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*Research Article*

# Solving Nonstiff Higher-Order Ordinary Differential Equations Using 2-Point Block Method Directly

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We describe the development of a 2-point block backward difference method (2PBBD) for solving system of nonstiff higher-order ordinary differential equations (ODEs) directly. The method computes the approximate solutions at two points simultaneously within an equidistant block. The integration coefficients that are used in the method are obtained only once at the start of the integration. Numerical results are presented to compare the performances of the method developed with 1-point backward difference method (1PBD) and 2-point block divided difference method (2PBDD). The result indicated that, for finer step sizes, this method performs better than the other two methods, that is, 1PBD and 2PBDD.

## 1. Introduction

In this paper, we consider the system of $d$th order ODEs of the form

$$y_i^{(d_i)} = f_i\left(x, \widetilde{Y}\right), \quad i = 1, 2, \ldots, s, \tag{1}$$

with $\widetilde{Y}(a) = \widetilde{\eta}$ in the interval $a \le x \le b$, where

$$\begin{aligned} \widetilde{Y}(x) &= \left(y_1, \ldots, y_1^{(d_1-1)}, \ldots, y_s, \ldots, y_s^{(d_s-1)}\right), \\ \widetilde{\eta} &= \left(\eta_1, \ldots, \eta_1^{(d_1-1)}, \ldots, \eta_s, \ldots, \eta_s^{(d_s-1)}\right). \end{aligned} \tag{2}$$

For simplicity of discussion and without loss of generality, we consider the single equation

$$y^{(d)} = f\left(x, \widetilde{Y}\right), \qquad \widetilde{Y}(a) = \widetilde{\eta}, \tag{3}$$

where

$$\widetilde{Y}^T = \left(y, y', \ldots, y^{(d-1)}\right), \qquad \widetilde{\eta}^T = \left(\eta, \eta', \ldots, \eta^{(d-1)}\right). \tag{4}$$

As shown in Figure 1, here the 2-point block method, the interval $[a, b]$, is divided into series of blocks with each block containing two points; that is, $x_{n-1}$ and $x_n$ is the first block while $x_{n+1}$ and $x_{n+2}$ is the second block, where solutions to (3) are to be computed.

Previous works on block method for solving (3) directly are given by Milne [1], Rosser [2], Shampine and Watts [3], and Chu and Hamilton [4]. According to Omar [5], both implicit and explicit block Adams methods in their divided difference form are developed for the solution of higher-order ODEs. Majid [6] has derived a code based on the variable step size and order of fully implicit block method to solve nonstiff higher-order ODEs directly. Ibrahim [7] has developed a new block backward differentiation formula method of variable step size for solving first- and second-order ODEs directly. Suleiman et al. [8] have introduced one-point backward difference methods for solving higher-order ODEs. Hence, this motivates us to extend the method to block method in solving nonstiff higher-order ODEs.

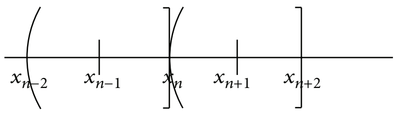


FIGURE 1: 2-point method.

## 2. The Formulation of the Predict-Evaluate-Correct-Evaluate (PECE) Multistep Block Method in Its Backward Difference Form (MSBBD) for Nonstiff Higher-Order ODEs

The code developed will be using the PECE mode with constant stepsize. The predictor and corrector for first and second point will have the following form.

Predictor:

$$^{\mathrm{pr}}y_{n+r}^{(d-t)} = \sum_{i=0}^{t-1}\frac{h^i}{i!}y_n^{(d-t+i)} + h^t\sum_{i=0}^{k-1}\gamma_{r,t,i}\nabla^i f_n, \tag{5}$$

where $\gamma_{r,t,i}$ is coefficient for predictor for $r = 1, 2$ and $t = 1, 2, \ldots, d$.

Corrector:

$$y_{n+r}^{(d-t)} = \sum_{i=0}^{t-1}\frac{h^i}{i!}y_n^{(d-t+i)} + h^t\sum_{i=0}^{k}\gamma_{r,t,i}^{*}\nabla^i f_{n+r}, \tag{6}$$

where $\gamma_{r,t,i}^{*}$ is coefficient for corrector for $r = 1, 2$ and $t = 1, 2, \ldots, d$.

We also formulate the corrector in terms of the predictor. Both points $y_{n+1}$ and $y_{n+2}$ can be written as

$$y_{n+1}^{(d-t)} = {}^{\mathrm{pr}}y_{n+1}^{(d-t)} + h\left[\gamma_{1,t,k}\nabla^k f_{n+1}\right], \tag{7}$$

$$y_{n+2}^{(d-t)} = {}^{\mathrm{pr}}y_{n+2}^{(d-t)} + h\left[\gamma_{2,t,k}\nabla^k f_{n+2} - \gamma_{2,t,k-1}\nabla^{k+1} f_{n+2}\right]. \tag{8}$$

We derived the formulation for both the predictor and corrector.

## 3. Derivation for Higher-Order Explicit Integration Coefficients

*3.1. For the First Point.* The derivation for up to third-order explicit integration coefficients for the first point $y_{n+1}$ has been given by Suleiman et al. [8].

*3.2. For the Second Point.* Integrating (3) once yields

$$y^{(d-1)}(x_{n+2}) = y^{(d-1)}(x_n) + \int_{x_n}^{x_{n+2}} f\left(x, y, y', y'', \ldots, y^{(d-1)}\right) dx. \tag{9}$$

Let $P_n(x)$ be the interpolating polynomial which interpolates the $k$ values $(x_n, f_n), (x_{n-1}, f_{n-1}), \ldots, (x_{n-k+1}, f_{n-k+1})$; then

$$P_n(x) = \sum_{i=0}^{k-1}(-1)^i\binom{-s}{i}\nabla^i f_n. \tag{10}$$

Approximating $f$ in (6) with $P_n(x)$ and letting

$$x = x_n + sh \quad \text{or} \quad s = \frac{x - x_n}{h} \tag{11}$$

gives

$$y^{(d-1)}(x_{n+2}) = y^{(d-1)}(x_n) + \int_0^2 \sum_{i=0}^{k-1}(-1)^i\binom{-s}{i}\nabla^i f_n h\, ds \tag{12}$$

or

$$y^{(d-1)}(x_{n+2}) = y^{(d-1)}(x_n) + h\sum_{i=0}^{k-1}\gamma_{2,1,i}\nabla^i f_n, \tag{13}$$

where

$$\gamma_{2,1,i} = (-1)^i\int_0^2\binom{-s}{i}ds. \tag{14}$$

Define the generating function $G_1(t)$ for the coefficient $\gamma_{2,1,i}$ as follows:

$$G_1(t) = \sum_{i=0}^{\infty}\gamma_{2,1,i}t^i. \tag{15}$$

Substituting $\gamma_{2,1,i}$ in (14) into $G_1(t)$ gives

$$\begin{aligned} G_1(t) &= \sum_{i=0}^{\infty}(-t)^i\int_0^2\binom{-s}{i}ds, \\ G_1(t) &= \int_0^2 (1-t)^{-s}ds, \\ G_1(t) &= \int_0^2 e^{-s\log(1-t)}ds, \end{aligned} \tag{16}$$

which leads to

$$G_1(t) = -\left[\frac{(1-t)^{-2}}{\log(1-t)} - \frac{1}{\log(1-t)}\right]. \tag{17}$$

Equation (17) can be written as

$$-\left(\sum_{i=0}^{\infty}\gamma_{2,1,i}t^i\right)\log(1-t) = (2-t)\left[\frac{t}{(1-t)^2}\right] \tag{18}$$

or

$$\begin{aligned} &\left(\gamma_{2,1,0} + \gamma_{2,1,1}t + \gamma_{2,1,2}t^2 + \cdots\right)\left(t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots\right) \\ &\quad = (2-t)\left(t + 2t^2 + 3t^3 + \cdots\right). \end{aligned} \tag{19}$$

Hence, the coefficients of $\gamma_{2,1,i}$ are given by

$$\sum_{i=0}^{k}\left(\frac{\gamma_{2,1,i}}{k-i+1}\right)=k+2,$$
$$\gamma_{2,1,k}=(k+2)-\sum_{i=0}^{k-1}\frac{\gamma_{2,1,i}}{(k-i+1)},\quad k=1,2,\ldots,\ \gamma_{2,1.0}=2. \tag{20}$$

Integrating (1) twice yields

$$y^{(d-2)}(x_{n+2})=y^{(d-2)}(x_n)+hy^{(d-1)}(x_n)+h^2\sum_{i=0}^{k-1}\gamma_{2,2,i}\nabla^i f_n. \tag{21}$$

Substituting $x$ with $s$ gives

$$\gamma_{2,2,i}=(-1)^i\int_0^2\frac{(2-s)}{1!}\binom{-s}{i}ds. \tag{22}$$

The generating function of the coefficient $\gamma_{2,2,i}$ is defined as follows:

$$G_2(t)=\sum_{i=0}^{\infty}\gamma_{2,2,i}t^i. \tag{23}$$

Substituting (22) into $G_2(t)$ above gives

$$G_2(t)=\int_0^2\frac{(2-s)}{1!}e^{-s\log(1-t)}ds. \tag{24}$$

Substituting $G_1(t)$ into (24) yields

$$G_2(t)=\frac{1}{1!}\left[\frac{2}{\log(1-t)}-\frac{1!G_1(t)}{\log(1-t)}\right]. \tag{25}$$

Equation (25) can be written as

$$\left(\sum_{i=0}^{\infty}\gamma_{2,2,i}t^i\right)\log(1-t)=\frac{1}{1!}\left[2-1!G_1(t)\right] \tag{26}$$

or

$$\left(\gamma_{2,2,0}+\gamma_{2,2,1}t+\gamma_{2,2,2}t^2+\cdots\right)\left(t+\frac{t^2}{2}+\frac{t^3}{3}+\cdots\right)$$
$$=\frac{1}{1!}\left[-2+1!\left(\gamma_{2,1,0}+\gamma_{2,1,1}t+\gamma_{2,1,2}t^2+\cdots\right)\right]. \tag{27}$$

Hence the coefficients of $\gamma_{2,2,k}$ in relation to coefficients of the previous order $\gamma_{2,1,k}$ are given by

$$\sum_{i=0}^{k}\frac{\gamma_{2,2,i}}{k-i+1}=\gamma_{2,1,k+1}, \tag{28}$$

$$\gamma_{2,2,0}=\gamma_{2,1,1},$$
$$\gamma_{2,2,k}=\gamma_{2,1,k+1}-\sum_{i=0}^{k-1}\frac{\gamma_{2,2,i}}{k-i+1},\quad k=1,2,\ldots. \tag{29}$$

By using the same process previously, we note that for integrating $(d-1)$ times yield

$$G_{(d-1)}(t)=\int_0^2\frac{(2-s)^{(d-2)}}{(d-2)!}e^{-s\log(1-t)}ds, \tag{30}$$

$$G_{(d-1)}(t)=\frac{1}{(d-2)!}\left[\frac{2^{(d-2)}}{\log(1-t)}-\frac{(d-2)!G_{(d-2)}(t)}{\log(1-t)}\right], \tag{31}$$

and, from (29), we get

$$\gamma_{2,(d-1),0}=\gamma_{2,(d-2),1},$$
$$\gamma_{2,(d-1),k}=\gamma_{2,(d-2),k+1}-\sum_{i=0}^{k-1}\frac{\gamma_{2,(d-1),i}}{k-i+1},\quad k=1,2,\ldots. \tag{32}$$

Integrating $(d)$ times yield

$$y(x_{n+2})=y(x_n)+hy'(x_n)+\cdots+\frac{h^{(d-1)}}{(d-1)!}y^{(d-1)}(x_n)$$
$$+\int_{x_n}^{x_{n+2}}\frac{(x_{n+2}-x)^{(d-1)}}{(d-1)!}$$
$$\times f\left(x,y,y',y'',\ldots,y^{(d-1)}\right)dx \tag{33}$$

or, in the backward difference formulation, given by

$$y(x_{n+2})=y(x_n)+hy'(x_n)+\cdots+\frac{h^{(d-1)}}{(d-1)!}y^{(d-1)}(x_n)$$
$$+h^{(d)}\sum_{i=0}^{k-1}\gamma_{2,(d),i}\nabla^i f_n, \tag{34}$$

where

$$\gamma_{2,(d),i}=(-1)^i\int_0^2\frac{(2-s)^{(d-1)}}{(d-1)!}\binom{-s}{i}ds. \tag{35}$$

The generating function

$$G_{(d)}(t)=\sum_{i=0}^{\infty}\gamma_{2,(d),i}t^i. \tag{36}$$

Substituting (35) into $G_{(d)}(t)$ above yields

$$G_{(d)}(t)=\int_0^2\frac{(2-s)^{(d-1)}}{(d-1)!}e^{-s\log(1-t)}ds. \tag{37}$$

As in (30), we now substitute $G_{(d-1)}(t)$ in (37) giving

$$G_{(d)}(t) = \frac{1}{(d-1)!}\left[\frac{2^{(d-1)}}{\log(1-t)} - \frac{(d-1)!G_{(d-1)}(t)}{\log(1-t)}\right]. \quad (38)$$

Equation (38) can be written as

$$\left(\sum_{i=0}^{\infty}\gamma_{2,(d),i}t^i\right)\log(1-t) = \frac{1}{(d-1)!}\left[2^{(d-1)} - (d-1)!G_{(d-1)}(t)\right] \quad (39)$$

or

$$\left(\gamma_{2,(d),0} + \gamma_{2,(d),1}t + \gamma_{2,(d),2}t^2 + \cdots\right)\left(t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots\right) = \frac{1}{(d-1)!} \times \left[\begin{matrix} -2^{(d-1)} \\ + (d-1)!\left(\gamma_{2,(d-1),0} + \gamma_{2,(d-1),1}t + \gamma_{2,(d-1),2}t^2 + \cdots\right)\end{matrix}\right]. \quad (40)$$

Hence the coefficients of $\gamma_{2,(d),k}$ in relation to coefficients of the previous order $\gamma_{2,(d-1),k}$ are given by

$$\sum_{i=0}^{k}\frac{\gamma_{2,(d),i}}{k-i+1} = \gamma_{2,(d-1),k+1},$$
$$\gamma_{2,(d),0} = \gamma_{2,(d-1),1}, \quad (41)$$
$$\gamma_{2,(d),k} = \gamma_{2,(d-1),k+1} - \sum_{i=0}^{k-1}\frac{\gamma_{2,(d),i}}{k-i+1}, \quad k = 1, 2, \ldots.$$

## 4. Derivation for Higher-Order Implicit Integration Coefficients

*4.1. For the First Point.* The derivation for up to third-order implicit integration coefficients for the first point $y_{n+1}$ has been given by Suleiman et al. [8].

*4.2. For the Second Point.* Integrating (3) once yields

$$y^{(d-1)}(x_{n+2}) = y^{(d-1)}(x_n) + \int_{x_n}^{x_{n+2}} f\left(x, y, y', y'', \ldots, y^{(d-1)}\right)dx. \quad (42)$$

Let $P_n(x)$ be the interpolating polynomial which interpolates the $k$ values $(x_n, f_n), (x_{n-1}, f_{n-1}), \ldots, (x_{n-k+1}, f_{n-k+1})$; then

$$P_n(x) = \sum_{i=0}^{k}(-1)^i\binom{-s}{i}\nabla^i f_{n+2}. \quad (43)$$

As in the previous derivation, we choose

$$x = x_{n+2} + sh \quad \text{or} \quad s = \frac{x - x_{n+2}}{h}. \quad (44)$$

Replacing $x$ by $s$ yields

$$y^{(d-1)}(x_{n+2}) = y^{(d-1)}(x_n) + \int_{-2}^{0}\sum_{i=0}^{k}(-1)^i\binom{-s}{i}\nabla^i f_{n+2}h\,ds. \quad (45)$$

Simplify

$$y^{(d-1)}(x_{n+2}) = y^{(d-1)}(x_n) + h\sum_{i=0}^{k}\gamma^*_{2,1,i}\nabla^i f_{n+2}, \quad (46)$$

where

$$\gamma^*_{2,1,i} = (-1)^i\int_{-2}^{0}\binom{-s}{i}ds. \quad (47)$$

Define the generating function $G_1^*(t)$ for the coefficient $\gamma^*_{2,1,i}$ as follows:

$$G_1^*(t) = \sum_{i=0}^{\infty}\gamma^*_{2,1,i}t^i. \quad (48)$$

or

$$G_1^*(t) = \sum_{i=0}^{\infty}(-t)^i\int_{-2}^{0}\binom{-s}{i}ds, \quad (49)$$

$$G_1^*(t) = \int_{-2}^{0}(1-t)^{-s}ds, \quad (50)$$

$$G_1^*(t) = \int_{-2}^{0}e^{-s\log(1-t)}ds, \quad (51)$$

which leads to

$$G_1^*(t) = -\left[\frac{1}{\log(1-t)} - \frac{(1-t)^2}{\log(1-t)}\right]. \quad (52)$$

For the case $t = 2$, the approximate solution of $y$ has the form

$$y^{(d-2)}(x_{n+2}) = y^{(d-2)}(x_n) + hy^{(d-1)}(x_n) + \int_{x_n}^{x_{n+2}}\frac{(x_{n+2}-x)^{(1)}}{1!} \times f\left(x, y, y', y'', \ldots, y^{(d-1)}\right)dx. \quad (53)$$

The coefficients are given by

$$\gamma^*_{2,2,i} = (-1)^i\int_{-2}^{0}\frac{(-s)}{1!}\binom{-s}{i}ds, \quad (54)$$

where $\gamma^*_{2,2,i}$ are the coefficients of the backward difference formulation of (54) which can be represented by

$$y^{(d-2)}(x_{n+2}) = y^{(d-2)}(x_n) + hy^{(d-1)}(x_n) + h^2\sum_{i=0}^{k}\gamma^*_{2,2,i}\nabla^i f_{n+2}. \quad (55)$$

Define the generating function of the coefficient $\gamma^*_{2,2,i}$ as follows:

$$G_2^*(t) = \sum_{i=0}^{\infty} \gamma^*_{2,2,i} t^i. \tag{56}$$

Substituting (54) into $G_2^*(t)$ above gives

$$G_2^*(t) = \int_{-2}^{0} \frac{(-s)}{1!} e^{-s\log(1-t)} ds. \tag{57}$$

Solving (57) with the substitution of (51) produces the relationship

$$G_2^*(t) = \frac{1}{1!}\left[\frac{2(1-t)^2}{\log(1-t)} - \frac{1!G_1^*(t)}{\log(1-t)}\right]. \tag{58}$$

By using the same process previously, we note that for integrating $(d-1)$ times yield

$$G_{(d-1)}^*(t) = \int_{-2}^{0} \frac{(-s)^{(d-2)}}{(d-2)!} e^{-s\log(1-t)} ds, \tag{59}$$

$$G_{(d-1)}^*(t) = \frac{1}{(d-2)!}\left[\frac{2^{(d-2)}(1-t)^2}{\log(1-t)} - \frac{(d-2)!G_{(d-2)}^*(t)}{\log(1-t)}\right]. \tag{60}$$

Integrating $(d)$ times yield

$$y(x_{n+2}) = y(x_n) + hy'(x_n) + \cdots + \frac{h^{(d-1)}}{(d-1)!} y^{(d-1)}(x_n) + \int_{x_n}^{x_{n+2}} \frac{(x_{n+2}-x)^{(d-1)}}{(d-1)!} \times f\left(x, y, y', y'', \ldots, y^{(d-1)}\right) dx. \tag{61}$$

The coefficients are given by

$$\gamma^*_{2,(d),i} = (-1)^i \int_{-2}^{0} \frac{(-s)^{(d-1)}}{(d-1)!} \binom{-s}{i} ds, \tag{62}$$

where $\gamma^*_{2,(d),i}$ are the coefficients of the backward difference formulation of (62) which can be represented by

$$y(x_{n+2}) = y(x_n) + hy'(x_n) + \cdots + \frac{h^{(d-1)}}{(d-1)!} y^{(d-1)}(x_n) + h^{(d)} \sum_{i=0}^{k} \gamma^*_{2,(d),i} \nabla^i f_{n+2}. \tag{63}$$

Define the generating function $G_{(d)}^*(t)$ of the coefficient $\gamma^*_{2,(d),i}$ as follows:

$$G_{(d)}^*(t) = \sum_{i=0}^{\infty} \gamma^*_{2,(d),i} t^i. \tag{64}$$

Substituting (62) into $G_{(d)}^*(t)$ above gives

$$G_{(d)}^*(t) = \int_{-2}^{0} \frac{(-s)^{(d-1)}}{(d-1)!} e^{-s\log(1-t)} ds. \tag{65}$$

Solving (65) with the substitution of (59) produces the relationship

$$G_{(d)}^*(t) = \frac{1}{(d-1)!}\left[\frac{2^{(d-1)}(1-t)^2}{\log(1-t)} - \frac{(d-1)!G_{(d-1)}^*(t)}{\log(1-t)}\right]. \tag{66}$$

## 5. The Relationship between the Explicit and Implicit Coefficients

*5.1. For the First Point.* Calculating the integration coefficients directly is time consuming when large numbers of integration are involved. An efficient technique of computing the coefficients is by formulating a recursive relationship between them. With this recursive relationship, we are able to obtain the implicit integration coefficient with minimal effort. The relationship between the explicit and implicit coefficients for the first point $y_{n+1}$ is already given by Suleiman et al. [8].

*5.2. For the Second Point.* For first-order coefficients,

$$G_1^*(t) = -\left[\frac{1}{\log(1-t)} - \frac{(1-t)^2}{\log(1-t)}\right]. \tag{67}$$

It can be written as

$$G_1^*(t) = -(1-t)^2\left[\frac{1}{(1-t)^2\log(1-t)} - \frac{1}{\log(1-t)}\right]. \tag{68}$$

By substituting

$$G_1(t) = -\left[\frac{1}{(1-t)^2\log(1-t)} - \frac{1}{\log(1-t)}\right] \tag{69}$$

into (68), we have

$$G_1^*(t) = (1-t)^2 G_1(t), \quad \left(\sum_{i=0}^{\infty} \gamma^*_{2,1,i} t^i\right) = (1-t)^2 \left(\sum_{i=0}^{\infty} \gamma_{2,1,i} t^i\right). \tag{70}$$

Expanding the equation yields

$$\left(\gamma^*_{2,1,0} + \gamma^*_{2,1,1} t + \gamma^*_{2,1,2} t^2 + \cdots\right) = \frac{1}{(1+2t+3t^2+\cdots)} \left(\gamma_{2,1,0} + \gamma_{2,1,1} t + \gamma_{2,1,2} t^2 + \cdots\right),$$

Table 1: The explicit integration coefficients for $k$ from 0 to 6 (for $y_{n+2}$).

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $\gamma_{2,1,k}$ | 2 | 2 | 7/3 | 8/3 | 269/90 | 33/10 | 13613/3780 |
| $\gamma_{2,2,k}$ | 2 | 4/3 | 4/3 | 62/45 | 43/30 | 94/63 | 1466/945 |
| $\gamma_{2,3,k}$ | 4/3 | 2/3 | 3/5 | 26/45 | 359/630 | 179/315 | 16159/28350 |

Table 2: The implicit integration coefficients for $k$ from 0 to 6 (for $y_{n+2}$).

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $\gamma^*_{2,1,k}$ | 2 | −2 | 1/3 | 0 | −1/90 | −1/90 | −8/945 |
| $\gamma^*_{2,2,k}$ | 2 | −8/3 | 2/3 | 4/90 | 1/90 | 1/315 | 1/1890 |
| $\gamma^*_{2,3,k}$ | 4/3 | −2 | 3/5 | 2/45 | 1/70 | 2/315 | 47/14175 |

$$\left(\gamma^*_{2,1,0} + \gamma^*_{2,1,1}t + \gamma^*_{2,1,2}t^2 + \cdots\right)\left(1 + 2t + 3t^2 + \cdots\right) = \left(\gamma_{2,1,0} + \gamma_{2,1,1}t + \gamma_{2,1,2}t^2 + \cdots\right). \quad (71)$$

This gives the recursive relationship

$$\sum_{i=0}^{k}(k - i + 1)\,\gamma^*_{2,1,i} = \gamma_{2,1,k}. \quad (72)$$

For second-order coefficient,

$$G^*_2(t) = \frac{1}{1!}\left[\frac{2(1-t)^2}{\log(1-t)} - \frac{1!G^*_1(t)}{\log(1-t)}\right]. \quad (73)$$

It can be written as

$$G^*_2(t) = \frac{(1-t)^2}{1!}\left[\frac{2}{\log(1-t)} - \frac{1!G^*_1(t)}{(1-t)^2\log(1-t)}\right]. \quad (74)$$

Substituting (70) into the equation above gives

$$G^*_2(t) = \frac{(1-t)^2}{1!}\left[\frac{2}{\log(1-t)} - \frac{1!(1-t)^2G_1(t)}{(1-t)^2\log(1-t)}\right] \quad (75)$$

or

$$G^*_2(t) = \frac{(1-t)^2}{1!}\left[\frac{2}{\log(1-t)} - \frac{1!G_1(t)}{\log(1-t)}\right]. \quad (76)$$

Substituting (25) into (76) gives

$$G^*_2(t) = (1-t)^2G_2(t), \quad \left(\sum_{i=0}^{\infty}\gamma^*_{2,2,i}t^i\right) = (1-t)^2\left(\sum_{i=0}^{\infty}\gamma_{2,2,i}t^i\right). \quad (77)$$

Expanding the equation yields

$$\left(\gamma^*_{2,2,0} + \gamma^*_{2,2,1}t + \gamma^*_{2,2,2}t^2 + \cdots\right) = \frac{1}{(1 + 2t + 3t^2 + \cdots)}\left(\gamma_{2,2,0} + \gamma_{2,2,1}t + \gamma_{2,2,2}t^2 + \cdots\right),$$

$$\left(\gamma^*_{2,2,0} + \gamma^*_{2,2,1}t + \gamma^*_{2,2,2}t^2 + \cdots\right)\left(1 + 2t + 3t^2 + \cdots\right) = \left(\gamma_{2,2,0} + \gamma_{2,2,1}t + \gamma_{2,2,2}t^2 + \cdots\right). \quad (78)$$

This gives the recursive relationship

$$\sum_{i=0}^{k}(k - i + 1)\,\gamma^*_{2,2,i} = \gamma_{2,2,k}. \quad (79)$$

By using the same process previously, we note that, for $(d-1)$-order coefficient, we have

$$G^*_{(d-1)}(t) = (1-t)^2G_{(d-1)}(t), \quad (80)$$

which leads to a recursive relationship

$$\sum_{i=0}^{k}(k - i + 1)\,\gamma^*_{2,(d-1),i} = \gamma_{2,(d-1),k}. \quad (81)$$

For $(d)$-order coefficient, we have

$$G^*_{(d)}(t) = \frac{1}{(d-1)!}\left[\frac{2^{(d-1)}(1-t)^2}{\log(1-t)} - \frac{(d-1)!G^*_{(d-1)}(t)}{\log(1-t)}\right]. \quad (82)$$

It can be written as

$$G^*_{(d)}(t) = \frac{(1-t)^2}{(d-1)!}\left[\frac{2^{(d-1)}}{\log(1-t)} - \frac{(d-1)!G^*_{(d-1)}(t)}{(1-t)^2\log(1-t)}\right]. \quad (83)$$

Substituting (80) into (83) gives

$$G^*_{(d)}(t) = \frac{(1-t)^2}{(d-1)!}\left[\frac{2^{(d-1)}}{\log(1-t)} - \frac{(d-1)!(1-t)^2G_{(d-1)}(t)}{(1-t)^2\log(1-t)}\right] \quad (84)$$

Table 3: List of test problems.

| | Problem | Initial value | Interval |
|---|---|---|---|
| 1 | $y' = y$<br>Exact solution: $y(x) = e^x$<br>Source: artificial problem | $y(0) = 1$ | $0 \le x \le 20$ |
| 2 | $y'' = -2y' + 3y$<br>Exact solution: $y(x) = e^x + e^{-3x}$<br>Source: Suleiman [9] | $y(0) = 2$<br>$y'(0) = -2$ | $0 \le x \le 64$ |
| 3 | $y_1'' = -\frac{y_1}{r^3}$<br>$y_2'' = -\frac{y_2}{r^3}$<br>$r = \left(y_1^2 + y_2^2\right)^2$<br>Exact solution:<br>$y_1(x) = \cos x$<br>$y_2(x) = \sin x$<br>Source: Shampine and Gordon [10] | $y_1(0) = 1$<br>$y_1'(0) = 0$<br>$y_2(0) = 0$<br>$y_2'(0) = 1$ | $0 \le x \le 16\pi$ |
| 4 | $y_1'' = -2y_1' - 5y_2 + 3$<br>$y_2' = y_1' + 2y_2$<br>Exact solution:<br>$y_1(x) = 2\cos x + 6\sin x - 6x - 2$<br>$y_1(x) = 2\sin x - 2\cos x + 3$<br>Source: Suleiman [9] | $y_1(0) = 0$<br>$y_1'(0) = 0$<br>$y_2(0) = 1$ | $0 \le x \le 16\pi$ |
| 5 | $y''' = -\frac{1}{x}y'' + \frac{1}{x^2}y' + \frac{1}{x}$<br>Exact solution:<br>$y(x) = \frac{x^2}{8}\left(2\ln\left(\frac{x}{2}\right) - \left(\frac{33}{13}\right) - \frac{2}{3}\ln(2)\right) + \left(\frac{1}{3} - \frac{26}{21}\ln\left(\frac{x}{2}\right)\right)\ln(2) + \frac{33}{26}$<br>Source: Russell and Shampine [11] | $y(1) = \frac{26}{21}\ln^2(2) + \frac{99}{104}$<br>$y'(1) = -\frac{40}{21}\ln(2) - \frac{5}{13}$<br>$y''(1) = \frac{3}{26} + \frac{4}{7}\ln(2)$ | $1 \le x \le 50$ |

Table 4: Numerical result for Problem 1.

| $H$ | Method | NS | $\text{Log}_{10}$(MAXE) | Time |
|---|---|---|---|---|
| $10^{-2}$ | 2PBBD | 1000 | −5.40549 | 6914 |
| | 2PBDD | 1000 | −5.88584 | 9509 |
| | 1PBD | 2000 | −6.77769 | 6912 |
| $10^{-3}$ | 2PBBD | 10000 | −7.34272 | 49146 |
| | 2PBDD | 10000 | −8.87606 | 54025 |
| | 1PBD | 20000 | −9.77432 | 50095 |
| $10^{-4}$ | 2PBBD | 100000 | −9.40044 | 195037 |
| | 2PBDD | 100000 | −10.67248 | 256339 |
| | 1PBD | 200000 | −10.48025 | 197800 |
| $10^{-5}$ | 2PBBD | 1000000 | −9.37408 | 1759047 |
| | 2PBDD | 1000000 | −9.37799 | 2055758 |
| | 1PBD | 2000000 | −9.23059 | 1638426 |
| $10^{-6}$ | 2PBBD | 10000000 | −8.81572 | 17500121 |
| | 2PBDD | 10000000 | −8.81572 | 19917218 |
| | 1PBD | 20000000 | −8.28209 | 14653209 |
| $10^{-7}$ | 2PBBD | 100000000 | −7.78104 | 176146062 |
| | 2PBDD | 100000000 | −7.78104 | 199668362 |
| | 1PBD | 200000000 | −7.39425 | 145626635 |

Table 5: Numerical result for Problem 2.

| $H$ | Method | NS | $\text{Log}_{10}$MAXE | Time |
|---|---|---|---|---|
| | 2PBBD | 3200 | −4.87654 | 21516 |
| $10^{-2}$ | 2PBDD | 3200 | −4.58643 | 23350 |
| | 1PBD | 6400 | −5.83463 | 22890 |
| | 2PBBD | 32000 | −7.86398 | 86824 |
| $10^{-3}$ | 2PBDD | 32000 | −7.56556 | 111374 |
| | 1PBD | 64000 | −8.80976 | 87426 |
| | 2PBBD | 320000 | −9.96338 | 776833 |
| $10^{-4}$ | 2PBDD | 320000 | −9.82447 | 828900 |
| | 1PBD | 640000 | −8.99888 | 725855 |
| | 2PBBD | 3200000 | −8.68100 | 7769459 |
| $10^{-5}$ | 2PBDD | 3200000 | −8.68099 | 8208180 |
| | 1PBD | 6400000 | −8.04024 | 6474481 |
| | 2PBBD | 32000000 | −7.64880 | 77008100 |
| $10^{-6}$ | 2PBDD | 32000000 | −7.64879 | 81918617 |
| | 1PBD | 64000000 | −7.15759 | 63364050 |
| | 2PBBD | 320000000 | −6.53269 | 773213500 |
| $10^{-7}$ | 2PBDD | 320000000 | −6.53269 | 817874121 |
| | 1PBD | 640000000 | −6.28394 | 616850084 |

or

$$G^*_{(d)}(t) = \frac{(1-t)^2}{(d-1)!}\left[\frac{2^{(d-1)}}{\log(1-t)} - \frac{(d-1)!G_{(d-1)}(t)}{\log(1-t)}\right]. \quad (85)$$

Substituting

$$G_{(d)}(t) = \frac{1}{(d-1)!}\left[\frac{2^{(d-1)}}{\log(1-t)} - \frac{(d-1)!G_{(d-1)}(t)}{\log(1-t)}\right] \quad (86)$$

into (85) leads to

$$\begin{gathered} G^*_{(d)}(t) = (1-t)^2 G_{(d)}(t), \\ \left(\sum_{i=0}^{\infty}\gamma^*_{2,(d),i}t^i\right) = (1-t)^2\left(\sum_{i=0}^{\infty}\gamma_{2,(d),i}t^i\right). \end{gathered} \quad (87)$$

Expanding the equation yields

$$\begin{aligned} &\left(\gamma^*_{2,(d),0} + \gamma^*_{2,(d),1}t + \gamma^*_{2,(d),2}t^2 + \cdots\right) \\ &\quad = \frac{1}{(1+2t+3t^2+\cdots)} \\ &\quad \times\left(\gamma_{2,(d),0} + \gamma_{2,(d),1}t + \gamma_{2,(d),2}t^2 + \cdots\right), \end{aligned}$$

$$\begin{aligned} &\left(\gamma^*_{2,(d),0} + \gamma^*_{2,(d),1}t + \gamma^*_{2,(d),2}t^2 + \cdots\right)\left(1 + 2t + 3t^2 + \cdots\right) \\ &\quad = \left(\gamma_{2,(d),0} + \gamma_{2,(d),1}t + \gamma_{2,(d),2}t^2 + \cdots\right), \end{aligned} \quad (88)$$

which leads to a recursive relationship

$$\sum_{i=0}^{k}(k-i+1)\,\gamma^*_{2,(d),i} = \gamma_{2,(d),k}. \quad (89)$$

Tables 1 and 2 are a few examples of the explicit and implicit integration coefficients.

## 6. Problem Tested

The problems shown in Table 3 are used to test the performance of the method.

## 7. Numerical Result

Tables 4, 5, 6, 7, and 8 give the numerical results for problems given in the previous section. The results for the 2PBBD are compared with those of 2PBDD and 1PBD according to Omar [5] and Suleiman et al. [8], respectively. Also given are graphs, where $\text{Log}_{10}$(MAXE) is plotted against $\text{Log}_{10}(H)$ and $\text{Log}_{10}$(Time). The following notations are used in the tables:

$H$: step size,

2PBBD: 2-point block backward difference method,

2PBDD: 2-point block divided difference method,

Table 6: Numerical result for Problem 3.

| $H$ | Method | NS | $\text{Log}_{10}$MAXE | Time |
|---|---|---|---|---|
| $10^{-2}$ | 2PBBD | 2513 | −6.17611 | 24465 |
| | 2PBDD | 2513 | −5.87541 | 24221 |
| | 1PBD | 5026 | −7.07922 | 28522 |
| $10^{-3}$ | 2PBBD | 25133 | −9.17248 | 152099 |
| | 2PBDD | 25133 | −8.87081 | 159389 |
| | 1PBD | 50265 | −10.01264 | 139073 |
| $10^{-4}$ | 2PBBD | 251328 | −10.04346 | 1418931 |
| | 2PBDD | 251328 | −10.0434 | 1366837 |
| | 1PBD | 502655 | −9.26354 | 1219971 |
| $10^{-5}$ | 2PBBD | 2513274 | −8.80560 | 14315026 |
| | 2PBDD | 2513274 | −8.80560 | 13192255 |
| | 1PBD | 5026548 | −8.32281 | 11525418 |
| $10^{-6}$ | 2PBBD | 25132742 | −7.99962 | 141938590 |
| | 2PBDD | 25132742 | −7.99962 | 130102816 |
| | 1PBD | 50265482 | −7.45728 | 106896913 |
| $10^{-7}$ | 2PBBD | 251327412 | −6.87690 | 1412455700 |
| | 2PBDD | 251327412 | −6.87690 | 1300828490 |
| | 1PBD | 502654824 | −6.44359 | 1063913703 |

Table 7: Numerical result for Problem 4.

| $H$ | Method | NS | $\text{Log}_{10}$MAXE | Time |
|---|---|---|---|---|
| $10^{-2}$ | 2PBBD | 2513 | −5.29159 | 23909 |
| | 2PBDD | 2513 | −5.69715 | 26684 |
| | 1PBD | 5026 | −6.58585 | 31772 |
| $10^{-3}$ | 2PBBD | 25133 | −7.74946 | 121769 |
| | 2PBDD | 25133 | −8.68299 | 131297 |
| | 1PBD | 50265 | −9.58464 | 121536 |
| $10^{-4}$ | 2PBBD | 251328 | −9.39201 | 1102741 |
| | 2PBDD | 251328 | −10.04970 | 1068239 |
| | 1PBD | 502655 | −9.26367 | 1046930 |
| $10^{-5}$ | 2PBBD | 2513274 | −8.80463 | 10004344 |
| | 2PBDD | 2513274 | −8.80548 | 9407995 |
| | 1PBD | 5026548 | −8.32278 | 10087395 |
| $10^{-6}$ | 2PBBD | 25132742 | −7.99900 | 99147571 |
| | 2PBDD | 25132742 | −7.99900 | 91521335 |
| | 1PBD | 50265482 | −7.45728 | 93207026 |
| $10^{-7}$ | 2PBBD | 251327412 | −6.87692 | 988919951 |
| | 2PBDD | 251327412 | −6.87692 | 911824508 |
| | 1PBD | 502654824 | −6.44359 | 922126253 |

1PBD: 1-point backward difference method,

NS: total number of steps,

MAXE: maximum error,

TIME: total execution times (in microsecond).

Two sets of scaled graphs were plotted, namely, (i) $\text{Log}_{10}$ (MAXE) against $\text{Log}_{10}(H)$ and (ii) $\text{Log}_{10}$(MAXE) against $\text{Log}_{10}$(TIME). For a particular abscissa, the lowest value of the ordinate is considered to be the more efficient at the abscissa considered. Hence, for the first set of graphs,

Table 8: Numerical result for Problem 5.

| $H$ | Method | NS | $\text{Log}_{10}$MAXE | Time |
|---|---|---|---|---|
| $10^{-2}$ | 2PBBD | 2450 | −3.65914 | 25345 |
| | 2PBDD | 2450 | −3.37200 | 26172 |
| | 1PBD | 4900 | −5.08070 | 27139 |
| $10^{-3}$ | 2PBBD | 24500 | −6.62483 | 106625 |
| | 2PBDD | 24500 | −6.32538 | 116848 |
| | 1PBD | 49000 | −8.06583 | 108650 |
| $10^{-4}$ | 2PBBD | 245000 | −9.64707 | 1021740 |
| | 2PBDD | 245000 | −9.30862 | 996446 |
| | 1PBD | 490000 | −10.55259 | 888813 |
| $10^{-5}$ | 2PBBD | 2450000 | −10.06216 | 9797882 |
| | 2PBDD | 2450000 | −10.06228 | 8683164 |
| | 1PBD | 4900000 | −9.42795 | 8211650 |
| $10^{-6}$ | 2PBBD | 24500000 | −8.99818 | 97582403 |
| | 2PBDD | 24500000 | −8.99818 | 85958740 |
| | 1PBD | 49000000 | −8.73189 | 80890328 |
| $10^{-7}$ | 2PBBD | 245000000 | −8.10589 | 978631746 |
| | 2PBDD | 245000000 | −8.10589 | 844019109 |
| | 1PBD | 490000000 | −7.70882 | 790057441 |

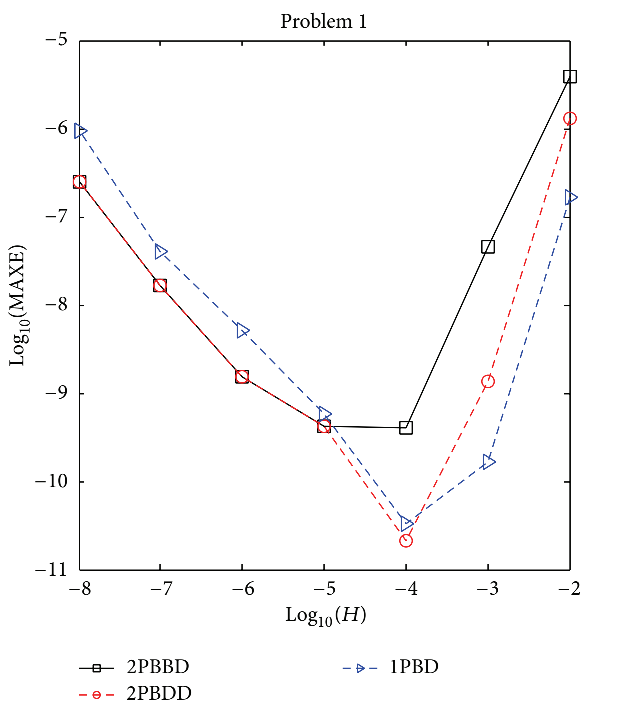


Figure 2: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(H)$ for Problem 1.

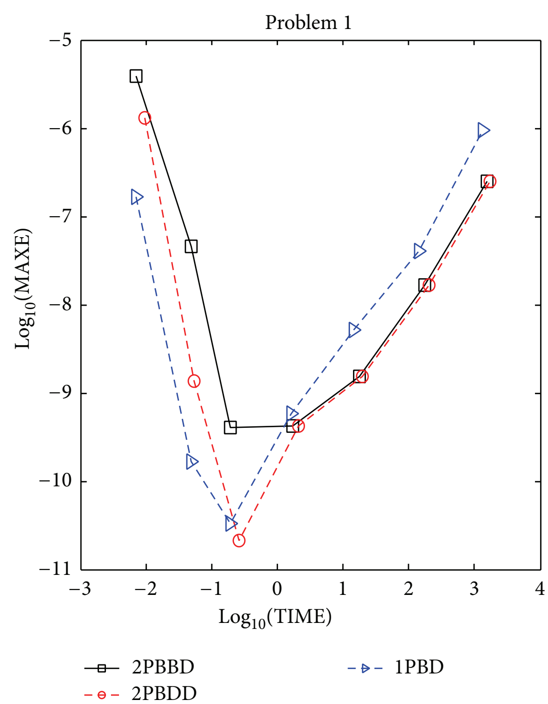


Figure 3: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(\text{TIME})$ for Problem 1.

that is, $\text{Log}_{10}(\text{MAXE})$ against $\text{Log}_{10}(H)$, the method 2PBBD is better when $\text{Log}_{10}(H) < -5$, and loses out for value of $\text{Log}_{10}(H) > -5$. For the second set of graphs, as the time increases, the 2PBBD is the method of choice since it is lowest for all five sets of problems (see Figures 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11). It gives us the impression of stability, where the errors

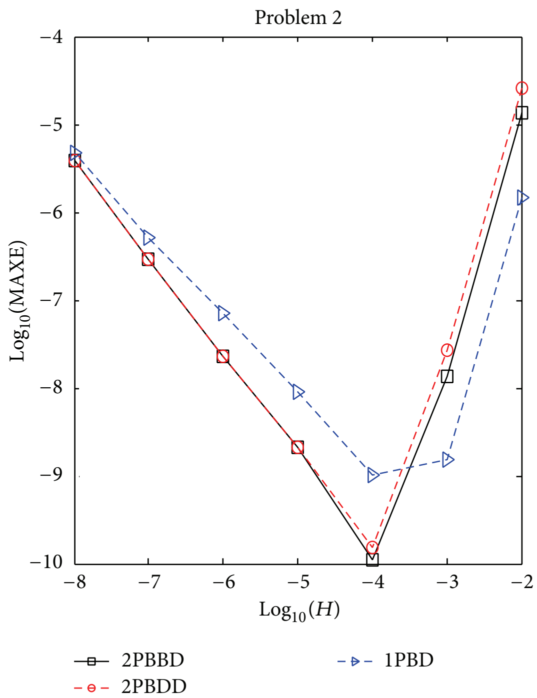


Figure 4: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(H)$ for Problem 2.

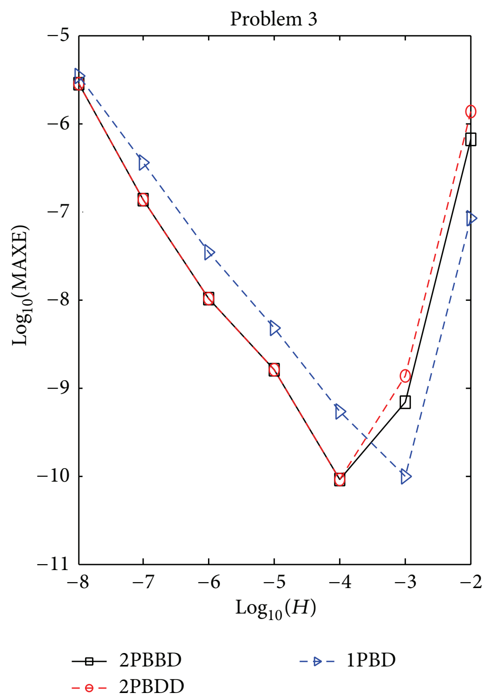


Figure 6: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(H)$ for Problem 3.

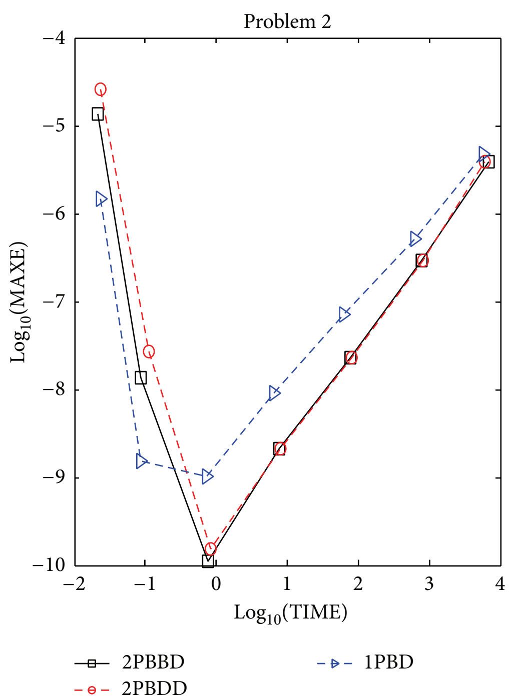


Figure 5: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(\text{TIME})$ for Problem 2.

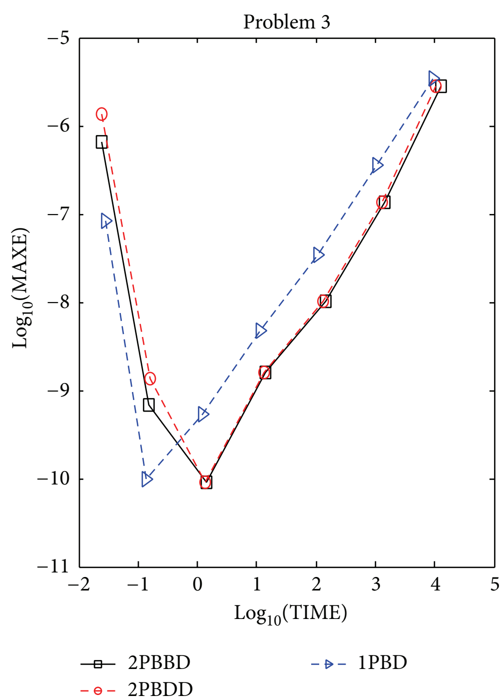


Figure 7: Graph of $\text{Log}_{10}(\text{MAXE})$ plotted against $\text{Log}_{10}(\text{TIME})$ for Problem 3.

grow most slowly compared with the other methods, 2PBDD and 1PBD.

## 8. Conclusion

Of the 3 methods, 2PBBD is therefore preferred as a general code and should be included as a collection of methods, as

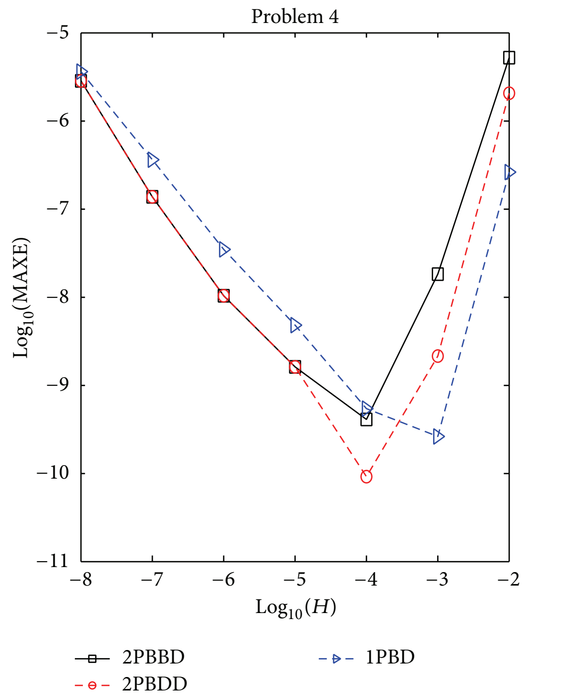


FIGURE 8: Graph of $\text{Log}_{10}$(MAXE) plotted against $\text{Log}_{10}(H)$ for Problem 4.

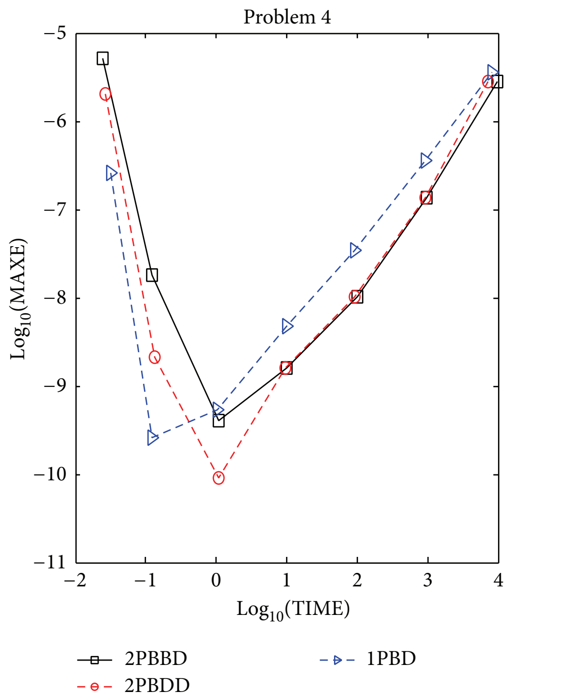


FIGURE 9: Graph of $\text{Log}_{10}$(MAXE) plotted against $\text{Log}_{10}$(TIME) for Problem 4.

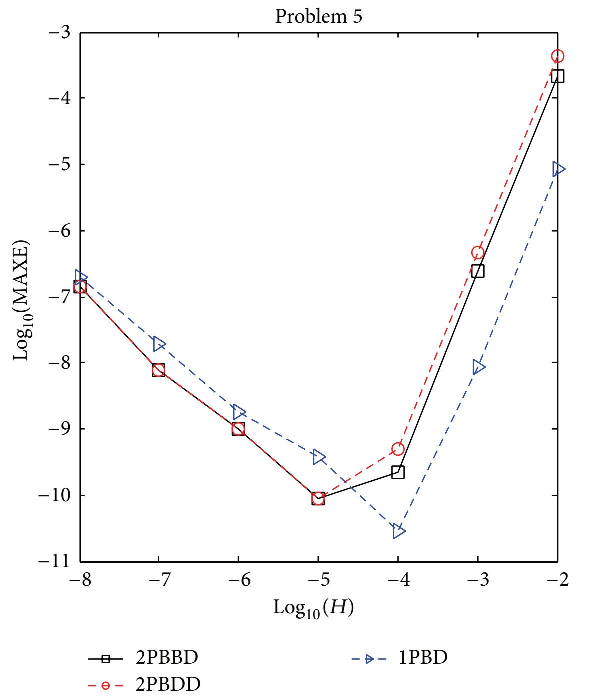


FIGURE 10: Graph of $\text{Log}_{10}$(MAXE) plotted against $\text{Log}_{10}(H)$ for Problem 5.

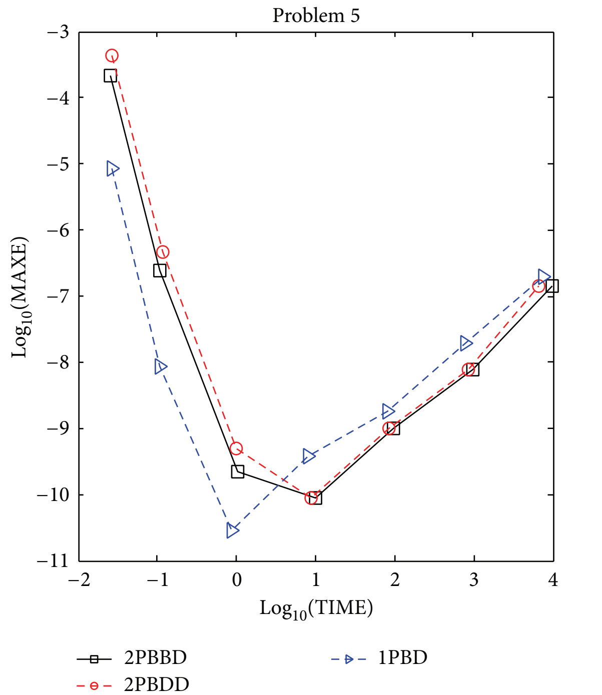


FIGURE 11: Graph of $\text{Log}_{10}$(MAXE) plotted against $\text{Log}_{10}$(TIME) for Problem 5.

a code for parallelization purposes, as an assembly of codes to be tested and studied, and as a code for solving ODEs.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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