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Research Article

Mathematical Analysis of HIV Models with Switching Nonlinear Incidence Functions and Pulse Control

Xiying Wang,[1] Wei Xu,[1] Yujun Cui,[2] and Xiaomei Wang[3]

[1] *Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, China*
[2] *Department of Applied Mathematics, Shandong University of Science and Technology, Qingdao, Shandong 266510, China*
[3] *School of Mathematics Science, University of Electronic Science and Technology of China, Chengdu, Sichuan 610054, China*

Correspondence should be addressed to Xiying Wang; wangxiying668@163.com

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This paper aims to study the dynamics of new HIV (the human immunodeficiency virus) models with switching nonlinear incidence functions and pulse control. Nonlinear incidence functions are first assumed to be time-varying functions and switching functional forms in time, which have more realistic significance to model infectious disease models. New threshold conditions with the periodic switching term are obtained to guarantee eradication of the disease, by using the novel type of common Lyapunov function. Furthermore, pulse vaccination is applied to the above model, and new sufficient conditions for the eradication of the disease are presented in terms of the pulse effect and the switching effect. Finally, several numerical examples are given to show the effectiveness of the proposed results, and future directions are put forward.

1. Introduction

Dynamical behavior of HIV infection models has been investigated to explain different phenomena with the help of the persistence of the disease and the global stability of the disease-free equilibrium. D'Onofrio [1] studied the global asymptotic stability of the disease-free equilibrium of the HIV infection model. It has been shown that the drug efficacy functions are bang-bang type, and the stability of the infection free steady state was studied by the basic reproduction number $R = (1-e_1)(1-e_2)N\beta S/ab$ with $\gamma = e_1$ and $\eta = e_2$ [2].

Assume that the parameter S is the total rate of production of healthy cells per unit time; b, a, and c are the per capita death rate of healthy cells, infected cells, and infective virus particles, respectively. Then, a basic mathematical model of HIV dynamics [1, 3–6], consisting of three state variables at time t corresponding to concentration of uninfected target cells $T(t)$, infected cells $I(t)$, and free virus particles $V(t)$, may be described by the following:

$$\begin{aligned} \dot{T} &= S - bT - f(T,V), \\ \dot{I} &= f(T,V) - aI, \\ \dot{V} &= g(I) - cV, \end{aligned} \tag{1}$$

where $f(T,V)$ and $g(I)$ denote incidence rate functions which are the average number of new infected cells and new virus particles per unite time, respectively. In infectious disease modeling, the incidence functions have become a crucial factor to ensure that the models have some realistic significance and may give some reasonable description. For example, in [7–9], authors assumed that $f(T,V) = \beta TV$ and $g(I) = KI$, where β is the transmission coefficient between uninfected cells and infective virus particles, and K is the average number of infective virus particles produced by an infected cell. Under the influence of HAART

(highly active antiviral therapy), which is generally a combination of RTI (reverse transcriptase inhibitors) drugs and PI (protease inhibitors) drugs, it is assumed that $f(T,V) = (1-\gamma)\beta TV$ and $g(I) = (1-\eta)NaI$ [1, 2], where N is the average number of infective virus particles produced by an infected cell in the absence of HAART; $1-\gamma$ $(0 < \gamma < 1)$ is the reverse transcriptase inhibitor drug effect; and $1-\eta$ $(0 < \eta < 1)$ is the protease inhibitor drug effect. Here, $f(T,V)$ is bilinear with respect to the number of uninfected cells and virus particles, and $g(I)$ is linear with respect to the number of infected cells. In fact, both bilinear incidence rate and linear incidence rate may be modified.

Recently, most of epidemic infection models with nonlinear incidence functions have been investigated by many authors. Korobeinikov [10] established global properties of two-dimension SIR and SIRS compartmental epidemic models with nonlinear transmission rate based on the method of Lyapunov functions. Liu and Stechlinski [11] analyzed infectious disease models with time-varying parameters and general nonlinear incidence rates and obtained some sufficient conditions to ensure the stability of the disease-free equilibrium. If the population is saturated with the infected individuals, the incidence rate may have nonlinear dependence on infective individuals [12]. In general, the HIV models' parameters (e.g., the contact rate, the effect of RTI drugs, and PI drugs) were assumed to be constant in time. However, these parameters may be time-varying, due to changes in host behavior. Take the effect of RTI drugs and PI drugs; for example, the effect of RTI drugs and PI drugs is usually characterized by a quick rise to a maximum soon after drug intake, followed by a slower decay within a cycle. A more realistic approach is to assume that these parameters are time-varying, which implies that two nonlinear incidence functions are time-varying functions.

A switched epidemic model is modeled by introducing switching functions into the HIV model. Moreover, according to the method of [3, 13], two switching nonlinear incidence functions are introduced into system (1) by replacing general nonlinear incidence functions.

Switched systems, consisting of continuous, discrete dynamics and logic based switching rule, have gained considerable attention by authors [14–18]. One main feature of the switched system is that the included switching law may induce stability of the switched system composed of two unstable subsystems. Switched systems have been applied in various areas, such as engine control systems, neural networks, ecosystems, mechanical systems, and even biological systems. Stability results have been derived by many researchers, via the methods of Lyapunov exponents, switched Lyapunov functions, and common Lyapunov functions (see [19–23]). Until now, there are few works about the switched HIV models. This paper mainly analyzes the HIV models subjected to switching nonlinear incidence functions. Using common Lyapunov functions method, the global stability of the disease-free equilibrium is discussed and new stability criteria are established to ensure eradication of the disease.

Moreover, pulse control is applied into the HIV models with switching nonlinear incidence functions. Due to switches of states and abrupt changes at the switching instants, switched systems exhibit impulsive effects, and they cannot be well described by using pure continuous or discrete switched systems [24]. Some results on the impulsive systems have been obtained [25, 26]. In this paper, new HIV models with switching nonlinear incidence functions and pulse control are developed, and the global asymptotic stability by the technique of common Lyapunov functions is analyzed.

The paper is organized as follows. The HIV model with switching nonlinear incidence functions is introduced, and global asymptotic stability of the disease-free equilibrium is presented in Section 2. In Section 3, pulse control is considered in the above HIV model, and sufficient conditions for the global asymptotic stability are obtained by the method of common Lyapunov functions. Numerical simulations are given in Section 4 to illustrate the threshold conditions established in the paper. Some conclusions and future directions are given in Section 5.

2. The HIV Model with Switching Nonlinear Incidence Functions

In general nonlinear incidence functions [3], the HIV models' parameters (e.g., the contact rate, the effect of RTI drugs, and PI drugs) are constants in time. In fact, the HIV models' parameters may be time-varying, due to changes in host behavior [11, 19]. Taking the contact rate, for example, it may be time-varying and switch in time under the influence of environmental factors or changes in host behavior. According to the above approach, assume that incidence functions are time-varying functions and may change functional forms in time. Consider families of m different incidence functions $f_i(T,V)$ and $g_i(I)$, for $i = 1, 2, \ldots, m$. Assume that the average number of new infected cells and new virus per unit time are modeled by the switching functions $f_{i_k}(T,V)$ and $g_{i_k}(I)$ on the interval $(t_{k-1}, t_k]$, $k = 1, 2, \ldots$. Suppose that these switching functions are governed by a switching rule $\sigma(t)$: $(t_{k-1}, t_k] \to i_k \in \{1, 2, \ldots, m\}$, $k = 1, 2, \ldots$, which is a piecewise constant function of time, continuous from the left. Assume that the switch time $\{t_k\}$ satisfies $t_k > t_{k-1}$ and $t_k \to \infty$ as $k \to \infty$. Denote set of all switching rules by $\mathcal{I}$. This leads to a new HIV model with switching nonlinear incidence functions:

$$\begin{aligned} \dot{T} &= S - bT - f_{i_k}(T,V), \quad t \in (t_{k-1}, t_k], \\ \dot{I} &= f_{i_k}(T,V) - aI, \\ \dot{V} &= g_{i_k}(I) - cV. \end{aligned} \tag{2}$$

The initial conditions are $T(0) = T_0$, $I(0) = I_0$, and $V(0) = V_0$. Here T, I, and $V \in \mathcal{R}^+$, where $\mathcal{R}^+$ denotes the set of nonnegative real numbers, and S, b, a, and c are in $\mathcal{R}^+$. From physical considerations, assume that $f_i(T,V)$ and $g_i(I)$ are continuous functions, which satisfy $f_i(T,V) > 0$ and $g_i(I) > 0$, for all $t \geq t_0$, $i = 1, 2, \ldots, m$, except along the boundaries $f_i(T,0) = f_i(0,V) = 0$ and $g_i(0) = 0$. Then, (2) has a disease-free equilibrium $E_0 = (S/b, 0, 0)$ and m endemic solutions

$E_i^*(T_i^*, I_i^*, V_i^*)$ for $i = 1, 2, \ldots, m$, where T_i^*, I_i^*, V_i^* satisfy the following:

$$S = bT_i^* + aI_i^*, \qquad aI_i^* = f_i(V_i^*)T_i^*, \qquad cV_i^* = g_i(I_i^*). \tag{3}$$

HIV drugs are most commonly prescribed to be taken on a fixed dose, fixed time interval basis. In fact, the fixed time interval between two dose can be seen as periodic and denoted by ω. Suppose that the switching rule σ satisfies $t_k - t_{k-1} = \tau_k$ with $\tau_{k+m} = \tau_k$, for $t \in (t_{k-1}, t_k]$. Suppose that $f_{i_k}(t) = f_k(t)$ and $g_{i_k}(t) = g_k(t)$. Suppose that $f_k(t) = f_k(t+\omega)$ and $g_k(t) = g_k(t+\omega)$ and $\beta_k(t) = \beta_k(t+\omega)$, where $\omega = \tau_1 + \tau_2 + \cdots + \tau_m$. Let $\mathcal{I}_{\text{Periodic}}$ be the set of periodic switching rule with $\mathcal{I}_{\text{Periodic}} \subset \mathcal{I}$.

Introducing some variables, $X_1 = T - S/b$, $X_2 = I$, and $X_3 = V$, and substituting the corresponding variables into system (2), we get

$$\begin{aligned} \dot{X}_1 &= -bX_1 - f_{i_k}\left(X_1 + \frac{S}{b}, X_3\right), \quad t \in (t_{k-1}, t_k], \\ \dot{X}_2 &= f_{i_k}\left(X_1 + \frac{S}{b}, X_3\right) - aX_2, \\ \dot{X}_3 &= g_{i_k}(X_2) - cX_3. \end{aligned} \tag{4}$$

For convenience of analysis, rewrite (4) in the following form:

$$\dot{X} = AX + F_{i_k}(X), \quad t \in (t_{k-1}, t_k], \tag{5}$$

where $X = (X_1, X_2, X_3)'$, $'$ denotes transposition, A is a 3×3 square matrix, and F_{i_k} is a column vector given by

$$A = \begin{pmatrix} -b & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix}, \qquad F_{i_k} = \begin{pmatrix} -f_{i_k}\left(X_1 + \frac{S}{b}, X_3\right) \\ f_{i_k}\left(X_1 + \frac{S}{b}, X_3\right) \\ g_{i_k}(X_2). \end{pmatrix}. \tag{6}$$

Assume that switching nonlinear functions $f_i(X)$ and $g_i(X)$ for $i = 1, 2, \ldots, m$ satisfy locally Lipschitz conditions; that is, for each $X \in \mathcal{R}_3^+$, there exist functions $L_i \geq 0$, $R_i \geq 0$, and $\delta = \delta(X) > 0$, such that $\|X - Y\| < \delta$ implies that

$$\begin{aligned} \|f_i(X) - f_i(Y)\| &\leq L_i \|X - Y\|, \\ \|g_i(X) - g_i(Y)\| &\leq R_i \|X - Y\|, \end{aligned} \tag{7}$$

in which $\|X\| = (X_1^2 + X_2^2 + X_3^2)^{1/2}$.

Obviously, due to switches, the characters of system (2) are different from most existing models (see [1, 8] and the reference therein). It is necessary to investigate the above HIV model. By transformation, the dynamics of disease-free equilibrium of system (2) are the same as the trivial solution of system (5). Via common Lyapunov functions, we consider the global asymptotic stability of the trivial solution of system (5).

The idea is that we first find a common Lyapunov function for each subsystem and then impose restrictions on switching to guarantee the stability of system (5). We have the following result regarding the global asymptotic stability of the trivial solution of system (5).

Theorem 1. *Assume that the switching rule σ is periodic and that $f_i(t, X) = f_i(t + \omega, X)$ and $g_i(t, X) = g_i(t + \omega, X)$, for $i = 1, 2, \ldots, m$, where $f_i(X)$ and $g_i(X)$ satisfy* (7)*. If there exists a constant $C > 0$, such that*

$$\tau_1\lambda_1 + \tau_2\lambda_2 + \cdots + \tau_m\lambda_m < -C, \tag{8}$$

where $\lambda_i = \lambda + 2L_i^2 + R_i^2 + 1$ with $\lambda = \max\{-2a, -2b, -2c\}$, then, the trivial solution of system (5) *is globally asymptotically stable, which implies that the disease-free equilibrium E_0 of system* (2) *is globally asymptotically stable.*

Proof. Assume that $i = i_k$, for $t \in (t_{k-1}, t_k]$, where i_k follows the switching rule $\sigma(t) \in \mathcal{I}_{\text{Periodic}}$. Define a common Lyapunov function U as $U = X'X$. Then, we have

$$\frac{dU(t)}{dt} = \dot{X}'X + X'\dot{X} = X'\left(A' + A\right)X + 2F_{i_k}'X. \tag{9}$$

Note that

$$\begin{aligned} X'\left(A' + A\right)X &\leq \lambda X'X, \\ 2F_{i_k}'X &\leq 2\left\|f_{i_k}(t, X)\right\|^2 + \left\|g_{i_k}(t, X)\right\|^2 \\ &\quad + X'X \leq \left(2L_{i_k}^2 + R_{i_k}^2 + 1\right)X'X. \end{aligned} \tag{10}$$

Substituting (10) into (9) yields

$$\dot{U} \leq \left(\lambda + 2L_{i_k}^2 + R_{i_k}^2 + 1\right)X'X = \lambda_{i_k}U. \tag{11}$$

For $t \in (t_{k-1}, t_k]$, it follows that

$$U(t) \leq U(t_{k-1}) \exp\left[\int_{t_{k-1}}^{t} \lambda_{i_k}(s)\, ds\right]. \tag{12}$$

Apply (12) on each subinterval. For $t \in (t_0, t_1]$, $U(t) \leq U(t_0)\exp[\int_{t_0}^{t} \lambda_{i_1}(s)ds]$. For $t \in (t_1, t_2]$, $U(t) \leq U(t_1)\exp[\int_{t_1}^{t} \lambda_{i_2}(s)ds] \leq U(t_0)\exp[\int_{t_0}^{t_1} \lambda_{i_1}(s)ds + \int_{t_1}^{t} \lambda_{i_2}(s)ds]$. In general, for $t \in (t_{k-1}, t_k]$, $U(t) \leq U(t_{k-1})\exp[\int_{t_{k-1}}^{t} \lambda_{i_k}(s)ds] \leq U(t_0)\exp[\int_{t_0}^{t_1} \lambda_{i_1}(s)ds + \int_{t_1}^{t_2} \lambda_{i_2}(s)ds + \cdots + \int_{t_{k-1}}^{t} \lambda_{i_k}(s)ds]$. For $t \in (\omega - t_m, \omega]$, $U(t) \leq U(t_0)\exp[\int_{t_0}^{t_1} \lambda_{i_1}(s)ds + \int_{t_1}^{t_2} \lambda_{i_2}(s)ds + \cdots + \int_{\omega - t_m}^{t} \lambda_{i_m}(s)ds]$. Then, $U(\omega) \leq U(t_0)\exp[\tau_1\lambda_1 + \tau_2\lambda_2 + \cdots + \tau_m\lambda_m]$. Define $\zeta = \exp[\tau_1\lambda_1 + \tau_2\lambda_2 + \cdots + \tau_m\lambda_m]$. By condition (8), we have $\zeta < 1$; furthermore, $U(\omega) \leq \zeta U(t_0) < U(t_0)$. Similarly, it can be shown that $U(h\omega) \leq \zeta U((h-1)\omega)$ for any integer $h = 1, 2, \ldots$ and hence

$$\begin{aligned} U(h\omega) &\leq \zeta U((h-1)\omega) \leq \zeta\left(\zeta U((h-2)\omega)\right) \\ &\leq \cdots \leq \zeta^h U(t_0); \end{aligned} \tag{13}$$

it follows that $\lim_{h\to\infty} U(h\omega) = \lim_{h\to\infty} \zeta^h U(t_0) = 0$. In general, for $t \in (t_{k-1}, t_k]$ and $h\omega < t_k \le (h+1)\omega$, we have

$$\begin{aligned} U(t) &\le U(h\omega) \exp\left[\int_{t_0}^{t_1} \lambda_{i_1}(s)\,ds + \int_{t_1}^{t_2} \lambda_{i_2}(s)\,ds \right. \\ &\qquad \left. + \cdots + \int_{t_{k-1}}^{t} \lambda_{i_k}(s)\,ds\right] \\ &\le MU(h\omega) \le MU(t_0)\,\zeta^h = MU(t_0) \exp[h \ln \zeta], \end{aligned} \tag{14}$$

where $M = \max_{t_{k-1}<t\le t_k} \exp[\int_{t_0}^{t_1} \lambda_{i_1}(s)ds + \int_{t_1}^{t_2} \lambda_{i_2}(s)ds + \cdots + \int_{t_{k-1}}^{t} \lambda_{i_k}(s)ds]$. Note that $\ln\zeta < 0$; then, $U(t)$ converges to zero as $t \to \infty$ and $h \to \infty$. Therefore, the trivial solution of system (5) is globally asymptotically stable, which implies that the disease-free equilibrium E_0 of system (2) is globally asymptotically stable. □

Remark 2. A general Lyapunov function is not used to deal with switched differential equations. In order to display the effect of switching, we consider the dynamics of the switched HIV models via the method of the common Lyapunov function.

Remark 3. In Theorem 1, general criteria are established to ensure that the disease will die out, no matter whether the subsystems are stable or unstable. Compared with [13], the stability of the disease-free equilibrium of system (2) is investigated by the common Lyapunov functions method.

Remark 4. Assume that $f_i(T,V) \equiv (1-\gamma)\beta TV$ and $g_i(I) \equiv NaI$; system (2) reduces to the system (1.1) in [5]. Compared with the results without the switching effect of [5], the results here are closer to the reality with practical significance, characterizing the switching effect $\tau_i\lambda_i$ in Theorem 1.

3. The Switched HIV Model with Pulse Vaccination

In this section, we investigate the dynamics of HIV model with switching nonlinear incidence functions and pulse control by common Lyapunov functions. Pulse control is strategy of periodically vaccinating the infectious disease in a relatively short time [11, 26]. Assume that a fraction p ($0 < p < 1$) of infected cells is impulsively treated every $\omega > 0$ time unit, moving infected cells $qI(t)$ ($0 \le q < p < 1$) to the various classes. This is reasonable from a physical perspective, since some of infected cells are affected badly by pulse vaccination, and they are becoming various particles. If $q = 0$ and $p > 0$, then the vaccine has efficacy. If $p = q = 1$, then vaccine is failing completely. Applying pulse vaccination to system (2), this gives a new switched HIV model

$$\begin{aligned} \dot{T} &= S - bT - f_{i_k}(T,V), \quad t \in (t_{k-1}, t_k], \\ \dot{I} &= f_{i_k}(T,V) - aI, \\ \dot{V} &= g_{i_k}(I) - cV, \\ T(t^+) &= T(t), \quad t = t_k, \\ I(t^+) &= (1-p)\,I(t), \\ V(t^+) &= V(t) + qI(t). \end{aligned} \tag{15}$$

The initial conditions are $T(t_0^+) = T_0 > 0$, $I(t_0^+) = I_0 \ge 0$, and $V(t_0^+) = V_0 \ge 0$. Note that system (15) has the same disease-free equilibrium E_0 as that of system (2). Under the transformation $X_1 = T - S/b$, $X_2 = I$, $X_3 = V$, system (15) has the following form:

$$\begin{aligned} \dot{X}_1 &= -bX_1 - f_{i_k}\left(X_1 + \frac{S}{b}, X_3\right), \quad t \in (t_{k-1}, t_k], \\ \dot{X}_2 &= f_{i_k}\left(X_1 + \frac{S}{b}, X_3\right) - aX_2, \\ \dot{X}_3 &= g_{i_k}(X_2) - cX_3, \\ X_1(t^+) &= X_1(t), \quad t = t_k, \\ X_2(t^+) &= (1-p)\,X_2(t), \\ X_3(t^+) &= X_3(t) + qX_1(t). \end{aligned} \tag{16}$$

The vector system

$$\begin{aligned} \dot{X} &= AX + F_{i_k}(X), \quad t \in (t_{k-1}, t_k], \\ X(t^+) &= BX, \quad t = t_k, \end{aligned} \tag{17}$$

where $X = (X_1, X_2, X_3)$, A and B are 3×3 matrices, and F_{i_k} is a column vector given by

$$A = \begin{pmatrix} -b & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -c \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-p & 0 \\ 0 & q & 1 \end{pmatrix},$$

$$F_{i_k} = \begin{pmatrix} -f_{i_k}\left(X_1 + \frac{S}{b}, X_3\right) \\ f_{i_k}\left(X_1 + \frac{S}{b}, X_3\right) \\ g_{i_k}(X_2). \end{pmatrix}. \tag{18}$$

Assume that switching nonlinear functions $f_i(X)$ and $g_i(X)$ (for $i = 1, 2, \ldots, m$) also satisfy (7). The following theorems give conditions for global asymptotic stability of the trivial solution of system (17) or the disease-free equilibrium of system (15).

Theorem 5. *Assume that the switching rule σ is periodic and that $f_i(t,X) = f_i(t+\omega, X)$ and $g_i(t,X) = g_i(t+\omega, X)$, for $i = 1, 2, \ldots, m$, where $f_i(X)$ and $g_i(X)$ satisfy* (7). *If there exists a constant $C > 0$, such that*

$$m \ln \rho + \tau_1\lambda_1 + \tau_2\lambda_2 + \cdots + \tau_m\lambda_m < -C, \tag{19}$$

where $\rho = \max\{1, ((1-p)^2 + q^2 + 1 + \sqrt{((1-p)^2 + q^2 + 1)^2 - 4(1-p)^2})/2\}$ *and* $\lambda_i = \lambda + 2L_i^2 + R_i^2 + 1$

with $\lambda = \max\{-2a, -2b, -2c\}$*, then the trivial solution of system* (17) *is globally asymptotically stable, which implies that the disease-free equilibrium* E_0 *of system* (15) *is globally asymptotically stable.*

Proof. Assume that $i = i_k$, for $t \in (t_{k-1}, t_k]$, where i_k follows the switching rule $\sigma(t) \in \mathcal{J}_{\text{Periodic}}$. Define the common Lyapunov function U as follows: $U = X'X$. According to the proof of Theorem 1, we have $\dot{U} \le \lambda_k U$. For $t \in (t_{k-1}, t_k]$, it follows that

$$U(t) \le U\left(t_{k-1}^{+}\right) \exp\left[\int_{t_{k-1}}^{t} \lambda_k(s)\, ds\right], \tag{20}$$

and immediately after the impulsive switch time $t = t_k$,

$$\begin{aligned} U\left(t_k^{+}\right) &= X\left(t_k^{+}\right)' X\left(t_k^{+}\right) = \left(BX\left(t_k\right)\right)' \left(BX\left(t_k\right)\right) \\ &= X\left(t_k\right)' \left(B'B\right) X\left(t_k\right) \le \rho X\left(t_k\right)' X\left(t_k\right) = \rho U\left(t_k\right). \end{aligned} \tag{21}$$

Apply (20) and (21) on each subinterval. For $t \in (t_0, t_1]$, $U(t) \le U(t_0)\exp[\int_{t_0}^{t} \lambda_1(s)ds]$. Further, $U(t_1^+) \le \rho U(t_1)$ and hence $U(t_1^+) \le \rho U(t_0)\exp[\int_{t_0}^{t_1} \lambda_1(s)ds]$. For $t \in (t_1, t_2]$, $U(t) \le U(t_1^+)\exp[\int_{t_1}^{t} \lambda_2(s)ds] \le \rho U(t_0)\exp[\int_{t_0}^{t_1} \lambda_1(s)ds + \int_{t_1}^{t} \lambda_2(s)ds]$. Then $U(t_2^+) \le \rho U(t_2) \le \rho^2 U(t_0)\exp[\int_{t_0}^{t_1} \lambda_1(s)ds + \int_{t_1}^{t_2} \lambda_2(s)ds]$. In general, for $t \in (t_{k-1}, t_k]$, $U(t) \le U(t_{k-1}^+)\exp[\int_{t_{k-1}}^{t} \lambda_k(s)ds] \le \rho^{k-1} U(t_0)\exp[\int_{t_0}^{t_1} \lambda_1(s)ds + \int_{t_1}^{t_2} \lambda_2(s)ds + \cdots + \int_{t_{k-1}}^{t} \lambda_k(s)ds]$. Then, for $t \in (\omega - t_m, \omega]$, $U(t) \le \rho^{m-1} U(t_0)\exp[\int_{t_0}^{t_1} \lambda_{i_1}(s)ds + \cdots + \int_{\omega - t_m}^{t} \lambda_{i_m}(s)ds]$ and immediately after the impulsive switch time $t = \omega$,

$$\begin{aligned} U\left(\omega^{+}\right) &= X\left(\omega^{+}\right)' X\left(\omega^{+}\right) = \left(BX(\omega)\right)' BX(\omega) \\ &= X(\omega)' B' BX(\omega) \le \rho X(\omega)' X(\omega) = \rho U(\omega) \\ &\le \rho^m U\left(t_0\right) \exp\left[\int_{t_0}^{t_1} \lambda_{i_1}(s)\, ds + \int_{t_1}^{t_2} \lambda_{i_2}(s)\, ds \right. \\ &\qquad \left. + \cdots + \int_{t_{m-1}}^{t_m} \lambda_{i_m}(s)\, ds\right] \\ &= U\left(t_0\right) \exp\left[m \ln \rho + \int_{t_0}^{t_1} \lambda_{i_1}(s)\, ds + \int_{t_1}^{t_2} \lambda_{i_2}(s)\, ds \right. \\ &\qquad \left. + \cdots + \int_{t_{m-1}}^{t_m} \lambda_{i_m}(s)\, ds\right] \\ &= U\left(t_0\right) \exp\left[m \ln \rho + \tau_1\lambda_1 + \tau_2\lambda_2 + \cdots + \tau_m\lambda_m\right] \\ &= \zeta U\left(t_0\right), \end{aligned} \tag{22}$$

where $\zeta = \exp[m \ln \rho + \tau_1\lambda_1 + \tau_2\lambda_2 + \cdots + \tau_m\lambda_m] < 1$ from (19). Similarly, it can be shown that $U(h\omega^+) \le \zeta U((h-1)\omega^+)$ for any integer $h = 1, 2, \ldots$ and so

$$\begin{aligned} U\left(h\omega^{+}\right) &\le \zeta U\left((h-1)\,\omega^{+}\right) \le \zeta\left(\zeta U\left((h-2)\,\omega^{+}\right)\right) \\ &\le \cdots \le \zeta^h U\left(t_0\right); \end{aligned} \tag{23}$$

it follows that

$$\lim_{h\to\infty} U\left(h\omega^{+}\right) = \lim_{h\to\infty} \zeta^h U\left(t_0\right) = 0. \tag{24}$$

In general, for $t \in (t_{k-1}, t_k]$ and $h\omega < t_k \le (h+1)\omega$, then $U(t) \le U(h\omega^+)\rho^{m-1}\exp[\int_{t_0}^{t_1} \lambda_{i_1}(s)ds + \int_{t_1}^{t_2} \lambda_{i_2}(s)ds + \cdots + \int_{t_{k-1}}^{t} \lambda_{i_k}(s)ds] \le MU(h\omega^+)\rho^{m-1} \le \rho^{m-1} MU(t_0)\exp[h\ln\zeta]$, where $M = \max_{t_{k-1}<t\le t_k} \exp[\int_{t_0}^{t_1} \lambda_{i_1}(s)ds + \int_{t_1}^{t_2} \lambda_{i_2}(s)ds + \cdots + \int_{t_{k-1}}^{t} \lambda_{i_k}(s)ds]$. Note that the $\ln\zeta < 0$; then, $U(t)$ converges to zero as $t \to \infty$ and $h \to \infty$. Thus, the trivial solution of system (17) is globally asymptotically stable, which implies that the disease-free equilibrium E_0 of system (15) is globally asymptotically stable. □

Remark 6. In Theorem 5, a general criterion, characterizing the switching term and the pulse term, is established to guarantee the global asymptotic stability of system (15).

Remark 7. As a special case, when $\rho = 1$, the dynamics of the disease-free equilibrium are mainly determined by the switching term $\tau_k\lambda_k$, for $k = 1, 2, \ldots$.

4. Numerical Simulations

In order to illustrate the effectiveness of the proposed results above, the stability of HIV models with switching nonlinear incidence functions and pulse control is presented. Moreover, the comparison between results in HIV models with and without the switching effect is presented. Here, we assume that $t_0 = 0$.

Example 1. Consider a HIV model with switching nonlinear incidence functions

$$\begin{aligned} \dot{T} &= S - bT - 10\left(1 - \gamma_{i_k}\right)\beta_{i_k} \sin(TV)\, V, \quad t \in \left(t_{k-1}, t_k\right], \\ \dot{I} &= 10\left(1 - \gamma_{i_k}\right)\beta_{i_k} \sin(TV)\, V - aI, \\ \dot{V} &= \left(1 - \eta_{i_k}\right) Na\, \sin(I)\, I - cV \end{aligned} \tag{25}$$

with $i_k \in \{1, 2\}$ following a periodic switching rule

$$\sigma(t) = \begin{cases} 1, & \text{if } t \in (k, k + 0.75], \\ 2, & \text{if } t \in (k + 0.75, k + 1], \end{cases} \quad k = 0, 1, 2, \ldots, \tag{26}$$

and a HIV model without switching nonlinear incidence functions

$$\begin{aligned} \dot{T} &= S - bT - 10\left(1 - \gamma\right)\beta \sin(TV)\, V, \\ \dot{I} &= 10\left(1 - \gamma\right)\beta \sin(TV)\, V - aI, \\ \dot{V} &= \left(1 - \eta\right) Na \sin(I)\, I - cV. \end{aligned} \tag{27}$$

By (26), we have $\tau_1 = 0.75$ and $\tau_2 = 0.25$. In this example, we first consider the global asymptotic stability of system

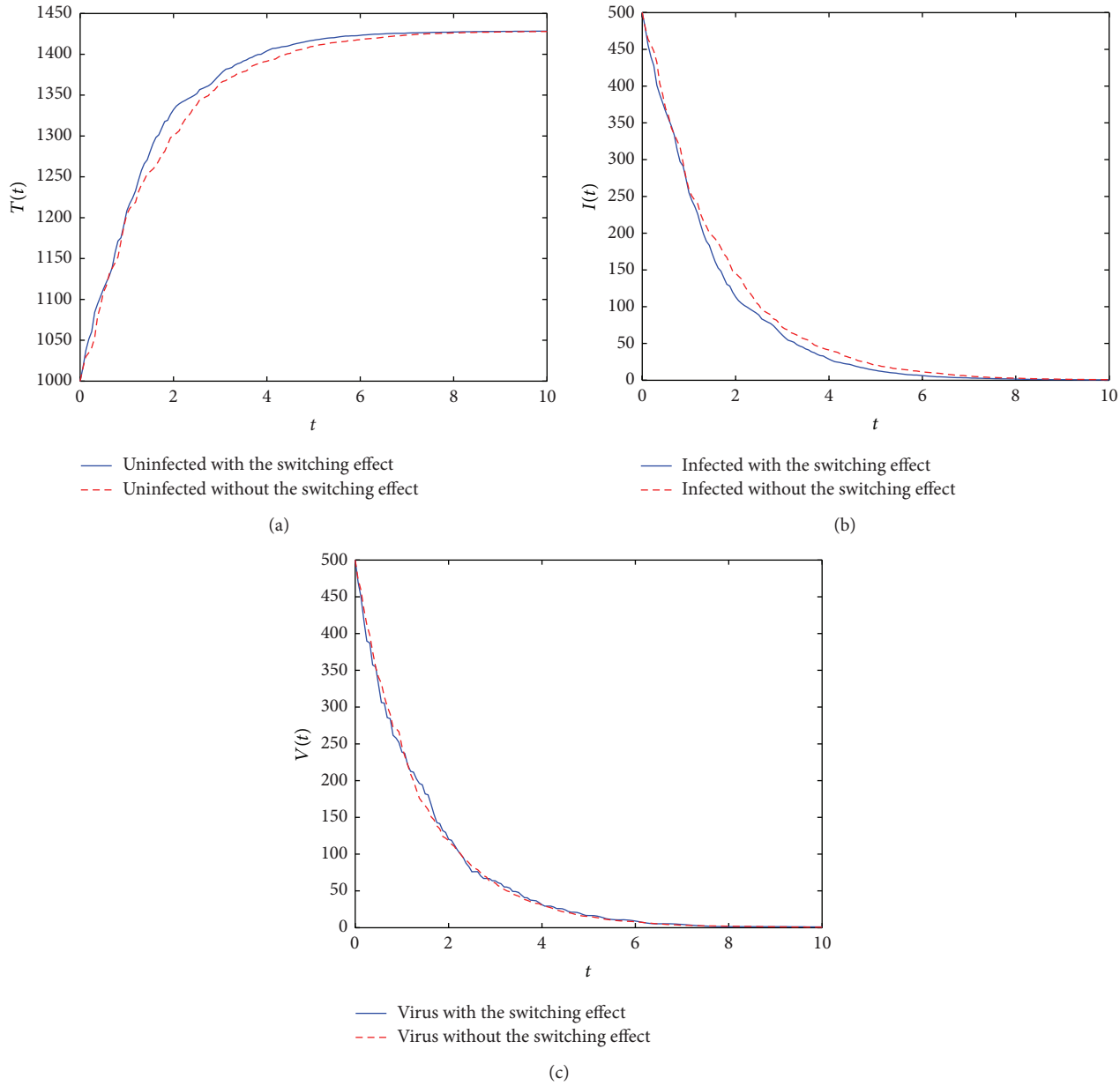

FIGURE 1: Numerical solutions of system (25) and system (27). The solutions with the switching effect (solid lines) were compared with the solutions without the switching effect (dashed lines).

(25) with initial values $T_0 = 1000$, $I_0 = 500$, and $V_0 = 500$. Take parameters $S = 10^3$, $b = 0.7$, $a = 0.7$, $c = 0.7$, and $N = 2$ and the switching parameters $\gamma_1 = 0.9$, $\gamma_2 = 0.7$, $\eta_1 = 0.8$, $\eta_2 = 0.6$, $\beta_1 = 0.32$, and $\beta_2 = 0.15$. For system (25) with these parameters specifications, we have $\|10(1-\gamma_{i_k})\beta_{i_k}\sin(TV)V\| \le 10(1-\gamma_{i_k})\beta_{i_k}\|V\|$ and $\|(1-\eta_{i_k})Na\sin(I)I\| \le (1-\eta_{i_k})Na\|I\|$ in (7) and $\tau_1\lambda_1 + \tau_2\lambda_2 = -0.0079$ with $\lambda_1 = -0.1168$ and $\lambda_2 = 0.3186$ in (8). Choosing $C = 0.0080$, the disease-free equilibrium $(T, I, V) = (S/b, 0, 0)$ is globally asymptotically stable by Theorem 1. The trajectory (solid lines) of system (25) is plotted in Figure 1, which indicates that this system is globally asymptotically stable. Furthermore, taking the parameters $\gamma = \gamma_2$, $\eta = \eta_2$, and $\beta = \beta_2$ of the system (27), and other parameters being the same as that of the system (25), we present the solutions (dashed lines) of the system (27) in Figure 1 for comparison purposes. In the first graph, we see that the total population of uninfected cells corresponding to the switching effect was much higher than the one without the switching effect, while, for infected cells and virus with the switching effect, the total populations were lower than the one without the switching effect (see the second graph and the third graph). Thus, computer simulations of these models agree well with mathematical theory.

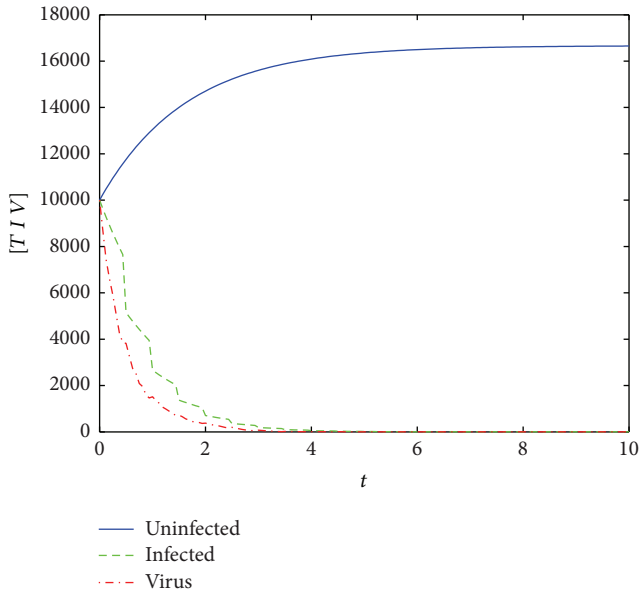

Figure 2: Numerical solutions of system (28).

Example 2. Consider a HIV model with switching nonlinear incidence functions and pulse control,

$$\begin{aligned}
\dot{T} &= S - bT - \left(1-\gamma_{i_k}\right)\beta_{i_k}\sin\left(TV\right)V, \quad t \in \left(t_{k-1}, t_k\right], \\
\dot{I} &= \left(1-\gamma_{i_k}\right)\beta_{i_k}\sin\left(TV\right)V - aI, \\
\dot{V} &= \left(1-\eta_{i_k}\right)Na\ \sin\left(I\right)I - cV, \\
T\left(t^+\right) &= T\left(t\right), \quad t = t_k, \\
I\left(t^+\right) &= \left(1-p\right)I\left(t\right), \\
V\left(t^+\right) &= V\left(t\right) + qI\left(t\right),
\end{aligned} \tag{28}$$

with $i_k \in \{1, 2\}$ following the periodic switching rule $\sigma(t)$: $\tau_1 = 0.5$, $\tau_2 = 0.5$, $m = 2$. According to Example 1, take $L_{i_k} = (1-\gamma_{i_k})\beta_{i_k}$ and $R_{i_k} = (1-\eta_{i_k})Na$ in (7). Take parameters $S = 10^4$, $b = 0.6$, $a = 0.6$, $c = 2$, and $N = 2$ and the switching parameters $\gamma_1 = 0.4$, $\gamma_2 = 0.3$, $\eta_1 = 0.7$, $\eta_2 = 0.6$, $\beta_1 = 10^{-4}$, and $\beta_2 = 10^{-5}$. Take the pulse control parameters $p = 0.3$ and $q = 0.1$. We can derive $\lambda_1 = -0.0704$, $\lambda_2 = 0.0304$, and $\rho = 1.0193$. Thus, (19) is satisfied with $C = 10^{-5}$. By Theorem 5, we can draw the conclusion that the disease-free equilibrium $(T, I, V) = (S/b, 0, 0)$ is globally asymptotically stable. Figure 2 shows a simulation result of this system. Moreover, it can be seen from Figure 2 that pulse vaccination is successful to cure the HIV infection.

5. Conclusions

In this paper, new HIV models with switching nonlinear incidence functions and pulse control are investigated. It is reasonable from a physical perspective that nonlinear incidence functions are assumed to be switching nonlinear incidence to incorporate into HIV models, since nonlinear incidence functions are changing in time, which may change functional form in time, due to changes in host behavior. For the periodic switching rule, some new sufficient conditions are established to ensure the global asymptotic stability of the disease-free equilibrium by constructing common Lyapunov functions. The obtained results have more advantages than those in [2, 7] and are very useful for a large class of infection disease models. The results indicated that the HIV model with the switching effect plays an important role in understanding the dynamics of the disease. Furthermore, taking pulse vaccination into the above model, a new HIV model with switching nonlinear incidence functions and pulse control is developed. Some sufficient conditions characterizing the pulse term and the switching term are derived to determine whether the pulse vaccination succeeded in preventing disease. Numerical examples are carried out to verify the proposed results. One future direction is to study multicity HIV infections models with switching parameters.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] A. d'Onofrio, "Periodically varying antiviral therapies: conditions for global stability of the virus free state," *Applied Mathematics and Computation*, vol. 168, no. 2, pp. 945–953, 2005.

[2] C. J. Browne and S. S. Pilyugin, "Periodic multidrug therapy in a within-host virus model," *Bulletin of Mathematical Biology*, vol. 73, no. 3, pp. 562–589, 2012.

[3] Z. Yuan, Z. Ma, and X. Tang, "Global stability of a delayed HIV infection model with nonlinear incidence rate," *Nonlinear Dynamics*, vol. 68, no. 1-2, pp. 207–214, 2012.

[4] H. Kwon, J. Lee, and S. Yang, "Optimal control of an age-structured model of HIV infection," *Applied Mathematics and Computation*, vol. 219, no. 5, pp. 2766–2779, 2012.

[5] T. Gao, W. Wang, and X. Liu, "Mathematical analysis of an HIV model with impulsive antiretroviral drug doses," *Mathematics and Computers in Simulation*, vol. 82, no. 4, pp. 653–665, 2011.

[6] H. Kwon, "Optimal treatment strategies derived from a HIV model with drug-resistant mutants," *Applied Mathematics and Computation*, vol. 188, no. 2, pp. 1193–1204, 2007.

[7] A. S. Perelson and P. W. Nelson, "Mathematical analysis of HIV-1 dynamics in vivo," *SIAM Review*, vol. 41, no. 1, pp. 3–44, 1999.

[8] P. De Leenheer and H. L. Smith, "Virus dynamics: a global analysis," *SIAM Journal on Applied Mathematics*, vol. 63, no. 4, pp. 1313–1327, 2003.

[9] R. M. Ribeiro, L. Qin, L. L. Chavez, D. Li, S. G. Self, and A. S. Perelson, "Estimation of the initial viral growth rate and basic

reproductive number during acute HIV-1 infection," *Journal of Virology*, vol. 84, no. 12, pp. 6096–6102, 2010.

[10] A. Korobeinikov, "Lyapunov functions and global stability for SIR and SIRS epidemiological models with non-linear transmission," *Bulletin of Mathematical Biology*, vol. 68, no. 3, pp. 615–626, 2006.

[11] X. Liu and P. Stechlinski, "Infectious disease models with time-varying parameters and general nonlinear incidence rate," *Applied Mathematical Modelling*, vol. 36, no. 5, pp. 1974–1994, 2012.

[12] Q. Yang, D. Jiang, N. Shi, and C. Ji, "The ergodicity and extinction of stochastically perturbed SIR and SEIR epidemic models with saturated incidence," *Journal of Mathematical Analysis and Applications*, vol. 388, no. 1, pp. 248–271, 2012.

[13] L. Cai, B. Guo, and X. Li, "Global stability for a delayed HIV-1 infection model with nonlinear incidence of infection," *Applied Mathematics and Computation*, vol. 219, no. 2, pp. 617–623, 2012.

[14] L. V. Hien, Q. P. Ha, and V. N. Phat, "Stability and stabilization of switched linear dynamic systems with time delay and uncertainties," *Applied Mathematics and Computation*, vol. 210, no. 1, pp. 223–231, 2009.

[15] P. Li, S. Zhong, and J. Cui, "Stability analysis of linear switching systems with time delays," *Chaos, Solitons & Fractals*, vol. 40, no. 1, pp. 474–480, 2009.

[16] H. Lin and P. J. Antsaklis, "Stability and stabilizability of switched linear systems: a survey of recent results," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 308–322, 2009.

[17] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, "Stability criteria for switched and hybrid systems," *SIAM Review*, vol. 49, no. 4, pp. 545–592, 2007.

[18] Z. Zhang and X. Liu, "Robust stability of uncertain discrete impulsive switching systems," *Computers & Mathematics with Applications*, vol. 58, no. 2, pp. 380–389, 2009.

[19] X. Liu and P. Stechlinski, "Pulse and constant control schemes for epidemic models with seasonality," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 2, pp. 931–946, 2011.

[20] E. Moulay, R. Bourdais, and W. Perruquetti, "Stabilization of nonlinear switched systems using control Lyapunov functions," *Nonlinear Analysis*, vol. 1, no. 4, pp. 482–490, 2007.

[21] X. Meng, Z. Li, and X. Wang, "Dynamics of a novel nonlinear SIR model with double epidemic hypothesis and impulsive effects," *Nonlinear Dynamics*, vol. 59, no. 3, pp. 503–513, 2010.

[22] Y. Xu, X. Wang, H. Zhang, and W. Xu, "Stochastic stability for nonlinear systems driven by Lévy noise," *Nonlinear Dynamics*, vol. 68, no. 1-2, pp. 7–15, 2012.

[23] X. Wang, Y. Xu, W. Xu, and H. Zhang, "Lyapunov exponents for nonlinear systems driven by Levy noise," *Journal of Dynamics Control*, vol. 9, pp. 135–138, 2011.

[24] Z. Guan, D. J. Hill, and X. Shen, "On hybrid impulsive and switching systems and application to nonlinear control," *IEEE Transactions on Automatic Control*, vol. 50, no. 7, pp. 1058–1062, 2005.

[25] X. Meng and Z. Li, "The dynamics of plant disease models with continuous and impulsive cultural control strategies," *Journal of Theoretical Biology*, vol. 266, no. 1, pp. 29–40, 2010.

[26] T. Zhang, X. Meng, Y. Song, and Z. Li, "Dynamical analysis of delayed plant disease models with continuous or impulsive cultural control strategies," *Abstract and Applied Analysis*, vol. 2012, Article ID 428453, 25 pages, 2012.