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Research Article

A Hybrid Mean Value Involving the Two-Term Exponential Sums and Two-Term Character Sums

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The main purpose of this paper is using the properties of Gauss sums and the estimate for character sums to study the hybrid mean value problem involving the two-term exponential sums and two-term character sums and give an interesting asymptotic formula for it.

1. Introduction

Let $q \geq 3$ be an integer and χ denotes a Dirichlet character mod$q$. For any integers m and n with $(mn, q) = 1$, we define the two-term exponential sum $C(m, n, k; q)$ and two-term character sum $N(m, n, \chi; q)$ as follows:

$$C(m,n,k;q) = \sum_{a=1}^{q} e\left(\frac{ma^k + na}{q}\right),$$
$$N(m,n,k,\chi;q) = \sum_{a=1}^{q} \chi\left(ma^k + na\right), \tag{1}$$

where $e(x) = e^{2\pi ix}$, χ denotes a nonprincipal Dirichlet character mod$q$, and k is a fixed positive integer.

These sums play a very important role in the study of analytic number theory, so they caused many number theorists' interest and favor. Some works related to $C(m, n, k; q)$ can be found in [1–5]. For example, Cochrane and Zheng [1] show that

$$\left|C(m,n,k;q)\right| \leq k^{\omega(q)} q^{1/2}, \tag{2}$$

where $\omega(q)$ denotes the number of all distinct prime divisors of q.

On the other hand, the sums $N(m, n, k, \chi; q)$ are a special case of the general character sums of the polynomials

$$\sum_{a=N+1}^{N+M} \chi\left(f(a)\right), \tag{3}$$

where M and N are any positive integers and $f(x)$ is a polynomial. If $q = p$ is an odd prime, then Weil (see [6]) obtained the following important conclusion.

Let χ be a qth-order character mod$p$; if $f(x)$ is not a perfect qth power mod p, then we have the estimate

$$\sum_{x=N+1}^{N+M} \chi\left(f(x)\right) \ll p^{1/2} \ln p, \tag{4}$$

where "$\ll$" constant depends only on the degree of $f(x)$. Some related results can also be found in [7–10].

Now we are concerned about whether there exists an asymptotic formula for the hybrid mean value

$$\sum_{m=1}^{q-1} \left|\sum_{a=1}^{q-1} \chi\left(ma^k + a\right)\right|^2 \cdot \left|\sum_{b=1}^{q-1} e\left(\frac{mb^k + b}{q}\right)\right|^2. \tag{5}$$

In this paper, we will use the analytic method and the properties of character sums to study this problem and give a sharp asymptotic formula for (5) with $q = p$, an odd prime. That is, we will prove the following.

Theorem 1. *Let p be an odd prime, let χ be any nonprincipal even character* mod p*, and let $\chi^3 \neq \chi_0$ be the principal character* mod p*. Then we have the asymptotic formula*

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(ma^3+a\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^2 = 2p^3+E\left(p\right), \tag{6}$$

where $E(p)$ satisfies the inequalities $-12p^2-2p \leq E(p) \leq 4p^2-2p$.

From this theorem we may immediately deduce the following.

Corollary 2. *For any odd prime p and any nonprincipal even character χ* mod p *with $\chi^3 \neq \chi_0$, one has*

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(ma^3+a\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^2 = 2p^3+O\left(p^2\right). \tag{7}$$

In the theorem, we only consider the polynomial $f(x) = mx^3+x$. For general polynomial $f(x) = mx^k+x^h$ with $k \geq 4$ and $1 \leq h < k$, whether there exists an asymptotic formula is complex problem for (5)*, it needs us to further study.*

For general positive integer $q \geq 4$, whether there exists an asymptotic formula for (5) *is also an interesting open problem.*

2. Several Lemmas

To complete the proof of our theorem, we need the following several lemmas.

Lemma 1. *Let p be an odd prime and let χ be any nonprincipal even character* mod p*. Then for any integer m with $(m,p)=1$, the identity*

$$\sum_{a=1}^{p-1}\chi\left(ma^3+a\right) = \frac{\tau\left(\chi_1\right)\tau\left(\overline{\chi_1}^3\right)\overline{\chi_1}\left(m\right)}{\tau\left(\overline{\chi}\right)} \times\left(1+\left(\frac{m}{p}\right)\frac{\tau\left(\chi_1\chi_2\right)\tau\left(\overline{\chi_1}^3\chi_2\right)}{\tau\left(\chi_1\right)\tau\left(\overline{\chi_1}^3\right)}\right), \tag{8}$$

where $(*/p)=\chi_2$ denotes the Legendre symbol and $\chi=\chi_1^2$.

Proof. Since $\chi(-1) = 1$, there exists one and only one character χ_1 mod p such that $\chi = \chi_1^2$. Thus, from the properties of Gauss sums we have

$$\begin{aligned}
\sum_{a=1}^{p-1}\chi\left(ma^3+a\right) &= \frac{1}{\tau\left(\overline{\chi}\right)}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}\left(b\right)e\left(\frac{b\left(ma^3+a\right)}{p}\right)\\
&= \frac{1}{\tau\left(\overline{\chi}\right)}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}\left(b\overline{a}\right)e\left(\frac{b\overline{a}\left(ma^3+a\right)}{p}\right)\\
&= \frac{1}{\tau\left(\overline{\chi}\right)}\sum_{b=1}^{p-1}\overline{\chi}\left(b\right)e\left(\frac{b}{p}\right)\sum_{a=1}^{p-1}\chi\left(a\right)e\left(\frac{bma^2}{p}\right)\\
&= \frac{1}{\tau\left(\overline{\chi}\right)}\sum_{b=1}^{p-1}\overline{\chi}\left(b\right)e\left(\frac{b}{p}\right)\sum_{a=1}^{p-1}\chi_1^2\left(a\right)e\left(\frac{bma^2}{p}\right)\\
&= \frac{1}{\tau\left(\overline{\chi}\right)}\sum_{b=1}^{p-1}\overline{\chi}\left(b\right)e\left(\frac{b}{p}\right) \times\sum_{a=1}^{p-1}\chi_1\left(a\right)\left(1+\left(\frac{a}{p}\right)\right)e\left(\frac{bma}{p}\right)\\
&= \frac{1}{\tau\left(\overline{\chi}\right)}\sum_{b=1}^{p-1}\overline{\chi}\left(b\right)e\left(\frac{b}{p}\right) \times\left(\overline{\chi_1}\left(bm\right)\tau\left(\chi_1\right) +\overline{\chi_1}\left(bm\right)\chi_2\left(bm\right)\tau\left(\chi_1\chi_2\right)\right)\\
&= \frac{\overline{\chi_1}\left(m\right)}{\tau\left(\overline{\chi}\right)}\left(\tau\left(\chi_1\right)\tau\left(\overline{\chi_1}^3\right) +\left(\frac{m}{p}\right)\tau\left(\chi_1\chi_2\right)\tau\left(\overline{\chi_1}^3\chi_2\right)\right)\\
&= \frac{\tau\left(\chi_1\right)\tau\left(\overline{\chi_1}^3\right)\overline{\chi_1}\left(m\right)}{\tau\left(\overline{\chi}\right)} \times\left(1+\left(\frac{m}{p}\right)\frac{\tau\left(\chi_1\chi_2\right)\tau\left(\overline{\chi_1}^3\chi_2\right)}{\tau\left(\chi_1\right)\tau\left(\overline{\chi_1}^3\right)}\right).
\end{aligned} \tag{9}$$

This proves Lemma 1. □

Lemma 2. *Let p be an odd prime, let χ be any nonprincipal even character* mod p*, $\chi = \chi_1^2$, and $\chi^3 \neq \chi_0$, the principal character* mod p*. Then for any integer m and any quadratic nonresidue r* mod p *with $(m,p)=1$, we have the identity*

$$\begin{aligned}
\left|\sum_{a=1}^{p-1}\chi\left(ma^3+a\right)\right|^2 &= 2p+\left(\frac{m}{p}\right)\frac{\tau^2\left(\chi_2\right)}{2p}\sum_{a=1}^{p-1}\left(\chi\left(a\right)+\overline{\chi}\left(a\right)\right) \times\sum_{b=1}^{p-1}\left(\frac{1-a^2b^3}{p}\right)\left(\frac{1-b}{p}\right)\\
&+\left(\frac{m}{p}\right)\frac{\tau^2\left(\chi_2\right)}{2p} \times\sum_{a=1}^{p-1}\left(\chi_1\left(r\right)\chi\left(a\right)+\overline{\chi_1}\left(r\right)\overline{\chi}\left(a\right)\right) \times\sum_{b=1}^{p-1}\left(\frac{1-ra^2b^3}{p}\right)\left(\frac{1-b}{p}\right).
\end{aligned} \tag{10}$$

Proof. From the properties of Gauss sums we have

$$\begin{aligned}\overline{\tau(\chi_1)}\tau(\chi_1\chi_2) &= \sum_{a=1}^{p-1}\overline{\chi}_1(a)\sum_{b=1}^{p-1}\chi_1(b)\chi_2(b)e\left(\frac{b-a}{p}\right) \\ &= \sum_{a=1}^{p-1}\overline{\chi}_1(a)\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{b(1-a)}{p}\right) \\ &= \tau(\chi_2)\sum_{a=1}^{p-1}\overline{\chi}_1(a)\left(\frac{1-a}{p}\right).\end{aligned} \tag{11}$$

So from (11) we have

$$\begin{aligned}&\frac{\tau(\chi_1\chi_2)\tau\left(\overline{\chi_1}^3\chi_2\right)}{\tau(\chi_1)\tau\left(\overline{\chi_1}^3\right)} \\ &= \frac{1}{p^2}\overline{\tau(\chi_1)\tau\left(\overline{\chi_1}^3\right)}\tau(\chi_1\chi_2)\tau\left(\overline{\chi_1}^3\chi_2\right) \\ &= \frac{\tau^2(\chi_2)}{p^2}\sum_{a=1}^{p-1}\overline{\chi}_1(a)\left(\frac{1-a}{p}\right)\sum_{b=1}^{p-1}\chi_1^3(b)\left(\frac{1-b}{p}\right) \\ &= \frac{\tau^2(\chi_2)}{p^2}\sum_{a=1}^{p-1}\overline{\chi}_1(a)\sum_{b=1}^{p-1}\left(\frac{1-ab^3}{p}\right)\left(\frac{1-b}{p}\right) \\ &= \frac{\tau^2(\chi_2)}{2p^2}\sum_{a=1}^{p-1}\overline{\chi}(a)\sum_{b=1}^{p-1}\left(\frac{1-a^2b^3}{p}\right)\left(\frac{1-b}{p}\right) \\ &\quad + \overline{\chi}_1(r)\frac{\tau^2(\chi_2)}{2p^2}\sum_{a=1}^{p-1}\overline{\chi}(a)\sum_{b=1}^{p-1}\left(\frac{1-ra^2b^3}{p}\right)\left(\frac{1-b}{p}\right).\end{aligned} \tag{12}$$

Note that $|\tau(\chi)| = |\tau(\chi_1)| = |\tau(\chi_1^3)| = \sqrt{p}$ and $\tau^2(\chi_2) = \pm p$; from (12) and Lemma 1 we may immediately deduce the identity

$$\begin{aligned}&\left|\sum_{a=1}^{p-1}\chi\left(ma^3+a\right)\right|^2 \\ &= p\cdot\left|1+\left(\frac{m}{p}\right)\frac{\tau(\chi_1\chi_2)\tau\left(\overline{\chi_1}^3\chi_2\right)}{\tau(\chi_1)\tau\left(\overline{\chi_1}^3\right)}\right|^2 \\ &= 2p+\left(\frac{m}{p}\right)\frac{\tau^2(\chi_2)}{2p}\sum_{a=1}^{p-1}\left(\chi(a)+\overline{\chi}(a)\right) \\ &\quad\times\sum_{b=1}^{p-1}\left(\frac{1-a^2b^3}{p}\right)\left(\frac{1-b}{p}\right)+\left(\frac{m}{p}\right)\frac{\tau^2(\chi_2)}{2p} \\ &\quad\times\sum_{a=1}^{p-1}\left(\chi_1(r)\chi(a)+\overline{\chi_1}(r)\overline{\chi}(a)\right) \\ &\quad\times\sum_{b=1}^{p-1}\left(\frac{1-ra^2b^3}{p}\right)\left(\frac{1-b}{p}\right).\end{aligned} \tag{13}$$

This proves Lemma 2. □

Lemma 3. *Let p be an odd prime, let χ be any nonprincipal even character* mod$p$, $\chi = \chi_1^2$, *and* $\chi^3 \neq \chi_0$, *the principal character* mod$p$. *Then for any integer m and any quadratic nonresidue r* mod p *with* $(m,p)=1$, *one has the estimate*

$$\begin{aligned}&\left|\sum_{a=1}^{p-1}\left(\chi_1(r)\chi(a)+\overline{\chi_1}(r)\overline{\chi}(a)\right)\sum_{b=1}^{p-1}\left(\frac{1-ra^2b^3}{p}\right)\left(\frac{1-b}{p}\right)\right. \\ &\left.+\sum_{a=1}^{p-1}\left(\chi(a)+\overline{\chi}(a)\right)\sum_{b=1}^{p-1}\left(\frac{1-a^2b^3}{p}\right)\left(\frac{1-b}{p}\right)\right| \leq 4p.\end{aligned} \tag{14}$$

Proof. Let n be any integer such that $(mn/p)=-1$ or $(m/p)+(n/p)=0$. Then from Lemma 2 we have

$$\left|\sum_{a=1}^{p-1}\chi\left(ma^3+a\right)\right|^2+\left|\sum_{a=1}^{p-1}\chi\left(na^3+a\right)\right|^2 = 4p. \tag{15}$$

Note that $|(m/p)(\tau^2(\chi_2)/p)| = 1$; applying (15) and Lemma 2 we have the estimate

$$\begin{aligned}&\left|\sum_{a=1}^{p-1}\left(\chi_1(r)\chi(a)+\overline{\chi_1}(r)\overline{\chi}(a)\right)\sum_{b=1}^{p-1}\left(\frac{1-ra^2b^3}{p}\right)\left(\frac{1-b}{p}\right)\right. \\ &\quad\left.+\sum_{a=1}^{p-1}\left(\chi(a)+\overline{\chi}(a)\right)\sum_{b=1}^{p-1}\left(\frac{1-a^2b^3}{p}\right)\left(\frac{1-b}{p}\right)\right| \\ &= \left|\left|\sum_{a=1}^{p-1}\chi\left(ma^3+a\right)\right|^2-\left|\sum_{a=1}^{p-1}\chi\left(na^3+a\right)\right|^2\right| \\ &\leq \left|\sum_{a=1}^{p-1}\chi\left(ma^3+a\right)\right|^2+\left|\sum_{a=1}^{p-1}\chi\left(na^3+a\right)\right|^2 \leq 4p.\end{aligned} \tag{16}$$

This proves Lemma 3. □

Lemma 4. *Let $p>3$ be a prime. Then we have the identity*

$$\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right|^2 = -\tau^2(\chi_2)\left(2+\left(\frac{3}{p}\right)\right), \tag{17}$$

where $(*/p)=\chi_2$ *denotes the Legendre symbol.*

Proof. For any odd prime p and integer n with $(n,p)=1$, from Hua's book [11] (Section 7.8, Theorem 8.2) we know that

$$\sum_{a=1}^{p}\left(\frac{a^2+n}{p}\right) = -1. \tag{18}$$

From this identity and the definition and properties of Gauss sums we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right|^2\\
&=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)e\left(\frac{m\left(a^3-b^3\right)+a-b}{p}\right)\\
&=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)e\left(\frac{mb^3\left(a^3-1\right)+b\left(a-1\right)}{p}\right)\\
&=\tau\left(\chi_2\right)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{b^3\left(a^3-1\right)}{p}\right)e\left(\frac{b\left(a-1\right)}{p}\right)\\
&=\tau\left(\chi_2\right)\sum_{a=1}^{p-1}\left(\frac{a^3-1}{p}\right)\sum_{b=1}^{p-1}\left(\frac{b}{p}\right)e\left(\frac{b\left(a-1\right)}{p}\right)\\
&=\tau^2\left(\chi_2\right)\sum_{a=1}^{p-1}\left(\frac{\left(a^3-1\right)\left(a-1\right)}{p}\right)\\
&=\tau^2\left(\chi_2\right)\left(\sum_{a=1}^{p}\left(\frac{4a^2+4a+4}{p}\right)-1-\left(\frac{3}{p}\right)\right)\\
&=\tau^2\left(\chi_2\right)\left(\sum_{a=1}^{p}\left(\frac{(2a+1)^2+3}{p}\right)-1-\left(\frac{3}{p}\right)\right)\\
&=\tau^2\left(\chi_2\right)\left(\sum_{a=1}^{p}\left(\frac{a^2+3}{p}\right)-1-\left(\frac{3}{p}\right)\right)\\
&=-\tau^2\left(\chi_2\right)\left(2+\left(\frac{3}{p}\right)\right).
\end{aligned}\tag{19}$$

This proves Lemma 4. □

3. Proof of the Theorem

In this section, we will complete the proof of our theorem. Note that the identities $|\tau(\chi_2)|^2 = p$ and

$$\begin{aligned}
&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right|^2\\
&=\sum_{m=1}^{p}\left|\sum_{a=1}^{p-1}e\left(\frac{ma^3+a}{p}\right)\right|^2-1\\
&=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=1}^{p}e\left(\frac{m\left(a^3-b^3\right)+a-b}{p}\right)-1\\
&=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=1}^{p}e\left(\frac{m\left(a^3-1\right)+b\left(a-1\right)}{p}\right)-1\\
&=\begin{cases}p^2-p-1, & \text{if } 3\dagger p-1,\\ p^2-3p-1, & \text{if } 3\mid p-1.\end{cases}
\end{aligned}\tag{20}$$

So from (20), Lemmas 2, 3, and 4, and noting that $|\tau(\chi_2)|^2 = p$ we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(ma^3+a\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^2\\
&=2p\cdot\sum_{m=1}^{p-1}\left|\sum_{c=1}^{p-1}e\left(\frac{mc^3+c}{p}\right)\right|^2\\
&\quad+\frac{\tau^2\left(\chi_2\right)}{2p}\sum_{a=1}^{p-1}\left(\chi\left(a\right)+\overline{\chi}\left(a\right)\right)\\
&\quad\times\sum_{b=1}^{p-1}\left(\frac{1-a^2b^3}{p}\right)\left(\frac{1-b}{p}\right)\\
&\quad\times\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left|\sum_{c=1}^{p-1}e\left(\frac{mc^3+c}{p}\right)\right|^2+\frac{\tau^2\left(\chi_2\right)}{2p}\\
&\quad\times\sum_{a=1}^{p-1}\left(\chi_1\left(r\right)\chi\left(a\right)+\overline{\chi}_1\left(r\right)\overline{\chi}\left(a\right)\right)\\
&\quad\times\sum_{b=1}^{p-1}\left(\frac{1-a^2b^3}{p}\right)\left(\frac{1-b}{p}\right)\\
&\quad\times\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)\left|\sum_{c=1}^{p-1}e\left(\frac{mc^3+c}{p}\right)\right|^2\\
&=2p^3+E\left(p\right),
\end{aligned}\tag{21}$$

where $E(p)$ satisfies the inequalities $-12p^2-2p \le E(p) \le 4p^2-2p$.

This completes the proof of our theorem.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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