Research Article

On PT Symmetry Systems: Invariance, Conservation Laws, and Reductions

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An analysis of a PT symmetric coupler with "gain in one waveguide and loss in another" is made; a transformation in the PT system and some assumptions results in a scalar cubic Schrödinger equation. We investigate the relationship between the conservation laws and Lie symmetries and investigate a Lagrangian, corresponding Noether symmetries, conserved vectors, and exact solutions via "double reductions."

1. Introduction

Physical systems exhibiting parity-time (PT) symmetry have been the subject of much investigation in recent years and are now extensively considered in diverse areas of physics, namely, quantum field theories, non-Hermitian Anderson models, and complex Lie algebras, just to name a few [1–10]. We know that even a single PT cell can exhibit unconventional features; it follows that one may wish to investigate what new behaviour and properties can be expected from PT symmetric lattices [11, 12].

In optics, it has recently been discovered that there is a class of optical systems, of which elements consist of gain and loss, that can be interpreted as an optics equivalent of the PT symmetry in quantum mechanics [11, 13]. The underlying equations describing the effects of pulse dispersion [13] have the following form:

$$iU_t + U_{xx} + 2|U|^2U = -V + i\gamma U,$$
$$iV_t + V_{xx} + 2|V|^2V = -U - i\gamma V. \tag{1}$$

To analyze the solutions of this equation we make a change of variables

$$U(x,t) = e^{i(\omega t - \theta)}\alpha(x,t), \qquad V(x,t) = e^{i\omega t}\beta(x,t), \tag{2}$$

where θ is a constant angle satisfying

$$\sin\theta = \gamma \tag{3}$$

and ω is an arbitrary real parameter. As a result of the transformation, (1) becomes

$$i\alpha_t + \alpha_{xx} - \omega\alpha + 2|\alpha|^2\alpha = -\cos\theta\beta + i\gamma(\alpha - \beta),$$
$$i\beta_t + \beta_{xx} - \omega\beta + 2|\beta|^2\beta = -\cos\theta\alpha + i\gamma(\alpha - \beta). \tag{4}$$

The system (4) admits a reduction $\alpha = \beta = q$ to the following scalar cubic Schrödinger equation:

$$iq_t + q_{xx} - a^2q + 2|q|^2q = 0, \tag{5}$$

where q is the complex valued dependent variable and $a^2 = \omega - \cos\theta$.

Equation (5) has a family of stationary soliton solutions; however, we will study the invariance, exact solutions, conservation laws, and double reductions. This will be done by decomposing (5) into real and imaginary parts to obtain the following system of partial differential equations (PDEs); if $q = u + iv$, then we have the following system:

$$u_t + v_{xx} - a^2v + 2v(u^2 + v^2) = 0,$$
$$-v_t + u_{xx} - a^2u + 2u(u^2 + v^2) = 0. \tag{6}$$

In light of symmetries, conservation laws, and double reduction to exact solutions, we will briefly consider the system of PDEs in (1) but the bulk of the analysis will centre around (5) via the system (6).

2. On the Conservation Laws of (1)

In order to determine conserved densities and fluxes, we resort to the invariance and multiplier approach based on the well-known result that the Euler-Lagrange operator annihilates a total divergence (see [14]). Firstly, if (T^t, T^x) is a conserved vector corresponding to a conservation law, then

$$D_t T^t + D_x T^x = 0 \tag{7}$$

along the solutions of the differential equation ($G(x, t, q, q_{(i)}, \ldots) = 0$, say).

If the system (1) is split into real and imaginary parts with $U = \mu + i\nu$ and $V = \epsilon + i\delta$ and replacing γ with "general" parameters, we get

$$
\begin{aligned}
\mu_t + \nu_{xx} + 2\left(\mu^2 + \nu^2\right)\nu + \delta - g_1\mu &= 0, \\
-\nu_t + \mu_{xx} + 2\left(\mu^2 + \nu^2\right)\mu + \epsilon + g_2\nu &= 0, \\
\epsilon_t + \delta_{xx} + 2\left(\epsilon^2 + \delta^2\right)\delta + \nu + g_3\epsilon &= 0, \\
\delta_t + \epsilon_{xx} + 2\left(\epsilon^2 + \delta^2\right)\epsilon + \mu - g_4\delta &= 0.
\end{aligned}
\tag{8}
$$

It turns out that the system (8) only admits nontrivial conservation laws (two) for $g_2 = -g_1$ and $g_4 = -g_3$ corresponding to *multipliers* (see below) $Q = (\nu_t, \mu_t, \delta_t, \epsilon_t)$ and $Q = (\nu_x, \mu_x, \delta_x, \epsilon_x)$ which in turn correspond to time and space translations, respectively. In this case, (8) becomes

$$
\begin{aligned}
\mu_t + \nu_{xx} + 2\left(\mu^2 + \nu^2\right)\nu + \delta - g_1\mu &= 0, \\
-\nu_t + \mu_{xx} + 2\left(\mu^2 + \nu^2\right)\mu + \epsilon - g_2\nu &= 0, \\
\epsilon_t + \delta_{xx} + 2\left(\epsilon^2 + \delta^2\right)\delta + \nu + g_3\epsilon &= 0, \\
\delta_t + \epsilon_{xx} + 2\left(\epsilon^2 + \delta^2\right)\epsilon + \mu + g_3\delta &= 0.
\end{aligned}
\tag{9}
$$

Thus, (1) has no nontrivial conservation laws even though the system is invariant under time and space translations.

We thus do a detailed study of the special case given in (6) instead.

3. Symmetries, Reductions, and Conservation Laws of (6)

The Lie symmetry approach on differential equations is well known; for details, see, for example, [15, 16]. In this section, we list a summary of these and explore the notion of a "double reduction" in order to obtain symmetry invariant (exact) solutions.

3.1. Symmetries and Reductions.
A one parameter Lie group of transformations that leave invariant (6) will be written as a vector field

$$
\begin{aligned}
X = {}& \tau(t, x, u, v)\,\partial_t + \xi(t, x, u, v)\,\partial_x \\
& + \eta^1(t, x, u, v)\,\partial_u + \eta^2(t, x, u, v)\,\partial_v.
\end{aligned}
\tag{10}
$$

This would be a generator of point symmetries of the system. We get the algebra generated by

$$
\begin{aligned}
X_1 &= \partial_t, \\
X_2 &= \partial_x, \\
X_3 &= u\partial_v - v\partial_u, \\
X_4 &= 2t\partial_x + ux\partial_v - vx\partial_u, \\
X_5 &= \left(-2a^2 tu - v\right)\partial_v + 2t\partial_t + \left(-u + 2a^2 tv\right)\partial_u + x\partial_x.
\end{aligned}
\tag{11}
$$

3.2. Conservation Laws.
In order to determine conserved densities and fluxes, we resort to the invariance and multiplier approach based on the well-known result that the Euler-Lagrange operator annihilates a total divergence (see [14]). Firstly, if (T^t, T^x) is a conserved vector corresponding to a conservation law, then

$$D_t T^t + D_x T^x = 0 \tag{12}$$

along the solutions of the differential equation ($G(x, t, q, q_{(i)}, \ldots) = 0$, say).

3.2.1. The Mutiplier Approach.
If there exists a nontrivial differential function Q, called a "multiplier," such that

$$E_q\left[QG\left(x, t, q, q_{(i)}, \ldots\right)\right] = 0, \tag{13}$$

then $Q(G(x, t, q, q_{(i)}, \ldots))$ is a total divergence; that is,

$$Q\left(G\left(x, t, q, q_{(i)}, \ldots\right)\right) = D_t T^t + D_x T^x, \tag{14}$$

for some (conserved) vector (T^t, T^x) and E_q is the respective Euler-Lagrange operator. Thus, knowledge of each multiplier Q leads to a conserved vector determined by, inter alia, a homotopy operator. See details and references in [14, 17].

For a system $G^1(x, t, u, v, u_{(i)}, v_{(i)}, \ldots) = 0$ and $G^2(x, t, u, v, u_{(i)}, v_{(i)}, \ldots) = 0$, $Q = (Q^1, Q^2)$, say, we get

$$
\begin{aligned}
Q^1 &\left(G^1\left(x, t, q, u_{(i)}, v_{(i)}, \ldots\right)\right) \\
&+ Q^2\left(G^2\left(x, t, q, u_{(i)}, v_{(i)}, \ldots\right)\right) = D_t T^t + D_x T^x, \quad (15) \\
& E_{(u,v)}\left[D_t T^t + D_x T^x\right] = 0.
\end{aligned}
$$

In each case, T^t is the *conserved density*.

The lengthy calculations for the system (6) lead to the following multipliers and corresponding conserved vectors.

(i) Consider $(Q^1, Q^2) = (v_t, u_t)$,

$$T^x = \frac{1}{2}\left(u_t u_x + v_t v_x - u u_{xt} - v v_{xt}\right),$$

$$T^t = \frac{1}{2}\left(u^4 - u^2\left(a^2 - 2v^2\right) + u u_{xx} + v\left(-a^2 v + v^3 + v_{xx}\right)\right). \tag{16}$$

The conserved density for the scalar equation is

$$\Phi^t = \frac{1}{2}|q|^4 + \frac{1}{2}(h-1)|q|^2 + q\bar{q}_{xx} + \bar{q}q_{xx}. \tag{17}$$

(ii) Consider $(Q^1, Q^2) = (v_x, u_x)$. Consider

$$T^x = \frac{1}{2}\left(u^4 - a^2 v^2 + v^4 - u^2\left(a^2 - 2v^2\right)\right.$$

$$\left. + v u_t - u v_t + u_x^2 + v_x^2\right), \tag{18}$$

$$T^t = \frac{1}{2}\left(-v u_x + u v_x\right).$$

The conserved density for the scalar equation is

$$\Phi^t = -\frac{i}{4}\left(\bar{q}q_x - q\bar{q}_x\right). \tag{19}$$

(iii) Consider $(Q^1, Q^2) = (u, -v)$,

$$T^x - v u_x + u v_x,$$

$$T^t = \frac{1}{2}\left(u^2 + v^2\right). \tag{20}$$

The conserved density for the scalar equation is

$$\Phi^t = -\frac{1}{2}|q|^2. \tag{21}$$

(iv) Consider $(Q^1, Q^2) = (-(1/2)xu + tv_x, (1/2)xv + tu_x)$,

$$T^x = \frac{1}{2}\left(t u^4 - a^2 t v^2 + t v^4 - t u^2\left(a^2 - 2v^2\right)\right.$$

$$\left. + v\left(t u_t + x u_x\right) - u\left(t v_t + x v_x\right) + t\left(u_x^2 + v_x^2\right)\right),$$

$$T^t = \frac{1}{4}\left(-x u^2 - v\left(xv + 2t u_x\right) + 2t u v_x\right). \tag{22}$$

The conserved density for the scalar equation is

$$\Phi^t = -\frac{1}{4}x|q|^2 - it\left(\bar{q}q_x - q\bar{q}_x\right). \tag{23}$$

3.2.2. A Lagrangian Formulation. The system (6) admits a Lagrangian

$$L = -\frac{1}{2}u_x^2 - \frac{1}{2}v_x^2 + \frac{1}{2}u_t v - \frac{1}{2}v_t u$$

$$- \frac{1}{2}a^2\left(u^2 + v^2\right) + \frac{1}{2}u^4 + \frac{1}{2}v^4 + u^2 v^2 \tag{24}$$

so that the corresponding Lagrangian for the Schrödinger equation (5) is $\mathscr{L} = -(1/2)|q_x|^2 - (1/2)a^2|q|^2 + (1/2)|q|^4 + (i/4)(\bar{q}q_t - q\bar{q}_t)$. The Noether symmetries, that is, the one parameter Lie groups of transformations, that leave invariant the functional $\iint L dx\, dt$ with zero gauges are the translations

$$X_1 = \partial_t, \qquad X_2 = \partial_x \tag{25}$$

with corresponding conserved vectors

$$T^x = v_x v_t + u_x u_t,$$

$$T^t = -\frac{1}{2}u_x^2 - \frac{1}{2}v_x^2 - \frac{1}{2}a^2 u^2 - \frac{1}{2}a^2 v^2$$

$$+ \frac{1}{2}u^4 + \frac{1}{2}v^4 + u^2 v^2, \tag{26}$$

with density of (5) given by

$$\Phi^t = -\frac{1}{2}|q_x|^2 - \frac{1}{2}a^2|q|^2 + \frac{1}{2}|q|^4 \tag{27}$$

and

$$T^x = \frac{1}{2}u_x^2 + \frac{1}{2}v_x^2 + \frac{1}{2}v u_t - \frac{1}{2}u v_t - \frac{1}{2}a^2 u^2$$

$$- \frac{1}{2}a^2 v^2 + \frac{1}{2}u^4 + \frac{1}{2}v^4 + u^2 v^2, \tag{28}$$

$$T^t = \frac{1}{2}\left(v_x u - u_x v\right),$$

with density of (5) given by

$$\Phi^t = -\frac{i}{4}\left(\bar{q}q_x - q\bar{q}_x\right), \tag{29}$$

respectively.

3.3. Double Reduction. To demonstrate how one uses symmetries and conservation laws to double reduce PDEs, we recall some definitions and theorems.

Definition 1 (see [18]). A Lie-Bäcklund symmetry generator X of the form (1) is associated with a conserved vector T of the system (6) if X and T satisfy the relations

$$X\left(T^i\right) + T^i D_k\left(\xi^k\right) - T^k D_k\left(\xi^i\right) = 0, \quad i = 1, \ldots, n. \tag{30}$$

Theorem 2 (see [19]). *Suppose that X is any Lie-Bäcklund symmetry of (6) and T^i, $i = 1, \ldots, n$, are the components of the conserved vector of (6). Then*

$$T^{*i} = \left[T^i, X\right] = X\left(T^i\right) + T^i D_j \xi^j - T^j D_j \xi^i, \quad i = 1, \ldots, n, \tag{31}$$

constitute the components of a conserved vector of (6); that is, $D_i T^{*i}|_{(6)} = 0$.

Theorem 3 (see [20]). *Suppose that $D_i T^i = 0$ is a conservation law of the partial differential equation system* (6). *Then under a contact transformation, there exist functions $\tilde{T}^i$ such that $J D_i T^i = \tilde{D}_i \tilde{T}^i$, where $\tilde{T}^i$ are given as*

$$
\begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix},
$$
$$
J \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} = A^T \begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix}
$$

(32)

in which

$$
A = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \cdots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \cdots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \cdots & \tilde{D}_n x_n \end{pmatrix},
$$
$$
A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \cdots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \cdots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \cdots & D_n \tilde{x}_n \end{pmatrix}
$$

(33)

and $J = \det(A)$.

Theorem 4 (fundamental theorem on double reduction [20]). *Suppose that $D_i T^i = 0$ is a conservation law of the partial differential equation system* (6). *Then under a similarity transformation of a symmetry X of the form* (1) *for the partial differential equation, there exist functions $\tilde{T}^i$ such that X is still a symmetry for the partial differential equation $\tilde{D}_i \tilde{T}^i = 0$ and*

$$
\begin{pmatrix} X\tilde{T}^1 \\ X\tilde{T}^2 \\ \vdots \\ X\tilde{T}^n \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} [T^1, X] \\ [T^2, X] \\ \vdots \\ [T^n, X] \end{pmatrix},
$$

(34)

where

$$
A = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \cdots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \cdots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \cdots & \tilde{D}_n x_n \end{pmatrix},
$$
$$
A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \cdots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \cdots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \cdots & D_n \tilde{x}_n \end{pmatrix}
$$

(35)

and $J = \det(A)$.

Our original system is equivalent to

$$
\text{sys}_1 = \begin{cases} q_1{}^1 G^1 + q_1{}^2 G^2 = 0, \\ q_1{}^1 G^1 - q_1{}^2 G^2 = 0. \end{cases}
$$

(36)

The system (36) can be rewritten as

$$
D_t T_1{}^t + D_x T_1{}^x = 0,
$$
$$
q_1{}^1 G^1 - q_1{}^2 G^2 = 0.
$$

(37)

3.3.1. A Double Reduction of (6) *by* $\langle X_2, X_3 \rangle$. We first show that X_2 and X_3 are associated with $T_1 = (T_2^t, T_2^x)$ using the following version of (6) for $i = 1, 2$:

$$
T^* = X \begin{pmatrix} T^t \\ T^x \end{pmatrix} - \begin{pmatrix} D_t \xi^t & D_x \xi^t \\ D_t \xi^x & D_x \xi^x \end{pmatrix} \begin{pmatrix} T^t \\ T^x \end{pmatrix}
$$
$$
+ \left(D_t \xi^t + D_x \xi^x\right) \begin{pmatrix} T^t \\ T^x \end{pmatrix}.
$$

(38)

We obtain

$$
\begin{pmatrix} T_2^{*t} \\ T_2^{*x} \end{pmatrix} = X_3^{[1]} \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} + (0) \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix}
$$
$$
= \begin{pmatrix} \left(-v v_x - u u_x + v v_x + u u_x\right) \\ \left(\dfrac{1}{2}\left(u_t u_x + v_t v_x - u u_{xt} - v v_{xt}\right)\right) \end{pmatrix}
$$
$$
= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$
$$
\begin{pmatrix} T_2^{*t} \\ T_2^{*x} \end{pmatrix} = X_2^{[1]} \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} + (0) \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix}
$$
$$
= \begin{pmatrix} \dfrac{\partial}{\partial x}\left[\dfrac{1}{2}\left(u v_x - v u_x\right)\right] \\ \dfrac{\partial}{\partial x}\left[\dfrac{1}{2}\left(u_t u_x + v_t v_x - u u_{xt} - v v_{xt}\right)\right] \end{pmatrix}
$$
$$
= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$

(39)

where

$$
X_3^{[1]} = -v \frac{\partial}{\partial u} + u \frac{\partial}{\partial v} - v_t \frac{\partial}{\partial u_t} - v_x \frac{\partial}{\partial u_x} + u_t \frac{\partial}{\partial v_t} + u_x \frac{\partial}{\partial v_x},
$$
$$
X_2^{[1]} = \frac{\partial}{\partial x}.
$$

(40)

Thus, X_2 and X_3 are both associated with T_2.

We can get a reduced conserved form for the first equation of the first system, sys_1 from (36), since X_2 and X_3 are both associated symmetries with T_2.

We consider a linear combination of X_2 and X_3, that is, of the form $X = X_2 + c X_3$, and transform this generator to its canonical form $Y = \partial/\partial s$, where we assume that this generator is of the form $Y = 0(\partial/\partial r) + \partial/\partial s + 0(\partial/\partial w) + 0(\partial/\partial p)$.

From $X(r) = 0$, $X(s) = 1$, $X(w) = 0$, and $X(p) = 0$, we obtain

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{-cv} = \frac{dv}{cu} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dp}{0}. \quad (41)$$

We solve (41) using the method of invariance, where the results are summarized as follows:

$$b_1 = t, \qquad b_2 = u^2 + v^2, \qquad b_3 = \arctan\left(\frac{v}{u}\right) - cx,$$

$$b_4 = r, \qquad b_5 = s - x, \qquad b_6 = w, \qquad b_7 = p, \quad (42)$$

where b_4, b_5, b_6, and b_7 are arbitrary functions all dependent on b_1, b_2, and b_3.

By choosing $b_4 = b_1$, $b_5 = 0$, $b_6 = \sqrt{b_2}$, and $b_7 = b_3$, we obtain the canonical coordinates

$$r = t, \qquad s = x, \qquad w = \sqrt{u^2 + v^2},$$

$$p = \arctan\left(\frac{v}{u}\right) - cs, \quad (43)$$

where $w = w(r)$ and $p = p(r)$, since $Y = \partial/\partial s$.

From (32), we compute A and $(A^{-1})^T$:

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} D_t r & D_t s \\ D_x r & D_x s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (A^{-1})^T, \quad (44)$$

where $J = \det(A) = 1$.

From (43), the inverse canonical coordinates are given by

$$t = r, \qquad x = s, \qquad u = w(r)\cos(p(r) + cs),$$

$$v = w(r)\sin(p(r) + cs). \quad (45)$$

We compute the first- and second-order partial derivatives of u and v from (45):

$$u_t = \left(\frac{d}{dr}w(r)\right)\cos(p(r) + cs)$$

$$- w(r)\left(\frac{d}{dr}p(r)\right)\sin(p(r) + cs),$$

$$u_x = -cw(r)\sin(p(r) + cs),$$

$$v_t = \left(\frac{d}{dr}w(r)\right)\sin(p(r) + cs)$$

$$+ w(r)\left(\frac{d}{dr}p(r)\right)\cos(p(r) + cs), \quad (46)$$

$$v_x = cw(r)\cos(p(r) + cs),$$

$$u_{xx} = -c^2 w(r)\cos(p(r) + cs),$$

$$v_{xx} = -c^2 w(r)\sin(p(r) + cs).$$

We now apply the formula from (33) with $i = 1, 2$ to obtain the reduced conserved form

$$\begin{pmatrix} T_2^r \\ T_2^s \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix}. \quad (47)$$

By substituting (45) and (46) into (47), we obtain

$$\begin{pmatrix} T_2^r \\ T_2^s \end{pmatrix}$$
$$= \begin{pmatrix} \dfrac{1}{2}cw(r)^2 \\ \dfrac{1}{2}\left[w(r)^4 + a^2 w(r)^2 - \left(\dfrac{d}{dr}p(r)\right)w(r)^2 + c^2 w(r)^2\right] \end{pmatrix}, \quad (48)$$

where the reduced conserved form is also given by

$$D_r T_2^r = 0. \quad (49)$$

The second step of double reduction can be given as

$$cw(r)^2 = k \quad (50)$$

or equivalently

$$w(r)^2 = k, \quad (51)$$

where k is a constant.

Differentiating (50) implicitly with respect to r results in

$$\frac{d}{dr}w(r) = 0. \quad (52)$$

The second equation of sys$_1$ from (36) is given by

$$v_x\left[u_t + v_{xx} - a^2 v + 2\left(u^2 + v^2\right)v\right]$$

$$- u_x\left[-v_t + u_{xx} - a^2 u + 2\left(u^2 + v^2\right)u\right] = 0. \quad (53)$$

After transforming (53) using (45) and (46), we obtain

$$-2cw(r)^2\frac{d}{dr}p(r)\left[\cos(p(r) + cs)\sin(p(r) + cs)\right],$$

$$+ 4w(r)^4\left[\cos(p(r) + cs)\sin(p(r) + cs)\right] = 0. \quad (54)$$

We now substitute (50) and (52) into (54) and simplify.

This results in the ordinary differential equation (ODE)

$$\frac{d}{dr}p(r) = \frac{2w(r)^2}{c}. \quad (55)$$

If we substitute (51) into (55) and then integrate both sides with respect to r, we obtain

$$p(r) = \frac{2kr}{c} + m, \quad (56)$$

where m is an integration constant.

Using (45), we obtain the final solution to our original system (6) as

$$u(t, x) = \sqrt{k}\cos\left(\frac{2kt}{c} + m + cx\right),$$

$$v(t, x) = \sqrt{k}\sin\left(\frac{2kt}{c} + m + cx\right). \quad (57)$$

Thus, $q = \sqrt{k}e^{i(2kt/c + m + cx)}$.

3.3.2. Case 2: A Reduction of (6) *by* X_4 *on* T_3. We now show that X_4 is associated with $T_3 = (T_3^t, T_3^x)$ using the formula (38).

We obtain

$$
\begin{pmatrix} T_3^{*t} \\ T_3^{*x} \end{pmatrix}
$$

$$
= X_4^{[1]} \begin{pmatrix} T_3^t \\ T_3^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} T_3^t \\ T_3^x \end{pmatrix} + (0) \begin{pmatrix} T_3^t \\ T_3^x \end{pmatrix}
$$

$$
= \begin{pmatrix} \dfrac{1}{2}\left(-2xuv + 2xuv\right) \\ \left(-xvv_x - xuu_x + xvv_x + v^2 + xuu_x + u^2 - u^2 - v^2\right) \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$

(58)

where

$$
X_4^{[1]} = 2t\frac{\partial}{\partial x} - xv\frac{\partial}{\partial u} + xu\frac{\partial}{\partial v} - \left(xv_t + 2u_x\right)\frac{\partial}{\partial u_t}
$$

$$
- \left(xv_x + v\right)\frac{\partial}{\partial u_x} + \left(xu_t - 2v_x\right)\frac{\partial}{\partial v_t} + \left(xu_x + u\right)\frac{\partial}{\partial v_x}.
$$

(59)

Thus, X_4 is associated with T_3.

We again transform the generator X_4 to its canonical form $Y = \partial/\partial s$.

From $X(r) = 0$, $X(s) = 1$, $X(w) = 0$, and $X(p) = 0$, we obtain

$$
\frac{dt}{0} = \frac{dx}{2t} = \frac{du}{-xv} = \frac{dv}{xu} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dp}{0}.
$$

(60)

The results from solving (60) are summarized as follows:

$$
b_1 = t, \qquad b_2 = u^2 + v^2, \qquad b_3 = \arctan\left(\frac{v}{u}\right) - \frac{x^2}{4t},
$$

(61)

$$
b_4 = r, \qquad b_5 = s - \frac{x}{2t}, \qquad b_6 = w, \qquad b_7 = p,
$$

where b_4, b_5, b_6, and b_7 are arbitrary functions all dependent on b_1, b_2, and b_3.

By choosing $b_4 = b_1$, $b_5 = 0$, $b_6 = \sqrt{b_2}$, and $b_7 = b_3$, we obtain the canonical coordinates

$$
r = t, \qquad s = \frac{x}{2t}, \qquad w = \sqrt{u^2 + v^2},
$$

(62)

$$
p = \arctan\left(\frac{v}{u}\right) - \frac{x^2}{4t}.
$$

From (32), we compute A and $(A^{-1})^T$:

$$
A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} 1 & 2s \\ 0 & 2r \end{pmatrix},
$$

(63)

$$
(A^{-1})^T = \begin{pmatrix} D_t r & D_t s \\ D_x r & D_x s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\dfrac{s}{r} & \dfrac{1}{2r} \end{pmatrix},
$$

where $J = \det(A) = 2r$.

From (62), the inverse canonical coordinates are given by

$$
t = r, \qquad x = 2rs, \qquad u = w(r)\cos\left(p(r) + rs^2\right),
$$

(64)

$$
v = w(r)\sin\left(p(r) + rs^2\right).
$$

We compute the first- and second-order partial derivatives of u and v from (64):

$$
u_t = \left(\frac{d}{dr}w(r)\right)\cos\left(p(r) + rs^2\right)
$$

$$
- w(r)\left(\frac{d}{dr}p(r)\right)\sin\left(p(r) + rs^2\right)
$$

$$
+ s^2 w(r)\sin\left(p(r) + rs^2\right),
$$

$$
u_x = -sw(r)\sin\left(p(r) + rs^2\right),
$$

$$
v_t = \left(\frac{d}{dr}w(r)\right)\sin\left(p(r) + rs^2\right)
$$

$$
+ w(r)\left(\frac{d}{dr}p(r)\right)\cos\left(p(r) + rs^2\right)
$$

(65)

$$
- s^2 w(r)\cos\left(p(r) + rs^2\right),
$$

$$
v_x = sw(r)\cos\left(p(r) + rs^2\right),
$$

$$
u_{xx} = -\frac{w(r)\sin\left(p(r) + rs^2\right)}{2r}
$$

$$
- s^2 w(r)\cos\left(p(r) + rs^2\right),
$$

$$
v_{xx} = \frac{w(r)\cos\left(p(r) + rs^2\right)}{2r}
$$

$$
- s^2 w(r)\sin\left(p(r) + rs^2\right).
$$

By substituting (64) and (65) into (47), we obtain

$$
\begin{pmatrix} T_3^r \\ T_3^s \end{pmatrix} = \begin{pmatrix} rw(r)^2 \\ 0 \end{pmatrix},
$$

(66)

where the reduced conserved form is also given by

$$
D_r T_3^r = 0.
$$

(67)

The second step of double reduction can be given as

$$
rw(r)^2 = k
$$

(68)

or equivalently

$$
w(r)^2 = \frac{k}{r},
$$

(69)

where k is a constant.

Differentiating (68) implicitly with respect to r results in

$$
2rw\frac{d}{dr}w(r) + w(r)^2 = 0
$$

(70)

or equivalently, after dividing both sides by $2r$,

$$w\frac{d}{dr}w(r) + \frac{w(r)^2}{2r} = 0. \tag{71}$$

The second equation of sys$_2$ from (36) is given by

$$uu_t - vv_t + uv_{xx} + vu_{xx}$$
$$- 2a^2uv + 4u^3v + 4v^3u = 0. \tag{72}$$

After transforming (72) using (64) and (65) and multiplying both sides by $2r$, we get the ODE

$$\frac{d}{dr}p(r) = 2k - a^2. \tag{73}$$

If we substitute (69) into (73) and then integrate both sides with respect to r,

$$p(r) = 2kr - ra^2 + m. \tag{74}$$

Using (64), we obtain the final solution to our original system (6) as

$$u(t,x) = \sqrt{\frac{k}{t}}\cos\left(2kt - ta^2 + m + \frac{x^2}{4t}\right),$$
$$v(t,x) = \sqrt{\frac{k}{t}}\sin\left(2kt - ta^2 + m + \frac{x^2}{4t}\right), \tag{75}$$

so that $q = \sqrt{k/t}e^{i(2kt-ta^2+m+x^2/4t)}$.

4. Conclusion

We have constructed conservation laws for the scalar cubic Schrödinger equation via the invariance and multiplier approach based on the well-known result that the Euler-Lagrange operator annihilates total divergence. Interestingly enough, the scalar cubic Schrödinger equation admits a Lagrangian resulting in Noether symmetries. Furthermore, two cases of double reduction were successfully performed and exact solutions were calculated and diagrammatically represented.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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