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Research Article

Hermite-Hadamard Type Inequalities for Superquadratic Functions via Fractional Integrals

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We use basic properties of superquadratic functions to obtain some new Hermite-Hadamard type inequalities via Riemann-Liouville fractional integrals. For superquadratic functions which are also convex, we get refinements of existing results.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$; then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2} \qquad (1)$$

is known as the Hermite-Hadamard inequality.

This remarkable result is well known in the literature as the Hermite-Hadamard inequality. Recently, the generalizations, refinements, and improvements of the classical Hermite-Hadamard inequality have been the subject of intensive research.

Definition 1 (see [1]). A function $f : [0,\infty) \to \mathbb{R}$ is superquadratic provided that for all $x \ge 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$f(y) \ge f(x) + C_x(y-x) + f(|y-x|), \qquad (2)$$

for all $y \ge 0$.

Theorem 2 (see [1]). *The inequality*

$$f\left(\int g\,d\mu\right) \le \int \left[f(g(s)) - f\left(\left|g(s) - \int g\,d\mu\right|\right)\right] d\mu(s) \qquad (3)$$

holds for all probability measures μ and all nonnegative, μ-integrable functions g, if and only if f is superquadratic.

The discrete version of the above theorem is also used in the sequel.

Lemma 3 (see [2]). *Suppose that f is superquadratic. Let $x_i \ge 0$, $1 \le i \le n$, and let $\overline{x} = \sum_{i=1}^n \lambda_i x_i$, where $\lambda_i \ge 0$ and $\sum_{i=1}^n \lambda_i = 1$. Then*

$$\sum_{i=1}^n \lambda_i f(x_i) \ge f(\overline{x}) + \sum_{i=1}^n \lambda_i f(|x_i - \overline{x}|). \qquad (4)$$

Nonnegative superquadratic functions are much better behaved as we see next.

Lemma 4 (see [1]). *Let f be a superquadratic function with $C_x \in \mathbb{R}$ as in Definition 1. Then one gets the following:*

(1) $f(0) \le 0$;

(2) *if $f(0) = f'(0) = 0$, then $C_x = f'(x)$ whenever f is differentiable at $x > 0$;*

(3) *if $f \ge 0$, then f is convex and $f(0) = f'(0) = 0$.*

The Hermite-Hadamard inequalities for superquadratic functions are established by Banic et al. in [3].

Theorem 5 (see [3]). *Let $f : [0, \infty) \to R$ be a superquadratic function and $0 \leq a < b$; then*

$$f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f\left(\left|x-\frac{a+b}{2}\right|\right)dx \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} - \frac{1}{(b-a)^2}\int_a^b \left[(b-x)f(x-a) + (x-a)f(b-x)\right]dx. \quad (5)$$

It is remarkable that Sarikaya et al. [4] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 6 (see [4]). *Let $f : [a, b] \to R$ be a positive function with $a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2}, \quad (6)$$

with $\alpha > 0$.

We remark that the symbols $J_{a^+}^{\alpha}$ and $J_{b^-}^{\alpha} f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \geq 0$ with $a \geq 0$ which are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a,$$
$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x < b, \quad (7)$$

respectively. Here, $\Gamma(\alpha)$ is the gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$.

Fractional integral operators are widely used to solve differential equations and integral equations. So a lot of work has been obtained on the theory and applications of fractional integral operators.

For more results concerning the fractional integral operators, we refer the reader to [5–10] and references cited therein.

In this paper, we establish some new Hermite-Hadamard type inequalities for superquadratic functions via Riemann-Liouville fractional integrals which refine the inequalities of (6) for superquadratic functions which are also convex.

2. Main Results

From Lemma 3 for $n = 2$, we get for $0 \leq m \leq z \leq M$, $m \leq M$, for the superquadratic function f that

$$f(z) \leq \frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M) - \frac{M-z}{M-m} f(z-m) - \frac{z-m}{M-m} f(M-z),$$
$$f(M+m-z) \leq \frac{z-m}{M-m} f(m) + \frac{M-z}{M-m} f(M) - \frac{M-z}{M-m} f(z-m) - \frac{z-m}{M-m} f(M-z) \quad (8)$$

hold, and therefore

$$f(z) + f(M+m-z) \leq f(m) + f(M) - 2\frac{M-z}{M-m} f(z-m) - 2\frac{z-m}{M-m} f(M-z). \quad (9)$$

Let $0 \leq a \leq x \leq (a+b)/2$; we get that

$$a \leq x \leq \frac{a+b}{2} \leq a+b-x \leq b. \quad (10)$$

Therefore, by replacing in (9)

$$z = \frac{a+b}{2}, \qquad M = a+b-x, \qquad m = x, \quad (11)$$

we get that

$$2f\left(\frac{a+b}{2}\right) \leq f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2} - x\right). \quad (12)$$

Theorem 7. *Let f be a superquadratic integrable function on $[0, b]$ with $0 \leq a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] - \frac{\alpha}{2(b-a)^{\alpha}}\int_a^b f\left|\frac{a+b}{2} - x\right| \times \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right)dx \quad (13)$$

with $\alpha > 0$.

Proof. By inequality (12), we get that

$$f\left(\frac{a+b}{2}\right)=f\left(\frac{a+b}{2}\right)\frac{\alpha}{2(b-a)^{\alpha}}\times\int_a^b\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx$$
$$=f\left(\frac{a+b}{2}\right)\frac{\alpha}{(b-a)^{\alpha}}\times\int_a^{(a+b)/2}\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx$$
$$\leq\frac{\alpha}{2(b-a)^{\alpha}}\int_a^{(a+b)/2}\left[f(x)+f(a+b-x)\right]\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx$$
$$-\frac{\alpha}{(b-a)^{\alpha}}\int_a^{(a+b)/2}f\left(\frac{a+b}{2}-x\right)\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx. \tag{14}$$

By the change of variable $x\rightarrow a+b-x$, we get

$$\frac{\alpha}{2(b-a)^{\alpha}}\int_a^{(a+b)/2}f(a+b-x)\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx=\frac{\alpha}{(b-a)^{\alpha}}\int_{(a+b)/2}^{b}f(x)\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx,$$
$$\frac{\alpha}{(b-a)^{\alpha}}\int_a^{(a+b)/2}f\left(\frac{a+b}{2}-x\right)\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx=\frac{\alpha}{(b-a)^{\alpha}}\int_{(a+b)/2}^{b}f\left(x-\frac{a+b}{2}\right)\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx. \tag{15}$$

Therefore

$$f\left(\frac{a+b}{2}\right)=f\left(\frac{a+b}{2}\right)\frac{\alpha}{2(b-a)^{\alpha}}\times\int_a^b\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx$$
$$=f\left(\frac{a+b}{2}\right)\frac{\alpha}{(b-a)^{\alpha}}\times\int_a^{(a+b)/2}\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx$$
$$\leq\frac{\alpha}{2(b-a)^{\alpha}}\int_a^b f(x)\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx-\frac{\alpha}{2(b-a)^{\alpha}}\int_a^b f\left|\frac{a+b}{2}-x\right|\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx$$
$$=\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-\frac{\alpha}{2(b-a)^{\alpha}}\int_a^b f\left|\frac{a+b}{2}-x\right|\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx. \tag{16}$$

We have completed the proof. □

Corollary 8. *Putting $\alpha=1$ in Theorem 7 gives*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{b-a}\int_a^b f\left|\frac{a+b}{2}-x\right|dx. \tag{17}$$

Let $0\leq a\leq x\leq(a+b)/2$; we get that

$$a\leq x\leq a+b-x\leq b. \tag{18}$$

Therefore, by replacing in (9)

$$z=x,\qquad M=b,\qquad m=a, \tag{19}$$

we get that

$$f(x)+f(a+b-x)\leq f(a)+f(b)-2\frac{b-x}{b-a}f(x-a)-2\frac{x-a}{b-a}f(b-x). \tag{20}$$

Theorem 9. *Let f be a superquadratic integrable function on $[0,b]$ with $0\leq a<b$. Then*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2}-\frac{\alpha}{2(b-a)^{\alpha}}\int_a^b\left[\frac{b-x}{b-a}f(x-a)+\frac{x-a}{b-a}f(b-x)\right]\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx \tag{21}$$

with $\alpha>0$.

Proof. By inequality (20), we get that

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} \\
&\quad= \frac{f(a)+f(b)}{2}\frac{\alpha}{2(b-a)^{\alpha}} \\
&\qquad\times\int_a^b\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx \\
&\quad= \left[f(a)+f(b)\right]\frac{\alpha}{2(b-a)^{\alpha}} \\
&\qquad\times\int_a^{(a+b)/2}\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx, \\
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right] \\
&\quad= \frac{\alpha}{2(b-a)^{\alpha}}\int_a^b f(x) \\
&\qquad\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx \\
&\quad= \frac{\alpha}{2(b-a)^{\alpha}}\int_a^{(a+b)/2}\left[f(x)+f(a+b-x)\right] \\
&\qquad\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx \\
&\quad\le \frac{\alpha}{2(b-a)^{\alpha}}\int_a^{(a+b)/2}\left[f(a)+f(b)\right] \\
&\qquad\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx \\
&\qquad- \frac{\alpha}{(b-a)^{\alpha}}\int_a^{(a+b)/2}\frac{b-x}{b-a}f(x-a) \\
&\qquad\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx \\
&\qquad- \frac{\alpha}{(b-a)^{\alpha}}\int_a^{(a+b)/2}\frac{x-a}{b-a}f(b-x) \\
&\qquad\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx \\
&\quad= \frac{f(a)+f(b)}{2} \\
&\qquad- \frac{\alpha}{2(b-a)^{\alpha}}\int_a^b\left[\frac{b-x}{b-a}f(x-a)\right. \\
&\qquad\left.+\frac{x-a}{b-a}f(b-x)\right] \\
&\qquad\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx.
\end{aligned} \tag{22}$$

The proof is completed. □

Corollary 10. *Choosing $\alpha = 1$ in Theorem 9, one has*

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)\,dx \le{}& \frac{f(a)+f(b)}{2} \\
&- \frac{1}{(b-a)^2}\int_a^b\left[(b-x)f(x-a)\right. \\
&\left.+(x-a)f(b-x)\right]dx.
\end{aligned} \tag{23}$$

Corollary 11. *Let f be defined as in Theorem 7; one gets*

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)+\frac{\alpha}{2(b-a)^{\alpha}} \\
&\quad\times\int_a^b f\left|\frac{a+b}{2}-x\right|\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx \\
&\le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right] \\
&\le \frac{f(a)+f(b)}{2}-\frac{\alpha}{2(b-a)^{\alpha}} \\
&\quad\times\int_a^b\left[\frac{b-x}{b-a}f(x-a)+\frac{x-a}{b-a}f(b-x)\right] \\
&\qquad\times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx.
\end{aligned} \tag{24}$$

Corollary 12. *Taking $\alpha = 1$ in Corollary 11, one obtains*

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f\left|\frac{a+b}{2}-x\right|dx \\
&\quad\le \int_a^b f(x)\,dx \\
&\quad\le \frac{f(a)+f(b)}{2}-\frac{1}{(b-a)^2} \\
&\qquad\times\int_a^b\left[(b-x)f(x-a)\right. \\
&\qquad\left.+(x-a)f(b-x)\right]dx.
\end{aligned} \tag{25}$$

3. Conclusion

In this note, we obtain some new Hermite-Hadamard type inequalities for superquadratic functions via Riemann-Liouville fractional integrals. For superquadratic functions which are also convex, we get refinements of known results. The concept of superquadratic functions in several variables is introduced in [11]. An interesting topic is whether we can use the methods in this paper to establish the Hermite-Hadamard inequalities for superquadratic functions in several variables via Riemann-Liouville integrals.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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