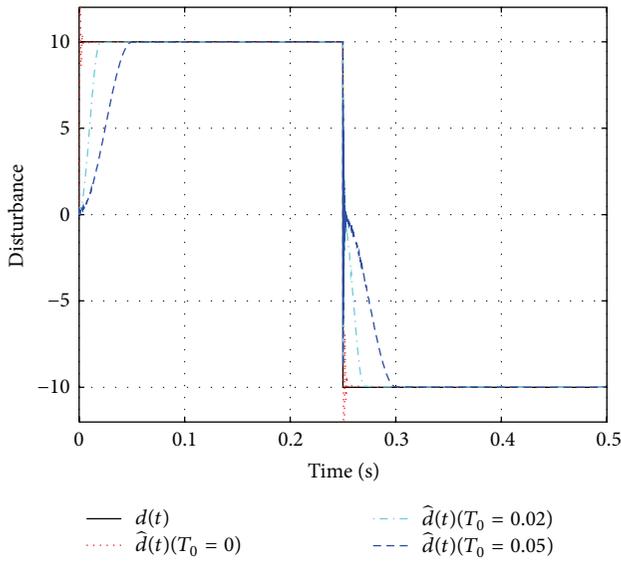


FIGURE 5: Results of Example 1 with different T_0 .

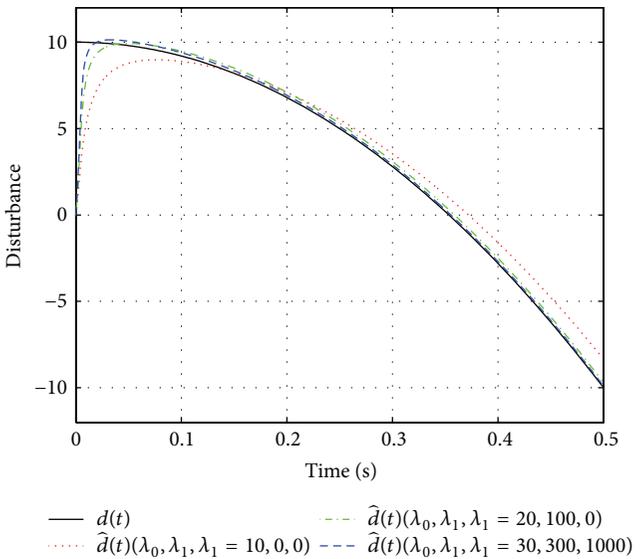


FIGURE 7: Results of Example 3 with FDOB0, FDOB1, and FDOB2.

the disturbance observer can be made dynamically fast by incorporating the higher order integrals.

Example 4. Considering the same system in (46) with a square wave disturbance, we design FDOB0 and state feedback controller, respectively, as follows:

$$\begin{aligned} \hat{d}(t) &= \lambda_0 g_0 (F^+ x(t) - z(t)), \\ \mathcal{D}^{0.8} z(t) &= F^+ f(x, u; t) + \frac{\hat{d}}{ts(t)}, \\ u(t) &= Kx(t) - F\hat{d}(t), \end{aligned} \tag{49}$$

where $K = \text{diag}(-10, -51, -13)$.

This simulation is implemented with $\sigma = 5$, $w = 0.3$, and $T_0 = 0.005$. The related results are shown in Figure 8, from which one can observe that the state feedback controller based on FDOB0 can stabilize the original nonlinear system well. Also, it explicitly shows that the FDOB0 plays a good role in observing and eliminating the system disturbance. Combined with the nominal outer-loop controller, the FDOB0 can be used as an inner-loop compensator to control the FOSs with such disturbance.

5. Conclusions

In this paper, the methods of observer for fractional order systems in time series expansion disturbance have been

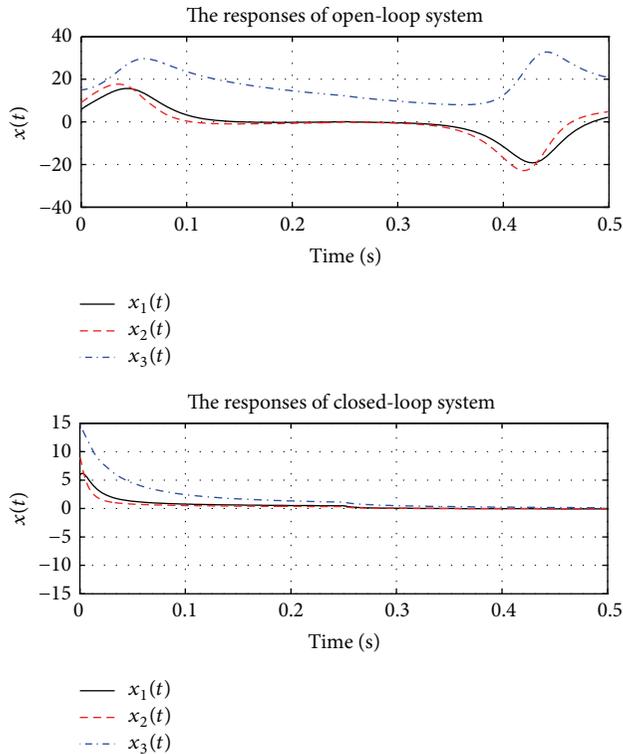


FIGURE 8: Results of Example 4 with control or not.

investigated. According to the maximum degree of disturbance polynomials, the disturbances are divided into three categories. And then, different observers have been designed for three disturbances, respectively. Compared to existing integer order results, the new proposed approaches have greater design freedom and the designed observers have a faster convergent speed. The numerical examples have shown the advantages and the efficiency of the proposed design methods. It is believed that the approaches provide a new avenue to solve such problems. The interesting future topics involve the following cases:

- (i) to study the problem of noise effect reduction in case the measured state is mixed with the measurement noise;
- (ii) to discuss the problem of output-based method when only partial states are measurable;
- (iii) to investigate the problem considering the missing measurement data.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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