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Research Article

On FS_+-Domains

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We introduce a new construction—FS_+-domain—and prove that the category with FS_+-domains as objects and Scott continuous functions as morphisms is a Cartesian closed category. We obtain that the Plotkin powerdomain $P^P(L)$ over an FS-domain L is an FS_+-domain.

1. Introduction

Powerdomains are very important structures in Domain theory, which play an important role in modeling the semantics of nondeterministic programming languages. Three classical powerdomains are the Hoare or lower powerdomain [1], the Smyth or upper powerdomain [2], and the Plotkin or convex powerdomain [3]. They are all free dcpo-algebras over (continuous) dcpos with special binary operators satisfying some equations and inequalities (see [4–12]).

In [13], Huth et al. concluded that the Hoare powerdomain $P^H(L)$ over a pointed domain L is a distributive $FS_\vee$-lattice. In [14], Meng and Kou obtained that the Smyth powerdomain $P^S(L)$ of a Lawson compact domain L is an $FS_\wedge$-domain. Then we have a problem whether the Plotkin powerdomain can be characterized by some special FS-domain. In this paper, we will introduce a new domain construction called the FS_+-domain which is a +-semilattice and there exists a directed family of finitely separated Scott continuous and +-semilattice homomorphisms which can approximate id_L, where the operation + is Scott continuous which satisfied the commutative, associative, and idempotency laws. And the category with FS_+-domains as objects and Scott continuous functions as morphisms is a Cartesian closed category. We will show that the Plotkin powerdomain $P^P(L)$ over an FS-domain L is an FS_+-domain, where the Plotkin powerdomain is the free dcpo-semilattice over a continuous dcpo.

Next, we collect some basic notions needed in this paper. The reader can also consult [4, 5, 15, 16]. A poset L is called a directed complete poset (a dcpo, for short) if any nonempty directed subset of L has a sup in L. For $x, y \in L$, x is way below y (denoted by $x \ll y$) if and only if, for all directed subsets $D \subseteq L$ for which sup D exists, the relation $y \leq \sup D$ implies the existence of a $d \in D$ with $x \leq d$. A dcpo L is called a continuous domain if, for all $x \in L$, $x = \bigvee^{\uparrow} \Downarrow x$; that is, the set $\Downarrow x = \{a \in L : a \ll x\}$ is directed and $x = \bigvee\{a \in L : a \ll x\}$. For a subset A of L, let $\uparrow A = \{x \in L : \exists a \in A,\ a \leq x\}$, $\downarrow A = \{x \in L : \exists a \in A,\ x \leq a\}$. We use $\uparrow a$ (resp., $\downarrow a$) instead of $\uparrow \{a\}$ (resp., $\downarrow \{a\}$) when $A = \{a\}$. A is called an upper (resp., a lower) set if $A = \uparrow A$ (resp., $A = \downarrow A$). If $(L, \leq)$ is a dcpo, we define the Scott topology, denoted by $\sigma(L)$, which has as its topology of closed sets all directed complete lower subsets, that is, lower sets closed under directed sups. A function f from a dcpo L into a dcpo P is continuous with respect to the Scott topologies if f preserves suprema of directed subsets.

Recall the definition of FS-domain: a dcpo L is called an FS-domain if id_L is approximated directly by a family of finitely separated Scott continuous functions. A Scott continuous function $f : L \to L$ is called finitely separated if there exists a finite set M_f such that, for each $x \in L$, there exists $m \in M_f$ such that $f(x) \leq m \leq x$.

2. FS_+-Domains

2.1. Categories of FS_+-Domains. For dcpos L and P, the function space $[L \to P]$ of all Scott continuous functions from L to P with the pointwise order is a dcpo. Then for dcpo +-semilattices D and E, we conclude that the function space $[D \rightarrow_+ E]$ of all the Scott continuous and +-semilattice homomorphisms from D to E with the pointwise order is a dcpo +-semilattice from the following theorem, where the operation + satisfies the commutative, associative, and idempotency laws.

Theorem 1. *Let D and E be dcpo +-semilattices; then $[D \rightarrow_+ E]$ is a dcpo +-semilattice.*

Proof. For any directed family $\{f_j \in [D \rightarrow_+ E] : j \in J\}$ and $x \in D$, set $f(x) = \bigvee_{j\in J} f_j(x)$. It is obvious that f is Scott continuous. Then

$$\begin{aligned} f(x+y) &= \bigvee_{j\in J} f_j(x+y) = \bigvee_{j\in J} \left(f_j(x) + f_j(y)\right) \\ &= \left(\bigvee_{j\in J} f_j(x)\right) + \left(\bigvee_{j\in J} f_j(y)\right) = f(x) + f(y). \end{aligned} \tag{1}$$

So f is also a Scott continuous and +-semilattice homomorphism. Hence $[D \rightarrow_+ E]$ is a dcpo.

For any $x \in D$, $f, g \in [D \rightarrow_+ E]$, we define $(f+g)(x) = f(x) + g(x)$. For a directed set $\{x_k \in D : k \in K\}$, we have

$$\begin{aligned} (f+g)\left(\bigvee_{k\in K}(x_k)\right) &= f\left(\bigvee_{k\in K}(x_k)\right) + g\left(\bigvee_{k\in K}(x_k)\right) \\ &= \bigvee_{k\in K} f(x_k) + \bigvee_{k\in K} g(x_k) \\ &= \bigvee_{k\in K}\bigvee_{k'\in K} [f(x_k) + g(x_{k'})] \\ &= \bigvee_{k\in K} [f(x_k) + g(x_k)] \\ &= \bigvee_{k\in K} [(f+g)(x_k)]. \end{aligned} \tag{2}$$

Then $f+g$ is Scott continuous.

For a pair of points x, y in D,

$$\begin{aligned} (f+g)(x+y) &= f(x+y) + g(x+y) \\ &= (f(x) + f(y)) + (g(x) + g(y)) \\ &= (f(x) + g(x)) + (f(y) + g(y)) \\ &= (f+g)(x) + (f+g)(y). \end{aligned} \tag{3}$$

That is, $f+g$ is a +-semilattice homomorphism. So $[D \rightarrow_+ E]$ is a +-semilattice.

Finally, by the Scott continuity of the operation +, we obtain the following conclusion. For the sup of the directed set $\{f_j \in [D \rightarrow_+ E] : j \in J\}$ and $g \in [D \rightarrow_+ E]$, if $x \in D$, then

$$\begin{aligned} \left[g + \left(\bigvee_{j\in J} f_j\right)\right](x) &= g(x) + \left(\bigvee_{j\in J} f_j(x)\right) \\ &= \bigvee_{j\in J}\left[g(x) + f_j(x)\right] \\ &= \bigvee_{j\in J}\left[\left(g + f_j\right)(x)\right] \\ &= \left[\bigvee_{j\in J}\left(g + f_j\right)\right](x). \end{aligned} \tag{4}$$

So $+ : [D \rightarrow_+ E] \times [D \rightarrow_+ E] \rightarrow [D \rightarrow_+ E]$ is Scott continuous.

We have obtained that $[D \rightarrow_+ E]$ is a dcpo +-semilattice. □

With respect to these special Scott continuous functions, we will introduce some new order structures.

Definition 2. A dcpo L is called an FS_+-domain if it is a +-semilattice and there exists a directed family of finitely separated Scott continuous and +-semilattice homomorphisms which can approximate id_L.

For example, an $FS_\wedge$-domain is a continuous dcpo $\wedge$-semilattice where id is approximated by a directed family of finitely separated Scott continuous functions preserving finite infs.

We know that an FS_+-domain is an FS-domain.

Theorem 3. *Let D and E be FS_+-domains; then $[D \rightarrow_+ E]$ and $[D \rightarrow E]$ are FS_+-domains.*

Proof. Suppose that $\mathscr{D}$ and $\mathscr{E}$ are approximate identities for D and E, respectively. Then we claim that the family

$$\mathscr{D} \otimes \mathscr{E} = \{\delta \otimes \epsilon : \delta \in \mathscr{D}, \epsilon \in \mathscr{E}\}, \tag{5}$$

defined by

$$f \longmapsto \epsilon^2 f \delta^2, \tag{6}$$

for $f \in [D \rightarrow_+ E]$ is an approximate identity for $[D \rightarrow_+ E]$ where $\delta \otimes \epsilon$ is finitely separated. The proof is similar to the case of FS-domains.

It suffices to show that $\delta \otimes \epsilon \in [D \rightarrow_+ E] \rightarrow_+ [D \rightarrow_+ E]$. Firstly, it is obvious that $\delta \otimes \epsilon$ is Scott continuous. Secondly, for a pair of points $f, g \in [D \rightarrow_+ E]$, we have for any $x \in D$

$$\begin{aligned} [(\delta \otimes \epsilon)(f+g)](x) &= \left[\epsilon^2 (f+g)\delta^2\right](x) \\ &= \epsilon^2\left[f\delta^2(x) + g\delta^2(x)\right] \end{aligned}$$

$$= \epsilon\left[\epsilon\left(f\delta^2(x) + g\delta^2(x)\right)\right]$$
$$= \epsilon^2 f\delta^2(x) + \epsilon^2 g\delta^2(x)$$
$$= \left[\epsilon^2 f\delta^2 + \epsilon^2 g\delta^2\right](x)$$
$$= \left[(\delta\otimes\epsilon)(f) + (\delta\otimes\epsilon)(g)\right](x). \tag{7}$$

So we conclude that $\delta\otimes\epsilon$ is a +-semilattice homomorphism. Then $[D \rightarrow_+ E]$ is an FS_+-domain. Similarly, $[D \rightarrow E]$ is also an FS_+-domain. □

Theorem 4. *The category with FS_+-domains as objects and Scott continuous functions as morphisms is a Cartesian closed category.*

Note that the category with FS_+-domains as objects and Scott continuous and +-semilattice homomorphisms as morphisms is not a Cartesian closed category generally, because the evaluation maps do not preserve the finite +-operation.

2.2. Classify the Powerdomains

Definition 5 (see [5]). Let L be a dcpo-algebra equipped with a Scott continuous binary operation + that satisfies the following equations: for any $a, b, c, \in L$

(1) $a + a = a$ (idempotency law);

(2) $a + b = b + a$ (commutative law);

(3) $a + (b + c) = (a + b) + c$ (associative law).

Then the dcpo-algebra is a commutative idempotent semigroup, called a dcpo-semilattice. The free dcpo-semilattice over a dcpo L is called the convex or Plotkin powerdomain of L and it is denoted by $P^P(L)$.

If the binary operation + satisfies the inequality $a + b \le a$, then we obtain the upper or Smyth powerdomain, and it is denoted by $P^S(L)$, where $a + b = a \wedge b$.

Similarly, if the binary operation + satisfies $a + b \ge a$, then it is called the lower or Hoare powerdomain, denoted by $P^H(L)$, where $a + b = a \vee b$.

Proposition 6 (see [5]). *For subsets C and D of a preordered set $(L, \le)$ one has*

(1) $C =_H \downarrow C$;

(2) $C \le_H D$ *iff* $\downarrow C \subseteq \downarrow D$;

(3) $C \ll_H D$ *iff there exists a finite subset* $F \subseteq L$ *such that* $C \subseteq \downarrow F \subseteq \Downarrow D$;

(4) $C =_S \uparrow C$;

(5) $C \le_S D$ *iff* $\uparrow D \subseteq \uparrow C$;

(6) $C \ll_S D$ *iff* $D \subseteq \operatorname{int}_\sigma(\uparrow C)$ *iff* $D \subseteq \Uparrow C$;

(7) $C =_P \downarrow C \cap \uparrow C = \sup\{\downarrow F \cap \uparrow F : F \prec C, F \subseteq_{fin} L\}$, *where* $F \prec C$ *iff* $F \subseteq \Downarrow C$ *and* $C \subseteq \Uparrow F$;

(8) $C \le_P D$ *iff* $\downarrow C \subseteq \downarrow D$ *and* $\uparrow D \subseteq \uparrow C$;

(9) $C \ll_P D$ *iff* $C \ll_H D$ *and* $C \ll_S D$.

Next, we draw the conclusion that some special FS-domain categories concerning the operation + can be used to classify the powerdomains.

Theorem 7. *If L is an FS-domain, then the convex powerdomain $P^P(L)$ is an FS_+-domain.*

Proof. Suppose that L is an FS-domain; then L is a Lawson compact domain. Thus, $P^P(L)$ is also a domain. Assume that $\mathcal{F} = \{f_i : L \rightarrow L\}_{i\in I}$ is the approximate identity for L, where $\mathcal{F}$ is a family of finitely separated Scott continuous functions; that is, for any f_i, there exists a finite set $M_i \subseteq L$ such that, for any $x \in L$, there exists some $m \in M_i$ such that $f_i(x) \le m \le x$. We claim that $\{P(f_i) : P^P(L) \rightarrow P^P(L)\}_{i\in I}$ is the approximate identity for $P^P(L)$. It suffices to consider four steps as follows.

(1) $P(f_i) \le P(id)$. For $A \in P^P(L)$, define $P(f_i)(A) = P(f_i(A)) = \downarrow f_i(A) \cap \uparrow f_i(A)$. By Proposition 6, $\downarrow f_i(A) \cap \uparrow f_i(A) \in P^P(L)$. For any $x \in A$, let $M_i(A) = \{m \in M_i : \exists x \in A, f_i(x) \le m \le x\}$; then $f_i(x) \le x$ implies $\downarrow f_i(A) \subseteq \downarrow M_i(A) \subseteq \downarrow A$ and $\uparrow A \subseteq \uparrow M_i(A) \subseteq \uparrow f_i(A)$. Hence $P(f_i)(A) = P(f_i(A)) \le_P A$.

(2) $\sup\{P(f_i) : i \in I\} = P(id)$. For any $A \in P^P(L)$, it is obvious that $\sup\{P(f_i)(A) : i \in I\} \le A$. Suppose $A \nleq \sup\{P(f_i)(A) : i \in I\}$. There is $B \in P^P(L)$ such that $B \ll_P A$ and $B \nleq \sup\{P(f_i)(A) : i \in I\}$. By $B \ll_P A$ and $A = \sup\{\downarrow F \cap \uparrow F : F \prec A, F \subseteq_{\text{fin}} L\}$, there is some finite set $F \prec A$ such that $B \le \downarrow F \cap \uparrow F$. But for any finite set $F \prec A$, we have $F = \sup\{f_i(F) : i \in I\}$, where $F \prec A$ iff $F \subseteq \Downarrow A$ and $A \subseteq \Uparrow F$. Then $\downarrow F \cap \uparrow F = \sup\{\downarrow f_i(F) \cap \uparrow f_i(F) : i \in I\} \le \sup\{\downarrow f_i(A) \cap \uparrow f_i(A) : i \in I\}$. This is a contradiction. Then we conclude that $\sup\{P(f_i) : i \in I\} = P(id)$.

(3) $P(f_i)$ is Scott continuous and finitely separated. For a directed family $\mathcal{D}$ in $P^P(L)$, we have

$$P(f_i)(\sup\mathcal{D}) = P(f_i(\sup\mathcal{D})) = P(\sup\{f_i(D) : D \in \mathcal{D}\}) = \sup\{P(f_i(D)) : D \in \mathcal{D}\}. \tag{8}$$

Then $P(f_i)$ is Scott continuous. For any $A \in P^P(L)$, $M_i(A)$ is a finite set. By $\downarrow f_i(A) \subseteq \downarrow M_i(A) \subseteq \downarrow A$ and $\uparrow A \subseteq \uparrow M_i(A) \subseteq \uparrow f_i(A)$, it follows that $\downarrow M_i(A) \cap \uparrow M_i(A) \in P^P(L)$. Let $\mathcal{M}_i = \{\downarrow M_i(A) \cap \uparrow M_i(A) : A \in P^P(L)\}$. Since $M_i(A) \subseteq M_i$ and M_i is finite, it follows that $\mathcal{M}_i$ is a finite family of $P^P(L)$. And we have that, for any A, there exists $\downarrow M_i(A) \cap \uparrow M_i(A) \in \mathcal{M}_i$ such that $P(f_i)(A) \le_P \downarrow M_i(A) \cap \uparrow M_i(A) \le_P A$; that is, $P(f_i)$ is finitely separated.

(4) $P(f_i)$ is a +-semilattice homomorphism. For $A, B \in P^P(L)$, since $P^P(L)$ is a +-semilattice, $A + B = \downarrow (A \cup B) \cap \uparrow (A \cup B) \in P^P(L)$:

$$\begin{aligned} P(f_i)(A+B) &= P(f_i)[\downarrow (A \cup B) \cap \uparrow (A \cup B)] \\ &= \downarrow f_i[\downarrow (A \cup B) \cap \uparrow (A \cup B)] \\ &\quad \cap \uparrow f_i[\downarrow (A \cup B) \cap \uparrow (A \cup B)] \\ &= \downarrow f_i(A \cup B) \cap \uparrow f_i(A \cup B) \\ &= \downarrow [(\downarrow f_i(A) \cap \uparrow f_i(A)) \\ &\qquad \cup (\downarrow f_i(B) \cap \uparrow f_i(B))] \\ &\quad \cap \uparrow [(\downarrow f_i(A) \cap \uparrow f_i(A)) \\ &\qquad \cup (\downarrow f_i(B) \cap \uparrow f_i(B))] \\ &= P(f_i)(A) + P(f_i)(B). \end{aligned} \tag{9}$$

Then we conclude that $\{P(f_i) : i \in I\}$ is the approximate identity for $P^P(L)$. Thus the convex powerdomain $P^P(L)$ is an FS_+-domain.

□

Combined with the work of Huth et al. [13] and Meng and Kou [14], we conclude the following theorem.

Theorem 8. *Let L be a domain. Then the following statements hold:*

(1) *if L is Lawson compact, then the Smyth powerdomain $P^S(L)$ is an $FS_\wedge$-domain (in [14]);*

(2) *if L has a least point, then the Hoare powerdomain $P^H(L)$ is a distributive $FS_\vee$-lattice (in [13]);*

(3) *if L is an FS-domain, then the Plotkin powerdomain $P^P(L)$ is an FS_+-domain.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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