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Research Article

Solutions of Second-Order m-Point Boundary Value Problems for Impulsive Dynamic Equations on Time Scales

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We study a general second-order m-point boundary value problems for nonlinear singular impulsive dynamic equations on time scales $u^{\Delta\nabla}(t) + a(t)u^{\Delta}(t) + b(t)u(t) + q(t)f(t,u(t)) = 0, t \in (0,1), t \neq t_k, u^{\Delta}(t_k^+) = u^{\Delta}(t_k) - I_k(u(t_k))$, and $k = 1,2,\ldots,n, u(\rho(0)) = 0, u(\sigma(1)) = \sum_{i=1}^{m-2}\alpha_i u(\eta_i)$. The existence and uniqueness of positive solutions are established by using the mixed monotone fixed point theorem on cone and Krasnosel'skii fixed point theorem. In this paper, the function items may be singular in its dependent variable. We present examples to illustrate our results.

1. Introduction

The theory of dynamic equations on time scales unifies the well-known analogies in the concept of difference equations and differential equations. In the past few years, the boundary value problems of dynamic equations on time scales have been studied by many authors (see [1–16] and references cited therein). Some classical tools have been used in the literature to study dynamic equations. These classical tools include the coincidence degree theory [11, 12], the method of upper and lower solutions [7, 10], and some fixed point theorems in cones for completely continuous operators [1–5, 9, 13–16]. Recently, multiple-point boundary value problems on time scale have been studied for instance [4, 5, 12].

In 2008, Lin and Du [5] studied the m-point boundary value problem for second-order dynamic equations on time scales:

$$u^{\Delta\nabla}(t) + f(t,u) = 0, \quad t \in (0,T) \in T,$$
$$u(0) = 0, \quad u(T) = \sum_{i=1}^{m-2} k_i u(\xi_i), \tag{1}$$

where T is a time scale. By using Green's function and the Leggett-Williams fixed point theorem in an appropriate cone, the existence of at least three positive solutions of the problem is obtained.

In 2009, Topal and Yantir [4] studied the general second-order nonlinear m-point boundary value problems:

$$u^{\Delta\nabla}(t) + a(t)u^{\Delta}(t) + b(t)u(t) + \lambda h(t) f(t,u(t)) = 0, \quad t \in (0,1),$$
$$u(\rho(0)) = 0, \qquad u(\sigma(1)) = \sum_{i=1}^{m-2}\alpha_i u(\eta_i) \tag{2}$$

with no singularity. The authors deal with the determining the value of λ, the existence of multiple positive solutions of (2) is obtained by using the Krasnosel'skii and Legget-William fixed point theorems.

Impulsive differential equations are now recognized as an excellent source of models for simulating processes and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnology, industrial robotics, optimal control, and so forth. In recent years, impulsive differential equations have become a very active area of research. In this paper, we consider the following impulsive singular dynamic equations on time scales:

$$\begin{gathered} u^{\Delta\nabla}(t) + a(t) u^{\Delta}(t) + b(t) u(t) + q(t) f(t, u(t)) = 0, \\ t \in (0,1),\ t \neq t_k, \\ u^{\Delta}(t_k^+) = u^{\Delta}(t_k) - I_k(u(t_k)), \quad k = 1, 2, \ldots, n, \\ u(\rho(0)) = 0, \qquad u(\sigma(1)) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{gathered} \tag{3}$$

where $\alpha \geq 0$, $0 < \eta_i < \eta_{i+1} < 1$, for all $i = 1, 2, \ldots, m-2$, I_k, f, q, a, and b satisfy the following:

(C1) $f \in C([\rho(0), \sigma(1)] \times (0, +\infty), [0, +\infty))$ and $f(t, u)$ may be singular at $u = 0$, $I_k \in C([0, +\infty), [0, +\infty))$;

(C2) $q \in C((0, 1), [0, +\infty))$ and there exists $t_0 \in (0, 1)$ such that $q(t_0) > 0$, $q(t)$ may be singular at $t = 0, 1$;

(C3) $a \in C([0, 1], [0, +\infty))$, $b \in C([0, 1], (-\infty, 0])$.

The main theorems of this paper complement the very little existence results devoted to impulsive dynamic equations on a time scale. We will prove our two existence results for the problem (3) by using mixed monotone fixed point theorem on cone [17] and Krasnosel'skii fixed point theorem [18]. This paper is organized as follows. In Section 2, starting with some preliminary lemmas, we state a mixed monotone fixed point theorem on cone and Krasnosel'skii fixed point theorem. In Section 3, we give the main result which states the sufficient conditions for the m-point boundary value problem (3) to have existence of positive solutions. We also present examples to illustrate our results work.

2. Preliminaries

In this section we state the preliminary information that we need to prove the main results. From Lemmas 2.1 and 2.2 in [4], we have the following lemma.

Lemma 1. *Assume that (C3) holds. Then the equations*

$$\begin{gathered} \phi_1^{\Delta\nabla}(t) + a(t) \phi_1^{\Delta}(t) + b(t) \phi_1(t) = 0, \quad t \in (0,1), \\ \phi_1(\rho(0)) = 0, \qquad \phi_1(\sigma(1)) = 1, \\ \phi_2^{\Delta\nabla}(t) + a(t) \phi_2^{\Delta}(t) + b(t) \phi_2(t) = 0, \quad t \in (0,1), \\ \phi_2(\rho(0)) = 1, \qquad \phi_2(\sigma(1)) = 0 \end{gathered} \tag{4}$$

have unique solutions ϕ_1 and ϕ_2, respectively, and

(a) *ϕ_1 is strictly increasing on $[\rho(0), \sigma(1)]$;* (b) *ϕ_2 is strictly decreasing on $[\rho(0), \sigma(1)]$.*

For the rest of the paper we need the following assumption:

(C4) $0 < \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i) < 1$.

Lemma 2. *Assume that (C3) and (C4) hold. Let $x \in C[\rho(0), \sigma(1)]$. Then boundary value problem*

$$\begin{gathered} x^{\Delta\nabla}(t) + a(t) x^{\Delta}(t) + b(t) x(t) + x(t) = 0, \quad t \in (0,1), \\ x(\rho(0)) = 0, \qquad x(\sigma(1)) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i) \end{gathered} \tag{5}$$

is equivalent to the integral equation

$$x(t) = \int_{\rho(0)}^{\sigma(1)} H(t,s) p(s) x(s) \nabla s + A\phi_1(t), \tag{6}$$

where

$$\begin{gathered} p(t) = e_a(\rho(t), \rho(0)), \\ A = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i \int_{\rho(0)}^{\sigma(1)} H(\eta_i, s) p(s) x(s) \nabla s, \\ H(t,s) = \frac{1}{\phi_1^{\Delta}(\rho(0))} \begin{cases} \phi_1(s) \phi_2(t), & s \leq t, \\ \phi_1(t) \phi_2(s), & t \leq s. \end{cases} \end{gathered} \tag{7}$$

Lemma 3. *Green's function $H(t,s)$ has the properties*

$$\begin{gathered} H(t,s) \leq H(t,t), \\ \frac{\phi_1^{\Delta}(\rho(0))}{\|\phi_1\| \|\phi_2\|} H(t,t) H(s,s) \leq H(t,s) \leq H(s,s), \\ H(t,t) \leq \phi_1(t) \frac{\|\phi_2\|}{\phi_1^{\Delta}(\rho(0))}. \end{gathered} \tag{8}$$

Lemma 4. *Assume that (C3) and (C4) hold. Let $u \in C[\rho(0), \sigma(1)]$ be a solution of the boundary value problem* (3) *if and only if u is a solution of the following impulsive integral equation:*

$$\begin{aligned} u(t) = & \int_{\rho(0)}^{\sigma(1)} G(t,s) p(s) q(s) f(s, u(s)) \nabla s \\ & + \sum_{k=1}^{n} G(t, t_k) I_k(u(t_k)), \end{aligned} \tag{9}$$

where

$$G(t,s) = H(t,s) + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i H(\eta_i, s) \phi_1(t). \tag{10}$$

Lemma 5. *Green's function $G(t,s)$ defined by* (10) *has the properties*

$$G_0(t) G^*(s) \leq G(t,s) \leq G^*(s), \qquad G(t,s) \leq \phi_1(t) C(s), \tag{11}$$

where

$$C(s)=\frac{\|\phi_2\|}{\phi_1^{\Delta}(\rho(0))}+\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\sum_{i=1}^{m-2}\alpha_i H(\eta_i,s),$$

$$G^*(s)=H(s,s)+\frac{\|\phi_1\|}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\sum_{i=1}^{m-2}\alpha_i H(\eta_i,s), \quad (12)$$

$$G_0(t)=\frac{\phi_1^{\Delta}(\rho(0))}{\|\phi_1\|\|\phi_2\|}H(t,t).$$

Lemma 6. *Assume that (C1)–(C4) hold. Then the solution of* (3) *satisfies* $u(t)\geq G_0(t)\|u\|$.

The proofs of the Lemmas 3–6 can be obtained easily by Lemmas 1 and 2.

For our constructions, we will consider the Banach space $E = C[\rho(0),\sigma(1)]$ equipped with standard norm $\|u\| = \max_{\rho(0)\leq t\leq\sigma(1)}|u(t)|$, $u\in E$. We define a cone K by

$$K=\{u\in X \mid u(t)\geq G_0(t)\|u\|,\ t\in[\rho(0),\sigma(1)]\}. \quad (13)$$

From Lemmas 4 and 5, we define the integral operator $T : K \to E$ by

$$Tu(t)=\int_{\rho(0)}^{\sigma(1)}G(t,s)p(s)q(s)f(s,u(s))\nabla s + \sum_{k=1}^{n}G(t,t_k)I_k(u(t_k)). \quad (14)$$

Then, it is clear that the solutions of (3) are the fixed points of the operator T.

Thus, from Lemma 4, standard arguments show that $T(K)\subset K$ and T is completely continuous.

The following content will play major role in our next analysis.

Let P be a normal cone of a Banach space E, and let $e\in P$ with $\|e\|\leq 1, e\neq\theta$. Define

$$Q_e=\{x\in P \mid x\neq\theta,\ \text{there exist constants } m,M>0 \text{ such that } me\leq x\leq M\,e\}. \quad (15)$$

Definition 7 (see [17]). Assume $S :\ Q_e\times Q_e \to Q_e$. S is said to be mixed monotone if $S(x,y)$ is nondecreasing in x and nonincreasing in y, that is, if $x_1\leq x_2$ $(x_1,x_2\in Q_e)$ implies $S(x_1,y)\leq S(x_2,y)$ for any $y\in Q_e$ and $y_1\leq y_2$ $(y_1,y_2\in Q_e)$ implies $S(x,y_1)\geq S(x,y_2)$ for any $x\in Q_e$. $x^*\in Q_e$ is said to be a fixed point of S if $S(x^*,x^*)=x^*$.

Theorem 8 (see [17, 19]). *Suppose that* $S : Q_e\times Q_e \to Q_e$ *is a mixed monotone operator and* $\exists$ *a constant* α, $0\leq\alpha<1$, *such that*

$$S\left(tx,\frac{1}{t}y\right)\geq t^{\alpha}S(x,y),\quad \forall x,y\in Q_e,\ 0<t<1. \quad (16)$$

Then S has a unique fixed point $x^*\in Q_e$. *Moreover, for any* $(x_0,y_0)\in Q_e\times Q_e$,

$$x_n=S(x_{n-1},y_{n-1}),\quad y_n=S(y_{n-1},x_{n-1}),\quad n=1,2,\ldots \quad (17)$$

satisfy

$$x_n\longrightarrow x^*,\qquad y_n\longrightarrow x^*, \quad (18)$$

where

$$\|x_n-x^*\|=o\left(1-r^{\alpha^n}\right),\qquad \|y_n-x^*\|=o\left(1-r^{\alpha^n}\right), \quad (19)$$

$0<r<1$, *and r is a constant from* (x_0,y_0).

Theorem 9 (see [18]). *Let X be a Banach space, and let* $P\subset X$ *be a cone in X. Assume that* Ω_1,Ω_2 *are open subsets of X with* $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, *and let* $S : P \to P$ *be a completely continuous operator such that either*

(1) $\|Sw\|\leq\|w\|$, $w\in P\cap\partial\Omega_1$, $\|Sw\|\geq\|w\|$, $w\in P\cap\partial\Omega_2$, *or*

(2) $\|Sw\|\geq\|w\|$, $w\in P\cap\partial\Omega_1$, $\|Sw\|\leq\|w\| w\in P\cap\partial\Omega_2$.

Then S has a fixed point in $P\cap\overline{\Omega}_2\setminus\Omega_1$.

3. Main Results

First, by using Theorem 8 we establish the following main result.

Theorem 10. *Suppose that conditions* (C1)–(C4) *hold and*

(C5) $f(t,u)=h_0(u)+g_0(u)$, $I_k(u)=h_k(u)+g_k(u)$ *(k = 0, 1, 2, …, n) and*

$$g_k:(0,+\infty)\longrightarrow(0,+\infty) \text{ is continuous and nonincreasing;}$$
$$h_k:[0,+\infty)\longrightarrow[0,+\infty) \text{ is continuous and nondecreasing} \quad (20)$$

for $k=0,1,2,\ldots,m$.

(C6) *There exists* $\alpha\in(0,1)$ *such that*

$$g_k\left(t^{-1}x\right)\geq t^{\alpha}g_k(x), \quad (21)$$

$$h_k(tx)\geq t^{\alpha}h_k(x), \quad (22)$$

for any $t\in(0,1)$ *and,* $x>0$, $k=0,1,2,\ldots,n$.

(C7) *Consider that* $q\in C((0,1),(0,\infty))$ *satisfies*

$$\int_0^1 C(s)\phi_1^{-\alpha}(s)p(s)q(s)\,ds<+\infty. \quad (23)$$

Then (3) *has an unique positive solution* $u^*(t)$.

Proof. Since (21) holds, let $t^{-1}x=y$; one has

$$g_k(y)\geq t^{\alpha}g_k(ty). \quad (24)$$

Then

$$g_k(ty)\leq\frac{1}{t^{\alpha}}g_k(y),\quad \text{for } t\in(0,1),\ y>0. \quad (25)$$

Let $y = 1$. The above inequality is

$$g_k(t) \le \frac{1}{t^\alpha} g_k(1), \quad \text{for } t \in (0,1). \tag{26}$$

From (21), (25), and (26), one has

$$\begin{gathered} g_k(t^{-1}x) \ge t^\alpha g_k(x), \qquad g_k\left(\frac{1}{t}\right) \ge t^\alpha g_k(1), \\ g_k(tx) \le \frac{1}{t^\alpha} g_k(x), \qquad g_k(t) \le \frac{1}{t^\alpha} g_k(1), \\ \text{for } t \in (0,1), \ x > 0. \end{gathered} \tag{27}$$

Similarly, from (22), one has

$$\begin{gathered} h_k(tx) \ge t^\alpha h_k(x), \qquad h_k(t) \ge t^\alpha h_k(1), \\ \text{for } t \in (0,1), \ x > 0, \ k = 0,1,\ldots,m. \end{gathered} \tag{28}$$

Let $t = 1/x, x > 1$; one has

$$h_k(x) \le x^\alpha h_k(1), \quad \text{for } x \ge 1, \ k = 0,1,\ldots,m. \tag{29}$$

Let $e(t) = \phi_1(t)/\|\phi_1\|$; it is clear that $\|e\| \le 1$. We define

$$Q_e = \left\{ x \in C[0,1] \mid \frac{1}{M} e(t) \le x(t) \le Me(t), t \in [0,1] \right\}, \tag{30}$$

where $M > 1$ is chosen such that

$$\begin{aligned} M > \max \Bigg\{ & \Bigg[\int_{\rho(0)}^{\sigma(1)} \|\phi_1\| C(s) p(s) q(s) \\ & \times \left(h_0(1) + e^{-\alpha}(s) g_0(1)\right) \nabla s + \sum_{k=1}^{n} \|\phi_1\| C(t_k) \\ & \times \left(h_k(1) + e^{-\alpha}(t_k) g_k(1)\right) \Bigg]^{1/(1-\alpha)}; \\ & \Bigg[\frac{\|\phi_1\|}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \\ & \times \sum_{i=1}^{m-2} \alpha_i \int_{\rho(0)}^{\sigma(1)} G(\eta_i, s) p(s) q(s) \\ & \times \left(e^\alpha(s) h_0(1) + g_0(1)\right) \nabla s \Bigg]^{-(1/(1-\alpha))} \Bigg\}. \end{aligned} \tag{31}$$

For any $x, y \in Q_e$, we define

$$\begin{aligned} T(x,y)(t) = & \int_{\rho(0)}^{\sigma(1)} G(t,s) p(s) q(s) \\ & \times \left(h_0(x(s)) + g_0(y(s))\right) \nabla s \\ & + \sum_{k=1}^{n} G(t,t_k) \left(h_k(x(t_k)) + g_k(x(t_k))\right). \end{aligned} \tag{32}$$

Let $x, y \in Q_e$. On one hand, from (27), (28), and (29), for $k = 0,1,2,\ldots,m$, we have

$$\begin{aligned} & g_k(y(t)) \le g_k\left(\frac{1}{M} e(t)\right) \le M^\alpha e^{-\alpha}(t) g_k(1), \\ & h_k(x(t)) \le h_k(Me(t)) \le h_k(M) \le M^\alpha h_k(1), \\ & g_k(y(t)) \ge g_k(Me(t)) \ge g_k(M) \\ & \qquad \ge g_k\left(\frac{1}{1/M}\right) \ge \frac{1}{M^\alpha} g_k(1), \\ & h_k(x(t)) \ge h_k\left(\frac{1}{M} e(t)\right) \ge e^\alpha(t) h_k\left(\frac{1}{M}\right) \\ & \qquad \ge e^\alpha(t) \frac{1}{M^\alpha} h_k(1); \end{aligned} \tag{33}$$

then

$$\begin{aligned} & \frac{1}{M^\alpha}\left(e^\alpha(t) h_k(1) + g_k(1)\right) \le h_k(x(t)) + g_k(y(t)) \\ & \quad \le M^\alpha \left(h_k(1) + e^{-\alpha}(t) g_k(1)\right), \quad t \in (0,1). \end{aligned} \tag{34}$$

Thus, for any $x, y \in Q_e$, we have

$$\begin{aligned} T(x,y)(t) \ge & e(t) \frac{\|\phi_1\|}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \\ & \times \sum_{i=1}^{m-2} \alpha_i \int_{\rho(0)}^{\sigma(1)} G(\eta_i, s) p(s) q(s) \\ & \times \left(h_0(x(s)) + g_0(y(s))\right) \nabla s \\ \ge & e(t) \frac{\|\phi_1\|}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \\ & \times \sum_{i=1}^{m-2} \alpha_i \int_{\rho(0)}^{\sigma(1)} G(\eta_i, s) p(s) q(s) \frac{1}{M^\alpha} \\ & \times \left(e^\alpha(s) h_0(1) + g_0(1)\right) \nabla s \\ \ge & \frac{1}{M^\alpha} e(t) \frac{\|\phi_1\|}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \\ & \times \sum_{i=1}^{m-2} \alpha_i \int_{\rho(0)}^{\sigma(1)} G(\eta_i, s) p(s) q(s) \\ & \times \left(e^\alpha(s) h_0(1) + g_0(1)\right) \nabla s \\ \ge & \frac{1}{M} e(t), \end{aligned}$$

$$
\begin{aligned}
T(x,y)(t) &\le e(t)\left[\int_{\rho(0)}^{\sigma(1)}\|\phi_1\| C(s)\,p(s)\,q(s)\left((h_0(x(s)) + g_0(x(s)))\nabla s + \sum_{k=1}^{n}\|\phi_1\| C(t_k)\right)\right.\\
&\qquad \left.\times(h_k(x(t_k)) + g_k(x(t_k)))\right]\\
&\le e(t)\left[\int_{\rho(0)}^{\sigma(1)}\|\phi_1\| C(s)\,p(s)\,q(s)\,M^{\alpha}\times\left(h_0(1)+e^{-\alpha}(s)\,g_0(1)\right)\nabla s\right.\\
&\qquad \left.+\sum_{k=1}^{n}\|\phi_1\| C(t_k)\,M^{\alpha}\times\left(h_k(1)+e^{-\alpha}(t_k)\,g_k(1)\right)\right]\\
&\le M^{\alpha}e(t)\left[\int_{\rho(0)}^{\sigma(1)}\|\phi_1\| C(s)\,p(s)\,q(s)\times\left(h_0(1)+e^{-\alpha}(s)\,g_0(1)\right)\nabla s\right.\\
&\qquad \left.+\sum_{k=1}^{n}\|\phi_1\| C(t_k)\times\left(h_k(1)+e^{-\alpha}(t_k)\,g_k(1)\right)\right]\\
&\le Me(t), \quad \text{for } t\in[0,1].
\end{aligned}
\tag{35}
$$

So, T is well defined and $T(Q_e\times Q_e)\subset Q_e$.

Next, for any $l\in(0,1)$, one has

$$
\begin{aligned}
T\left(lx,l^{-1}y\right)(t) &= \int_{\rho(0)}^{\sigma(1)} G(t,s)\,p(s)\,q(s)\times\left(h_0(lx(s)) + g_0\left(l^{-1}y(s)\right)\right)\nabla s\\
&\quad+\sum_{k=1}^{n} G(t,t_k)\left(h_k(lx(t_k)) + g_k\left(l^{-1}y(t_k)\right)\right)\\
&\ge \int_{\rho(0)}^{\sigma(1)} G(t,s)\,p(s)\,q(s)\times\left(l^{\alpha}h_0(x(s)) + l^{\alpha}g_0(y(s))\right)\nabla s\\
&\quad+\sum_{k=1}^{n} G(t,t_k)\left(l^{\alpha}h_k(x(t_k)) + l^{\alpha}g_k(y(t_k))\right)\\
&\ge l^{\alpha}\left\{\int_{\rho(0)}^{\sigma(1)} G(t,s)\,p(s)\,q(s)\times\left(h_0(x(s)) + g_0(y(s))\right)\nabla s\right.\\
&\quad\left.+\sum_{k=1}^{n} G(t,t_k)\left(h_k(x(t_k)) + g_k(y(t_k))\right)\right\}\\
&= l^{\alpha}T(x,y)(t), \quad \text{for } t\in[0,1].
\end{aligned}
\tag{36}
$$

So the conditions of Theorems 8 hold. Therefore there exists a unique $u^{*}\in Q_e$ such that $T(u^{*},u^{*})=u^{*}$. This completes the proof of Theorem 10. □

Example 11. Consider the following singular boundary value problem:

$$
\begin{aligned}
&u^{\Delta\nabla}(t)+a(t)\,u^{\Delta}(t)+b(t)\,u(t)+\left(\mu x^{\alpha}+x^{-\beta}\right)=0,\\
&\qquad\qquad t\in(0,1),\ t\neq t_k,\\
&u^{\Delta}\left(t_k^{+}\right)=u^{\Delta}(t_k)-a_k\left(u^{a}(t_k)\right),\quad k=1,2,\ldots,n;\\
&u(\rho(0))=0,\qquad u(\sigma(1))=\sum_{i=1}^{m-2}\alpha_i u(\eta_i),
\end{aligned}
\tag{37}
$$

where $\alpha,\beta>0$, $\max\{\alpha,\beta\}<1$, $\mu\ge 0$, $0<\eta_i<\eta_{i+1}<1$, for all $i=1,2,\ldots,m-2$, and a and b satisfy $a\in C([0,1],[0,+\infty))$, $b\in C([0,1],(-\infty,0])$.

Let $\alpha_0=\max\{\alpha,\beta\}$, $h(x)=\mu x^{a}$, $g(x)=x^{-b}$, and $q(t)=1$. Thus

$$
\begin{aligned}
&h(lx)=\mu l^{\alpha}x^{\alpha}\ge l^{\alpha_0}h(x),\\
&g\left(l^{-1}x\right)=l^{\beta}x^{-\beta}\ge l^{\alpha_0}g(x),\\
&\int_0^1 G^{-\alpha_0}(s,s)\,ds<+\infty.
\end{aligned}
\tag{38}
$$

Applying Theorem 10, we can find that the above equation has a unique solution $u^{*}(t)$.

In the next, using Theorem 9 we establish the following main result.

Theorem 12. *Suppose that conditions* (C1)–(C4) *hold and the following conditions are satisfied:*

$$f(t,u) \le g_0(u) + h_0(u), \quad I_k(u) \le g_k(u) + h_k(u) \quad (k = 1,2,\dots,n) \text{ on } [0,1]\times(0,\infty) \text{ with } g_i > 0 \quad (i = 0,1,2,\dots,n) \text{ continuous and nonincreasing on } (0,\infty), \tag{39}$$

$$h_i \ge 0 \quad (i = 0,1,2,\dots,n) \text{ continuous on } [0,\infty),$$

$$\frac{h_i}{g_i} \quad (i = 0,1,2,\dots,n) \text{ nondecreasing on } (0,\infty),$$

$$\exists K_0 \text{ with } g_i(xy) \le K_0 g_i(x) g_i(y), \quad \forall x > 0,\ y > 0, \quad i = 0,1,2,\dots,n, \tag{40}$$

$$a_0 = \int_{\rho(0)}^{\sigma(1)} G^*(s)\,p(s)\,q(s)\,g_0(G_0(s))\nabla s + \sum_{k=1}^{n} G^*(t_k)\,g_k(G_0(t_k)) < \infty, \tag{41}$$

$$\exists r > 0 \text{ with } \frac{r}{\max_{0\le i\le n}\{g_i(r) + h_i(r)\}} > K_0 a_0; \tag{42}$$

$$\text{there exists } \rho(0) < \theta < \frac{\sigma(1) - \rho(0)}{2} \text{ (choose and fix it) and a continuous, nonincreasing function } \overline{g}_i : (0,\infty) \longrightarrow (0,\infty), \text{ a continuous function } \overline{h}_i : [0,\infty) \longrightarrow (0,\infty) \text{ with } \frac{\overline{h}_i}{\overline{g}_i} \text{ nondecreasing on } (0,\infty) \ (i = 0,1,2,\dots,n), \text{ and with } f(t,u) \ge \overline{g}_0(u) + \overline{h}_0(u), \quad I_k(u) \ge \overline{g}_k(u) + \overline{h}_k(u) \quad (k = 1,\dots,n) \text{ for } (t,u) \in [\theta, \sigma(1)-\theta]\times(0,\infty), \tag{43}$$

$$\exists 0 < R_1 < r < R_2 \text{ with } (j = 1,2),$$

$$R_j \left(\min_{0\le i\le n} \left\{ \overline{g}_i(R_j) \times \left[1 + \left(\overline{h}_i \left(\frac{\phi_1^\Delta(\rho(0))}{\|\phi_1\|\,\|\phi_2\|} R_j \phi_1(\theta)\,\phi_2(\sigma(1)-\theta) \right) \times \left(\overline{g}_i \left(\frac{\phi_1^\Delta(\rho(0))}{\|\phi_1\|\,\|\phi_2\|} R_j\, \phi_1(\theta)\,\phi_2 \times (\sigma(1)-\theta) \right) \right)^{-1} \right) \right] \right\} \right)^{-1} < \mu \overline{b}_0,$$
$$\overline{b}_0 = \int_{\theta}^{\sigma(1)-\theta} G(\sigma,s)\,p(s)\,q(s)\nabla s + \sum_{k=1}^{n} G(\sigma,t_k). \tag{44}$$

Here $G(t,s)$ *is Green's function and*

$$\int_{\theta}^{\sigma(1)-\theta} G(\sigma,s)\,p(s)\,q(s)\,ds = \sup_{t\in[0,1]} \int_{\theta}^{\sigma(1)-\theta} G(t,s)\,p(s)\,q(s)\,ds. \tag{45}$$

Then (3) *has two nonnegative solutions* u_i *with* $R_1 < \|u_1\| < r < \|u_2\| < R_2$ *and* $u_i(t) > 0$ *for* $t \in (0,1)$, $i = 1,2$.

Proof. First we will show that there exists a solution u_2 to (3) with $u_2(t) > 0$ for $t \in (0,1)$ and $r < \|u_2\| < R_2$. Let

$$\Omega_1 = \{u \in E : \|u\| < r\}, \qquad \Omega_2 = \{u \in E : \|u\| < R_2\}. \tag{46}$$

We now show that

$$\|Tu\| < \|u\| \quad \text{for } K \cap \partial\Omega_1. \tag{47}$$

To see this, let $u \in K \cap \partial\Omega_1$. Then $\|u\| = \|u\|_{[0,1]} = r$ and $u(t) \ge G_0(t) r$ for $t \in [0,1]$. So for $t \in [0,1]$ we have

$$\begin{aligned} g_i(u) + h_i(u) &= g_i(u)\left(1 + \frac{h_i(u)}{g_i(u)}\right) \\ &\le g_i(G_0(t) r)\left(1 + \frac{h_i(r)}{g_i(r)}\right) \\ &\le K_0 g_i(r)\left(1 + \frac{h_i(r)}{g_i(r)}\right) g_i(G_0(t)) \\ &\le K_0 \left(g_i(r) + h_i(r)\right) g_i(G_0(t)) \\ &\le K_0 \max_{0\le i\le m} \{g_i(r) + h_i(r)\}\, g_i(G_0(t)), \\ &\quad \text{for } i = 0,1,2,\dots,n. \end{aligned} \tag{48}$$

Then

$$\begin{aligned}(Tu)(t) &= \int_{\rho(0)}^{\sigma(1)} G(t,s)\,p(s)\,q(s)\,f((s,u(s)))\,\nabla s \\ &\quad + \sum_{k=1}^{n} G(t,t_k)\,I_k(u(t_k)) \\ &\le \int_{\rho(0)}^{\sigma(1)} G^*(s)\,p(s)\,q(s)\,(g_0(u(s)) + h_0(u(s)))\,\nabla s \\ &\quad + \sum_{k=1}^{n} G^*(t_k)\,(g_k(u(s)) + h_k(u(s))) \\ &\le K_0\left[\int_{\rho(0)}^{\sigma(1)} G^*(s)\,p(s)\,q(s)\,g_0(G_0(s))\,\nabla s \right.\\ &\quad \left. + \sum_{k=1}^{n} G^*(t_k)\,g_k(G_0(t_k))\right] \\ &\quad \times \max_{0\le i\le m}\{g_i(r) + h_i(r)\}.\end{aligned} \tag{49}$$

This together with (42) yields

$$\|Tu\| = \|Tu\|_{[0,1]} < r = \|u\|, \tag{50}$$

so (47) is satisfied.

Next we show

$$\|Tu\| > \|u\| \quad \text{for } K\cap\partial\Omega_2. \tag{51}$$

To see this let $u \in K\cap\partial\Omega_2$ so $\|u\| = \|u\|_{[0,1]} = R_2$ and $u(t) \ge G_0(t)R_2$ for $t\in[0,1]$.

We have

$$\begin{aligned}&\overline{g}_i(u(s)) + \overline{h}_i(u(s)) \\ &\quad = \overline{g}_i(u(s))\left[1 + \frac{\overline{h}_i(u(s))}{\overline{g}_i(u(s))}\right] \\ &\quad \ge \overline{g}_i(R_2)\left[1 + \frac{\overline{h}_i(G_0(s)R_2)}{\overline{g}_i(G_0(s)R_2)}\right] \\ &\quad \ge \overline{g}_i(R_2) \\ &\qquad \times \left[1 + \left(\overline{h}_i\left(\frac{\phi_1^{\Delta}(\rho(0))}{\|\phi_1\|\,\|\phi_2\|}\;R_2\phi_1\right.\right.\right. \\ &\qquad\qquad \left. \times(\theta)\,\phi_2(\sigma(1)-\theta)\right) \\ &\qquad\qquad \times\left(\overline{g}_i\left(\frac{\phi_1^{\Delta}(\rho(0))}{\|\phi_1\|\,\|\phi_2\|}R_2\phi_1(\theta)\,\phi_2\right.\right. \\ &\qquad\qquad \left.\left.\left.\left. \times(\sigma(1)-\theta)\right)\right)^{-1}\right)\right] \\ &\qquad\qquad \text{for } s\in[\theta,\sigma(1)-\theta].\end{aligned} \tag{52}$$

Then

$$\begin{aligned}&(Tu)(\sigma) \\ &\quad \ge \int_{\rho(0)}^{\sigma(1)} G(\sigma,s)\,p(s)\,q(s)\left[\overline{g}_0(u(s)) + \overline{h}_0(u(s))\right]\nabla s \\ &\qquad + \sum_{k=1}^{n} G(\sigma,t_k)\left[\overline{g}_k(u(t_k)) + \overline{h}_k(u(t_k))\right] \\ &\quad \ge \left\{\int_{\theta}^{\sigma(1)-\theta} G(\sigma,s)\,p(s)\,q(s)\,\nabla s + \sum_{k=1}^{n} G(\sigma,t_k)\right\} \\ &\qquad \times \min_{0\le i\le m}\left\{\overline{g}_i(R_2)\left[1 + \left(\overline{h}_i\left(\frac{\phi_1^{\Delta}(\rho(0))}{\|\phi_1\|\,\|\phi_2\|}\;R_2\phi_1(\theta)\right.\right.\right.\right. \\ &\qquad\qquad \left.\times\phi_2(\sigma(1)-\theta)\right) \\ &\qquad\qquad \times\left(\overline{g}_i\left(\frac{\phi_1^{\Delta}(\rho(0))}{\|\phi_1\|\,\|\phi_2\|}\;R_2\phi_1(\theta)\,\phi_2\right.\right. \\ &\qquad\qquad \left.\left.\left.\left.\left. \times(\sigma(1)-\theta)\right)\right)^{-1}\right)\right]\right\}.\end{aligned} \tag{53}$$

This together with (44) yields

$$(Tu)(\sigma) > R_2 = \|u\|. \tag{54}$$

Thus $\|Tu\| > \|u\|$, so (51) holds.

Now Theorem 9 implies that T has a fixed point $u_2 \in K\cap(\bar{\Omega}_2\backslash\Omega_1)$; that is, $r \le \|u_2\| = \|u_2\|_{[0,1]} \le R$ and $u_2(t) \ge q(t)r$ for $t\in[0,1]$. It follows from (47) and (51) that $\|u_2\| \ne r$, $\|u_2\| \ne R_2$, so we have $r < \|u_2\| < R_2$.

Similarly, if we put

$$\Omega_1 = \{u\in E : \|u\| < R_1\}, \qquad \Omega_2 = \{u\in E : \|u\| < r\}, \tag{55}$$

we can show that there exists a solution u_1 to (3) with $u_1(t) > 0$ for $t\in(0,1)$ and $R_1 < \|u_1\| < r$. This completes the proof of Theorem 12. □

Similar to the proof of Theorem 12, we have the following result.

Theorem 13. *Suppose that conditions* (C1)–(C4) *and* (39)–(43) *hold. In addition suppose that*

$$\exists 0 < R_1 < r \text{ with}$$

$$R_1 \left(\min_{0\le i\le n} \left\{ \overline{g}_i (R_1) \times \left[1 + \left(\overline{h}_i \left(\frac{\phi_1^\Delta (\rho (0))}{\|\phi_1\| \|\phi_2\|} \times R_1 \phi_1 (\theta) \phi_2 (\sigma (1) - \theta) \right) \times \left(\overline{g}_i \left(\frac{\phi_1^\Delta (\rho (0))}{\|\phi_1\| \|\phi_2\|} \times R_1 \phi_1 (\theta) \phi_2 \times (\sigma (1) - \theta) \right) \right)^{-1} \right) \right] \right\} \right)^{-1}$$
$$< \mu \overline{b}_0,$$
$$\overline{b}_0 = \int_\theta^{\sigma(1)-\theta} G(\sigma, s) p(s) q(s) \nabla s + \sum_{k=1}^{n} G(\sigma, t_k). \tag{56}$$

Then (3) *has a nonnegative solution* u_1 *with* $R_1 < \|u_1\| < r$ *and* $u_1(t) > 0$ *for* $t \in (0, 1)$.

Remark 14. If in (56) we have $R_1 > r$, then (3) has a nonnegative solution u_2 with $r < \|u_2\| < R_2$ and $u_2(t) > 0$ for $t \in (0, 1)$.

It is easy to use Theorem 12 and Remark 14 to write theorems which guarantee the existence of more than two solutions to (3). We state one such result.

Theorem 15. *Suppose that conditions (C1)–(C4);* (39)–(41)*; and* (43) *hold. Assume that* $\exists m \in \{1, 2, \dots\}$ *and constants* R_j, r_j $(j = 1, \dots, m)$, *with* $r_1 > b_0$, *and*

$$0 < R_1 < r_1 < R_2 < r_2 < \cdots < R_m < r_m. \tag{57}$$

In addition suppose for each $j = 1, \dots, m$ *that*

$$\frac{r_j}{\max_{0\le i\le n} \{ g_j (r_j) + h_j (r_j) \}} > K_0^2 a_0,$$

$$R_j \left(\min_{0\le i\le n} \left\{ \overline{g}_i (R_j) \times \left[1 + \left(\overline{h}_i \left(\frac{\phi_1^\Delta (\rho (0))}{\|\phi_1\| \|\phi_2\|} \times R_j \phi_1 (\theta) \times \phi_2 (\sigma (1) - \theta) \right) \times \left(\overline{g}_i \left(\frac{\phi_1^\Delta (\rho (0))}{\|\phi_1\| \|\phi_2\|} \times R_j \phi_1 (\theta) \phi_2 \times (\sigma (1) - \theta) \right) \right)^{-1} \right) \right] \right\} \right)^{-1}$$
$$< \mu \overline{b}_0,$$
$$\overline{b}_0 = \int_\theta^{\sigma(1)-\theta} G(\sigma, s) p(s) q(s) \nabla s + \sum_{k=1}^{n} G(\sigma, t_k) \tag{58}$$

hold. Then (3) *has nonnegative solutions* $y_1, \dots, y_m$ *with* $y_j(t) > 0$ *for* $t \in (0, 1)$.

Example 16. Consider the boundary value problem

$$\begin{gathered} u^{\Delta\nabla} (t) + a(t) u^\Delta (t) + b(t) u(t) + \mu_0 \left(u^{-\alpha_0} (t) (1 + |\sin t|) + u^{\beta_0} (t) (1 + |\cos t|) \right) = 0, \quad t \in (0, 1), \ t \neq t_k, \\ u^\Delta (t_k^+) = u^\Delta (t_k) - \mu_k \left(u^{-\beta_k} (t_k) + u^{\beta_k} (t_k) \right), \\ k = 1, 2, \dots, n; \\ u(\rho(0)) = 0, \qquad u(\sigma(1)) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{gathered} \tag{59}$$

where $0 < \alpha_i < 1 < \beta_i$ $(i = 0, 1, 2, \dots, n)$, and $\max_{0\le i\le n}\{\mu_i\} \in (0, \mu)$ is such that

$$\mu \le \frac{1}{4 a_1}, \tag{60}$$

and here

$$a_1 = \int_{\rho(0)}^{\sigma(1)} G^* (s) p(s) g_0 (G_0 (s)) \nabla s + \sum_{k=1}^{n} G^* (t_k) g_k (G_0 (t_k)) < \infty. \tag{61}$$

Then (59) has two solutions u_1, u_2 with $u_1(t) > 0$, $u_2(t) > 0$ for $t \in (0, 1)$.

To see this we will apply Theorem 12 with (here $0 < R_1 < 1 < R_2$ will be chosen below)

$$\begin{gathered} g_i(u) = \frac{1}{2}\overline{g}_i(u) = 2u^{-\alpha_i}, \\ h_i(u) = \frac{1}{2}\overline{h}_i(u) = 2u^{\beta_i}, \\ i = 0, 1, 2, \ldots, n, \\ \theta = \frac{3\rho(0) + \sigma(1)}{4}, \qquad q(t) = \mu_0, \quad K_0 = 1. \end{gathered} \tag{62}$$

Clearly (39)–(41) and (43) hold, and

$$\begin{aligned} a_0 = \mu_0 \int_{\rho(0)}^{\sigma(1)} G^*(s)\, p(s)\, g_0(G(s,s)) \nabla s \\ + \sum_{k=1}^{n} \mu_k G^*(t_k)\, g_k(G_0(t_k)) < \infty. \end{aligned} \tag{63}$$

Now (42) holds with $r = 1$ since

$$\frac{r}{\max_{0\le i\le n}\{g_i(r) + h_i(r)\}} = \frac{1}{4} \ge \mu a_1 > K_0^2 a_0. \tag{64}$$

Finally, let

$$\begin{aligned} &\overline{g}_{i^*}(R_j)\Bigg[1 + \Bigg(\overline{h}_{i^*}\Bigg(\frac{\phi_1^\Delta(\rho(0))}{\|\phi_1\|\|\phi_2\|} \times R_j\phi_1(\theta)\,\phi_2(\sigma(1)-\theta)\Bigg) \\ &\quad \times \Bigg(\overline{g}_{i^*}\Bigg(\frac{\phi_1^\Delta(\rho(0))}{\|\phi_1\|\|\phi_2\|} \times R_j\phi_1(\theta)\,\phi_2(\sigma(1)-\theta)\Bigg)\Bigg)^{-1}\Bigg)\Bigg] \\ &= \min_{0\le i\le n}\Bigg\{\overline{g}_i(R_j)\Bigg[1 + \Bigg(\overline{h}_i\Bigg(\frac{\phi_1^\Delta(\rho(0))}{\|\phi_1\|\|\phi_2\|} \times R_j\phi_1(\theta)\,\phi_2(\sigma(1)-\theta)\Bigg) \\ &\quad \times \Bigg(\overline{g}_i\Bigg(\frac{\phi_1^\Delta(\rho(0))}{\|\phi_1\|\|\phi_2\|} R_j\phi_1(\theta)\,\phi_2 \times (\sigma(1)-\theta)\Bigg)\Bigg)^{-1}\Bigg)\Bigg]\Bigg\} \end{aligned} \tag{65}$$

and notice that (44) is satisfied for R_1 small and R_2 large since

$$\begin{aligned} &R_j\Bigg(\min_{0\le i\le n}\Bigg\{\overline{g}_i(R_j) \times \Bigg[1 + \Bigg(\overline{h}_i\Bigg(\frac{\phi_1^\Delta(\rho(0))}{\|\phi_1\|\|\phi_2\|} R_j\phi_1(\theta) \times \phi_2(\sigma(1)-\theta)\Bigg) \\ &\quad \times \Bigg(\overline{g}_i\Bigg(\frac{\phi_1^\Delta(\rho(0))}{\|\phi_1\|\|\phi_2\|} R_j\phi_1(\theta)\,\phi_2 \times (\sigma(1)-\theta)\Bigg)\Bigg)^{-1}\Bigg)\Bigg]\Bigg\}\Bigg)^{-1} \\ &\le R_j^{1+\alpha_{i^*}} \times \Bigg(1 + \Bigg(\frac{\phi_1^\Delta(\rho(0))}{\|\phi_1\|\|\phi_2\|}\phi_1(\theta) \times \phi_2(\sigma(1)-\theta)\Bigg)^{(\alpha_{i^*}+\beta_{i^*})} \times R_j^{(\alpha_{i^*}+\beta_{i^*})}\Bigg)^{-1} \longrightarrow 0, \end{aligned} \tag{66}$$

as $R_1 \to 0, R_2 \to \infty$, since $b > 1$. Thus all the conditions of Theorem 12 are satisfied so existence is guaranteed.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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