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Research Article

A Generalized Henry-Type Integral Inequality and Application to Dependence on Orders and Known Functions for a Fractional Differential Equation

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We discuss on integrable solutions for a generalized Henry-type integral inequality in which weak singularity and delays are involved. Not requiring continuity or differentiability for some given functions, we use a modified iteration argument to give an estimate of the unknown function in terms of the multiple Mittag-Leffler function. We apply the result to give continuous dependence of solutions on initial data, derivative orders, and known functions for a fractional differential equation.

1. Introduction

Since Gronwall [1] and Bellman [2] discussed the integral inequalities

$$0 \le u(t) \le \int_\alpha^t (a + bu(s))\, ds, \tag{1}$$

$$u(t) \le a + \int_\alpha^t f(s)\, u(s)\, ds, \tag{2}$$

respectively, there have been made many generalizations, some of which were applied to existence, uniqueness, boundedness, and stability of solutions and invariant manifolds for differential equations and integral equations. In 1956 Bihari [3] discussed a nonlinear version of the integral inequality

$$u(t) \le a + \int_0^t f(s)\, w(u(s))\, ds, \quad t \ge 0, \tag{3}$$

where the given function w is continuous, nondecreasing, and positive definite (i.e., $w(u) \ge 0$ for all u and $w(u) = 0$ if and only if $u = 0$) on $[0, \infty)$. In 2000 this result was generalized by Lipovan [4] to the delay case

$$\begin{aligned} u(t) \le{}& a + \int_{t_0}^t f(s)\, w(u(s))\, ds \\ &+ \int_{b(t_0)}^{b(t)} g(s)\, w(u(s))\, ds, \quad t_0 \le t < t_1, \end{aligned} \tag{4}$$

where b is a continuously differentiable and nondecreasing function from $[t_0, t_1)$ to $[t_0, t_1)$ such that $b(t) \le t$. In 2005 Agarwal et al. [5] investigated the integral inequality of a finite sum

$$\begin{aligned} u(t) \le a(t) + \sum_{i=1}^n \int_{b_i(t_0)}^{b_i(t)} f_i(t, s)\, w_i(u(s))\, ds, \\ t_0 \le t < t_1, \end{aligned} \tag{5}$$

where the monotonicity of a is not required but w_{i+1} has stronger monotonicity than w_i for each i. In recent years some new generalizations were given in, for example, [6–8]. Many results on integral inequalities can be found in Pachpatte's monograph (see [9]).

Among various types of integral inequalities, integral inequalities with weakly singular kernels are important tools in the discussion of reaction-diffusion equations and fractional differential equations. As shown in [10], the integral $\int_{t0}^{t} K(t,s)$ is singular on the line $s = t$; it is referred to be *weakly singular* if it is singular and $\int_{t0}^{t} |f(t,s)|ds < \infty$ for all $t_0 \le t < T \le \infty$. In fractional differential equations we need to consider the following Riemann-Liouville derivative operator and integral operator (see [11])

$$D_{t_0}^{\alpha} f(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \times \int_{t_0}^{x} (x-t)^{-\alpha} f(t)\, dt, \quad 0 < \alpha < 1, \tag{6}$$

$$I_{t_0}^{\beta} f(x) := \frac{1}{\Gamma(\beta)} \int_{t_0}^{x} (x-t)^{\beta-1} f(t)\, dt, \quad \beta > 0,$$

in which the singular kernels $(x-t)^{-\alpha}$ and $(x-t)^{\beta-1}$ are included, respectively. Another example can be seen from the Cauchy problem of the evolution equation

$$\dot{x} + A(t)x = f(t), \qquad x(t_0) = x_0, \tag{7}$$

in a Banach space X, where A is a sectorial operator and f is locally Hölder continuous with Hölder index $0 < \theta < 1$. From the variation-constant formula (see [12]), we encounter the following:

$$\begin{gathered} \left\| A e^{-A(t-s)} \right\| = O\left((t-s)^{-1}\right), \\ \| f(t) - f(s) \| = O\left((t-s)^{\theta}\right), \end{gathered} \tag{8}$$

and therefore the singular kernel $(t-s)^{\theta-1}$ is the estimation of its solution. For this reason the integral inequality with a weakly singular kernel

$$u(t) \le a(t) + b \int_0^t (t-s)^{\beta-1} u(s)\, ds, \quad t \ge 0, \tag{9}$$

where $b \ge 0$, $0 < \beta < 1$ and both a and u are nonnegative and locally integrable, was considered in Henry's book [12] in 1981 and the estimate

$$u(t) \le a(t) + \int_0^t \left\{ \sum_{n=1}^{\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right\} ds, \quad t \ge 0, \tag{10}$$

where Γ is the Gamma function, was given by an iteration approach. Following Henry's idea, in 1994 Sano and Kunimatsu [13] extended Henry's result to a more general integral inequality

$$\begin{aligned} u(t) \le{}& c_1 + c_2 t^{\beta-1} + c_3 \int_0^t u(s)\, ds \\ &+ c_4 \int_0^t (t-s)^{\beta-1} u(s)\, ds, \quad t \ge 0, \end{aligned} \tag{11}$$

where $c_1, c_2, c_3, c_4 \ge 0$, $0 < \beta < 1$. Inequality (9) was also extended by Ye et al. [14] in 2007 by replacing the constant b with a nonnegative, nondecreasing, bounded, and continuous function $g(t)$.

Another idea for inequalities with a weakly singular kernel was introduced by Medved [15] in 1997 for the following Henry-Bihari type integral inequality:

$$u(t) \le a(t) + \int_0^t (t-s)^{\beta-1} F(s)\, w(u(s))\, ds, \quad t \ge 0, \tag{12}$$

where $0 < \beta < 1$. This inequality is of Bihari's form (3) with a weakly singular kernel. He applied the well-known Hölder inequality to separate the unknown u from the singular kernel, that is,

$$\begin{aligned} u(t) &\le a(t) + \int_0^t (t-s)^{\beta-1} e^{s} F(s)\, e^{-s} w(u(s))\, ds \\ &\le a(t) + \left(\int_0^t (t-s)^{p(\beta-1)} e^{ps} ds \right)^{1/p} \\ &\quad \times \left(\int_0^t F(s)^q e^{-qs} w(u(s))^q ds \right)^{1/q}, \end{aligned} \tag{13}$$

where $p, q > 1$ are certain constants such that $1/p + 1/q = 1$, so that the inserted exponential factor e^s makes the singular integral $\int_0^t (t-s)^{p(\beta-1)} e^{ps} ds$ convergent as $s \to t$ and the inequality is reduced to the classic Bihari's form (3). In 2002 Ma and Yang [16] improved Medved's method and gave an estimation to the Volterra-type integral inequality with a more general form of weakly singular kernel

$$\begin{aligned} u(t) \le{}& a(t) + b(t) \int_0^t \left(t^{\sigma} - s^{\sigma}\right)^{\mu-1} s^{\tau-1} g(s)\, u(s)\, ds \\ &+ c(t) \int_0^t \left(t^{\alpha} - s^{\alpha}\right)^{\beta-1} s^{\gamma-1} f(s)\, w(u(s))\, ds, \\ & t \ge 0, \end{aligned} \tag{14}$$

where $\alpha, \beta, \gamma, \sigma, \mu, \tau \ge 0$. Recently Ma and Pečarić [17] also employed the separation approach to discuss another weakly singular integral inequality

$$u^p(t) \le a(t) + b(t) \int_0^t \left(t^{\alpha} - s^{\alpha}\right)^{\beta-1} s^{\gamma-1} f(s)\, u^q(s)\, ds, \quad t \ge 0, \tag{15}$$

under the condition $p \ge q \ge 0$.

In this paper we investigate the following integral inequality of finite sum:

$$u(t) \le a(t) + \sum_{i=1}^{m} \int_{b_i(t_0)}^{b_i(t)} \left(t - f_i(s)\right)^{\beta_i} h_i(t,s)\, u(s)\, ds, \quad t \ge 0, \tag{16}$$

in which weakly singular kernels and delays are involved. Not requiring continuity of a or differentiability of b_i, we give an estimate for locally integrable u in terms of the multiple Mittag-Leffler function (see [18]). We prefer Henry's iteration approach in our proof because the approach does not reduce the problem to the classic one so that no more continuity and differentiability are required. Our result generalizes the works made in [14–17] in some sense because of the more general form (16). Finally, we apply the result to give continuous dependence of solutions on initial data, derivative orders, and known functions for a fractional differential equation.

2. Main Results

For constants t_0, T such that $t_0 < T \le +\infty$, consider inequality (16), where a, b_i, f_i and h_i are given nonnegative functions and satisfy the following hypotheses:

(H_1) a is a locally integrable function on $[t_0, T)$, that is, Lebesgue integrable on every compact subset of $[t_0, T)$, and the integrations in (16) are bounded by replacing u with a in (16);

(H_2) every h_i, $i = 1, \dots, m$, is continuous on $[t_0, T) \times [t_0, T)$;

(H_3) every $b_i : [t_0, T) \to [t_0, T)$, $i = 1, \dots, m$, is continuous and strictly increasing such that $b_i(t) \le t$ for all $t \in [t_0, T)$;

(H_4) every f_i, $i = 1, \dots, m$, is continuously differentiable on $[t_0, T)$ such that $f_i'(t) > 0$ and $t \le f_i(t) \le b_i^{-1}(t)$ for all $t \in [t_0, T)$.

Theorem 1. *Suppose that (H_1)–(H_4) hold and $\beta_i > -1$, $i = 1, \dots, m$. Then, every nonnegative and locally integrable function $u(t)$ which satisfies* (16) *on $[t_0, T)$ and the integrations in* (16) *bounded has the estimate*

$$u(t) \le a(t) + \sum_{n=1}^{\infty} \left\{ \sum_{1 \le k_1, \dots, k_n \le m} \frac{\prod_{i=1}^{n} \Gamma\left(\beta_{k_i} + 1\right) \widetilde{h}_{k_i}(t)}{\Gamma\left(\sum_{i=1}^{n} \beta_{k_i} + n\right)} \cdot \int_{b_{k_1}(t_0)}^{b_{k_1}(t)} \left(t - f_{k_1}(s)\right)^{\sum_{i=1}^{n} \beta_{k_i} + n - 1} \times f_{k_1}'(s)\, a(s)\, ds \right\}, \quad t \in [t_0, T), \tag{17}$$

where $\widetilde{h}_i(t) = \max_{(s,\tau) \in [t_0, t] \times [t_0, t]} h_i(s, \tau) / f_i'(\tau)$, $i = 1, \dots, m$.

We leave the proof of this theorem to next section. In what follows, we express the estimate of series form in terms of the multiple Mittag-Leffler function (see [18]).

The original Mittag-Leffler function

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad z \in \mathbb{C}, \ \alpha > 0, \tag{18}$$

was proposed as an extension of the exponential function by Mittag-Leffler ([19]) in 1903. An extension of two parameters

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad \alpha, \beta, z \in \mathbb{C}, \Re(\alpha), \Re(\beta) > 0, \tag{19}$$

was proposed by Wiman [20] in 1905. Later, two Mittag-Leffler functions with three parameters were given separately by Prabhakar [21] and Kilbas and Saigo [22]. In 1996 Hadid and Luchko [18] generalized the function into the multiple form

$$E_{(\alpha_1,\dots,\alpha_n),\beta}(z_1, \dots, z_n) := \sum_{k=0}^{\infty} \sum_{\substack{l_1 + \cdots + l_n = k \\ l_1, \dots, l_n \ge 0}} \frac{(k; l_1, \dots, l_n) \prod_{i=1}^{n} z_i^{l_i}}{\Gamma\left(\sum_{i=1}^{n} l_i \alpha_i + \beta\right)}, \tag{20}$$

where $\alpha_i, \beta, z_i \in \mathbb{C}$, $\Re(\alpha_i)$, $\Re(\beta) > 0$, $i = 1, \dots, n$ and

$$(k; l_1, \dots, l_n) := \frac{k!}{l_1! \times \cdots \times l_n!}. \tag{21}$$

These generalized Mittag-Leffler functions have been treated as significant special functions since they played an important role in computing fractional calculus and solving fractional differential and integral equations modeled in physics, chemistry, biology, engineering, and applied sciences (see monographs [11, 23]). We have the following inequality:

$$E_{(\alpha_1,\dots,\alpha_n),1}(z_1, \dots, z_n) \le \prod_{i=1}^{n} E_{\alpha_i}(z_i), \quad \alpha_i, z_i \in \mathbb{R}_+, \quad i = 1, \dots, n. \tag{22}$$

In fact, we can show that

$$\Gamma\left(\sum_{i=1}^{m} a_i + 1\right) \ge \prod_{i=1}^{m} \Gamma(a_i + 1), \quad a_i > 0, \ i = 1, \dots, m. \tag{23}$$

By Euler's definition on Gamma function (see [24]), we have

$$\Gamma(a_i + 1) = \frac{1}{a_i + 1} \prod_{n=1}^{\infty} \frac{(1 + (1/n))^{a_i + 1}}{1 + ((a_i + 1)/n)},$$
$$\Gamma\left(\sum_{i=1}^{m} a_i + 1\right) = \frac{1}{\sum_{i=1}^{m} a_i + 1} \prod_{n=1}^{\infty} \frac{(1 + (1/n))^{\sum_{i=1}^{m} a_i + 1}}{1 + \left(\sum_{i=1}^{m} (a_i + 1)/n\right)}. \tag{24}$$

It follows that

$$\frac{\Gamma\left(\sum_{i=1}^{m} a_i + 1\right)}{\prod_{i=1}^{m} \Gamma(a_i + 1)} = \frac{\prod_{i=1}^{m} (a_i + 1)}{\sum_{i=1}^{m} a_i + 1} \prod_{n=1}^{\infty} \frac{\prod_{i=1}^{m} \left(1 + ((a_i + 1)/n)\right)}{(1 + (1/n))^{m-1} \left(1 + \left(\sum_{i=1}^{m} (a_i + 1)/n\right)\right)}. \tag{25}$$

It is easy to know that $\prod_{i=1}^{m} (a_i + 1) \ge \sum_{i=1}^{m} a_i + 1$. Then, we can prove by induction that

$$\prod_{i=1}^{m} \left(1 + \frac{a_i + 1}{n}\right) \ge \left(1 + \frac{1}{n}\right)^{m-1} \left(1 + \frac{\sum_{i=1}^{m} a_i + 1}{n}\right). \tag{26}$$

Obviously, (26) is true for $m = 1$. Assume that (26) is true for $m = l$. We get

$$\prod_{i=1}^{l+1}\left(1+\frac{a_i+1}{n}\right) = \left(1+\frac{1}{n}\right)^{l-1}\left(1+\frac{\sum_{i=1}^{l} a_i+1}{n}\right) \times\left(1+\frac{a_{l+1}+1}{n}\right) \geq \left(1+\frac{1}{n}\right)^{l}\left(1+\frac{\sum_{i=1}^{l+1} a_i+1}{n}\right), \tag{27}$$

since

$$\left(1+\frac{\sum_{i=1}^{l} a_i+1}{n}\right)\left(1+\frac{a_{l+1}+1}{n}\right) \geq \left(1+\frac{1}{n}\right)\left(1+\frac{\sum_{i=1}^{l+1} a_i+1}{n}\right), \tag{28}$$

implying that (26) holds for $m = l+1$. Thus, from (26) we see that (23) holds. By (23), we have

$$E_{(\alpha_1,\ldots,\alpha_n),1}(z_1,\ldots,z_n) = \sum_{m=0}^{\infty}\sum_{1\leq k_1,\ldots,k_m\leq n}\frac{\prod_{i=1}^{m} z_{k_i}}{\Gamma\left(\sum_{i=1}^{m}\alpha_{k_i}+1\right)} \leq \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\frac{\prod_{i=1}^{n} z_i^{k_i}}{\prod_{i=1}^{n}\Gamma\left(k_i\alpha_i+1\right)} = \prod_{i=1}^{n} E_{\alpha_i}(z_i). \tag{29}$$

Therefore, (22) is proved.

Corollary 2. *Suppose that the hypotheses of Theorem 1 hold and additionally that $a(t)$ is continuous on $[t_0, T)$. Then,*

$$u(t) \leq \tilde{a}(t) E_{(\beta_1+1,\ldots,\beta_m+1),1} \times\left(\tilde{h}_1(t)\Gamma(\beta_1+1)(t-t_0)^{\beta_1+1}, \ldots, \tilde{h}_m(t)\Gamma(\beta_m+1)(t-t_0)^{\beta_m+1}\right) \leq \tilde{a}(t)\prod_{i=1}^{m} E_{\beta_i+1}\left(\tilde{h}_i(t)\Gamma(\beta_i+1)(t-t_0)^{\beta_i+1}\right), \tag{30}$$

where $\tilde{a}(t) = \max_{t_0\leq\tau\leq t} a(\tau)$ and $E_\alpha(z)$ and $E_{(\alpha_1,\ldots,\alpha_n),1}$ are defined in (18) *and* (20).

Proof. Starting from (17), we have

$$\begin{aligned} u(t) &\leq \tilde{a}(t)\left\{1+\sum_{n=1}^{\infty}\left\{\sum_{1\leq k_1,\ldots,k_n\leq m}\frac{\prod_{i=1}^{n}\Gamma\left(\beta_{k_i}+1\right)\tilde{h}_{k_i}(t)}{\Gamma\left(\sum_{i=1}^{n}\beta_{k_i}+n\right)} \times\int_{b_{k_1}(t_0)}^{b_{k_1}(t)}\left(t-f_{k_1}(s)\right)^{\sum_{i=1}^{n}\beta_{k_i}+n-1} \times f'_{k_1}(s)\,ds\right\}\right\} \\
&\leq \tilde{a}(t)\left\{1+\sum_{n=1}^{\infty}\left(\sum_{1\leq k_1,\ldots,k_n\leq m}\frac{\prod_{i=1}^{n}\Gamma\left(\beta_{k_i}+1\right)\tilde{h}_{k_i}(t)}{\Gamma\left(\sum_{i=1}^{n}\beta_{k_i}+n+1\right)} \times(t-t_0)^{\sum_{i=1}^{n}(\beta_{k_i}+1)}\right)\right\} \\
&= \tilde{a}(t)\sum_{n=0}^{\infty}\sum_{1\leq k_1,\ldots,k_n\leq m}\left(\prod_{i=1}^{n}\Gamma\left(\beta_{k_i}+1\right)\tilde{h}_{k_i}(t) \times(t-t_0)^{\beta_{k_i}+1}\right) \times\left(\Gamma\left(\sum_{i=1}^{n}\beta_{k_i}+n+1\right)\right)^{-1} \\
&= \tilde{a}(t) \times\sum_{k=0}^{\infty}\sum_{\substack{k_1+\cdots+k_m=k\\ k_1,\ldots,k_m\geq 0}}\left((k;k_1,\ldots,k_m) \times\prod_{i=1}^{m}\left(\Gamma(\beta_i+1)\tilde{h}_i(t) \times(t-t_0)^{\beta_i+1}\right)^{k_i}\right) \times\left(\Gamma\left(\sum_{i=1}^{m}k_i(\beta_i+1)+1\right)\right)^{-1} \\
&= \tilde{a}(t) E_{(\beta_1+1,\ldots,\beta_m+1),1} \times\left(\tilde{h}_1(t)\Gamma(\beta_1+1)(t-t_0)^{\beta_1+1}, \ldots, \tilde{h}_m(t)\Gamma(\beta_m+1)(t-t_0)^{\beta_m+1}\right).
\end{aligned} \tag{31}$$

It follows from (22) that

$$\tilde{a}(t) E_{(\beta_1+1,\ldots,\beta_m+1),1} \times\left(\tilde{h}_1(t)\Gamma(\beta_1+1)(t-t_0)^{\beta_1+1}, \ldots, \tilde{h}_m(t)\Gamma(\beta_m+1)(t-t_0)^{\beta_m+1}\right) \leq \tilde{a}(t)\prod_{i=1}^{m} E_{\beta_i+1}\left(\tilde{h}_i(t)\Gamma(\beta_i+1)(t-t_0)^{\beta_i+1}\right). \tag{32}$$

The corollary is proved. □

3. Proof of Theorem

Let $L^1_{\mathrm{loc},w}[t_0, T)$ consist of all locally integrable nonnegative functions ϕ on $[t_0, T)$ such that $\sum_{i=1}^{m}\int_{b_i(t_0)}^{b_i(t)}(t-f_i(s))^{\beta_i} h_i(t,s)\phi(s)ds < \infty$ for arbitrarily given $t \in [t_0, T)$. We can verify that $L^1_{\mathrm{loc},w}[t_0,T)$ is a linear space, due to the linearity of integration. Define a linear operator B on $L^1_{\mathrm{loc},w}[t_0,T)$ as

$$B\phi(t) := \sum_{i=1}^{m} \int_{b_i(t_0)}^{b_i(t)} (t-f_i(s))^{\beta_i} h_i(t,s)\phi(s)\,ds, \quad t\in[t_0,T), \tag{33}$$

where $\phi \in L^1_{\mathrm{loc},w}[t_0,T)$. We claim that B is self-mapping on $L^1_{\mathrm{loc},w}[t_0,T)$. In fact, for all $\phi \in L^1_{\mathrm{loc},w}[t_0,T)$ and given $t \in [t_0,T)$, $\sum_{i=1}^{m}\int_{b_i(t_0)}^{b_i(t)}(t-f_i(s))^{\beta_i}h_i(t,s)\phi(s)ds < \infty$, implying that $B\phi \in C_+[t_0,T)$, the set of all continuous and nonnegative functions on $[t_0,T)$. We know that $C_+[t_0,T) \subset L^1_{\mathrm{loc}}[t_0,T)$. For all $\psi \in C_+[t_0,T)$, from the continuity of ψ and h_i, we know that they are all locally bounded. We also know that $(t-f_i(s))^{\beta_i}$ are integrable on $s \in [b_i(t_0), b_i(t)]$ by (H_3), (H_4) and $\beta_i > -1$. It follows that $\sum_{i=1}^{m}\int_{b_i(t_0)}^{b_i(t)}(t-f_i(s))^{\beta_i}h_i(t,s)\psi(s)ds < \infty$, implying that $C_+[t_0,T) \subset L^1_{\mathrm{loc},w}[t_0,T)$. Then, inequality (16) can be simplified as $u(t) \le a(t) + Bu(t)$, from which we can prove by induction that

$$u(t) \le \sum_{l=0}^{n-1} B^l a(t) + B^n u(t). \tag{34}$$

We claim that for all integers $n \ge 1$,

$$B^n u(t) \le \sum_{1\le k_1,\ldots,k_n\le m} \frac{\prod_{i=1}^{n}\Gamma(\beta_{k_i}+1)\tilde{h}_{k_i}(t)}{\Gamma\left(\sum_{i=1}^{n}\beta_{k_i}+n\right)} \cdot \int_{b_{k_n}(t_0)}^{b_{k_n}(t)} \left(t-f_{k_n}(s)\right)^{\sum_{i=1}^{n}\beta_{k_i}+n-1} f'_{k_n}(s)\,u(s)\,ds, \tag{35}$$

where $\tilde{h}_{k_i}$ are well-defined on $[t_0,T)$ since h_{k_i} and f'_{k_i} are continuous and $f'_{k_i}(t) > 0$ on $[t_0,T)$. In fact, (35) is true for $n = 1$. Assume that (35) is true for $n = l$. Then

$$\begin{aligned} B^{l+1}u(t) &= B^l(Bu(t)) \\ &\le \sum_{1\le k_1,\ldots,k_l\le m} \frac{\prod_{i=1}^{l}\Gamma(\beta_{k_i}+1)\tilde{h}_{k_i}(t)}{\Gamma\left(\sum_{i=1}^{l}\beta_{k_i}+l\right)} \times \int_{b_{k_l}(t_0)}^{b_{k_l}(t)} \left(t-f_{k_l}(s)\right)^{\sum_{i=1}^{l}\beta_{k_i}+l-1} f'_{k_l}(s) \\ &\quad \cdot \sum_{k_{l+1}=1}^{m} \left\{ \int_{b_{k_{l+1}}(t_0)}^{b_{k_{l+1}}(s)} h_{k_{l+1}}(s,\tau) \times \left(s-f_{k_{l+1}}(\tau)\right)^{\beta_{k_{l+1}}} \times u(\tau)\,d\tau \right\} ds. \end{aligned} \tag{36}$$

It follows that

$$B^{l+1}u(t) \le \sum_{1\le k_1,\ldots,k_l\le m} \frac{\prod_{i=1}^{l}\Gamma(\beta_{k_i}+1)\tilde{h}_{k_i}(t)}{\Gamma\left(\sum_{i=1}^{l}\beta_{k_i}+l\right)} \times \sum_{k_{l+1}=1}^{m} \tilde{h}_{k_{l+1}}(t)\,\Phi(t), \tag{37}$$

where

$$\Phi(t) := \int_{b_{k_l}(t_0)}^{b_{k_l}(t)} \left(t-f_{k_l}(s)\right)^{\sum_{i=1}^{l}\beta_{k_i}+l-1} f'_{k_l}(s)\,\varphi(s)\,ds, \tag{38}$$

$$\varphi(s) := \int_{b_{k_{l+1}}(t_0)}^{b_{k_{l+1}}(s)} \left(s-f_{k_{l+1}}(\tau)\right)^{\beta_{k_{l+1}}} f'_{k_{l+1}}(\tau)\,u(\tau)\,d\tau. \tag{39}$$

Note that $f_{k_i}(b_{k_i}(t)) \le t$ by (H_4). It implies that $s - f_{k_{l+1}}(\tau) \ge 0$ for all $\tau \in [b_{k_{l+1}}(t_0), b_{k_{l+1}}(s)]$ by the monotonicity of $f_{k_{l+1}}$. It follows from (39) that φ is a nondecreasing function. Hence, $\varphi(f_{k_l}^{-1}(t)) \le \varphi(t)$ by (H_4). On the other hand, from (H_3) and (H_4) we see that $b_i(t_0) = f_i(t_0) = t_0$, implying that $f_{k_l}(b_{k_l}(t_0)) = t_0$. With the change of variables $\xi = f_{k_l}(s)$, it follows from (38) that

$$\begin{aligned} \Phi(t) &= \int_{f_{k_l}(b_{k_l}(t_0))}^{f_{k_l}(b_{k_l}(t))} (t-\xi)^{\sum_{i=1}^{l}\beta_{k_i}+l-1}\varphi\left(f_{k_l}^{-1}(\xi)\right)d\xi \\ &\le \int_{t_0}^{t} (t-\xi)^{\sum_{i=1}^{l}\beta_{k_i}+l-1}\varphi\left(f_{k_l}^{-1}(\xi)\right)d\xi \\ &\le \int_{t_0}^{t} (t-\xi)^{\sum_{i=1}^{l}\beta_{k_i}+l-1}\varphi(\xi)\,d\xi. \end{aligned} \tag{40}$$

Letting s denote ξ and substituting (39) in (40), we get

$$\begin{aligned} \Phi(t) &\le \int_{t_0}^{t}\int_{b_{k_{l+1}}(t_0)}^{b_{k_{l+1}}(s)} (t-s)^{\sum_{i=1}^{l}\beta_{k_i}+l-1}\left(s-f_{k_{l+1}}(\tau)\right)^{\beta_{k_{l+1}}} \times f'_{k_{l+1}}(\tau)\,u(\tau)\,d\tau\,ds \\ &= \int_{b_{k_{l+1}}(t_0)}^{b_{k_{l+1}}(t)}\int_{b_{k_{l+1}}^{-1}(\tau)}^{t} (t-s)^{\sum_{i=1}^{l}\beta_{k_i}+l-1} \times \left(s-f_{k_{l+1}}(\tau)\right)^{\beta_{k_{l+1}}} ds \times f'_{k_{l+1}}(\tau)\,u(\tau)\,d\tau, \end{aligned} \tag{41}$$

by interchanging the order of integration. In (41) we observe that

$$\begin{aligned} &\int_{b_{k_{l+1}}^{-1}(\tau)}^{t} (t-s)^{\sum_{i=1}^{l}\beta_{k_i}+l-1}\left(s-f_{k_{l+1}}(\tau)\right)^{\beta_{k_{l+1}}} ds \\ &= \left(t-f_{k_{l+1}}(\tau)\right)^{\sum_{i=1}^{l+1}\beta_{k_i}+l} \end{aligned}$$

$$\times \int_{b_{k_{l+1}}^{-1}(\tau)}^{t} \left(\frac{t-s}{t-f_{k_{l+1}}(\tau)}\right)^{\sum_{i=1}^{l}\beta_{k_i}+l-1} \times \left(\frac{s-f_{k_{l+1}}(\tau)}{t-f_{k_{l+1}}(\tau)}\right)^{\beta_{k_{l+1}}} d\left(\frac{s-f_{k_{l+1}}(\tau)}{t-f_{k_{l+1}}(\tau)}\right)$$
$$= \left(t-f_{k_{l+1}}(\tau)\right)^{\sum_{i=1}^{l+1}\beta_{k_i}+l} \times \int_{z_1}^{1} (1-z)^{\sum_{i=1}^{l}\beta_{k_i}+l-1} z^{\beta_{k_{l+1}}} dz$$
$$\leq \left(t-f_{k_{l+1}}(\tau)\right)^{\sum_{i=1}^{l+1}\beta_{k_i}+l} \times \int_{0}^{1} (1-z)^{\sum_{i=1}^{l}\beta_{k_i}+l-1} z^{\beta_{k_{l+1}}} dz$$
$$= \left(t-f_{k_{l+1}}(\tau)\right)^{\sum_{i=1}^{l+1}\beta_{k_i}+l} B\left(\sum_{i=1}^{l}\beta_{k_i}+l, \beta_{k_{l+1}}+1\right)$$
$$= \frac{\Gamma\left(\sum_{i=1}^{l}\beta_{k_i}+l\right)\Gamma\left(\beta_{k_{l+1}}+1\right)}{\Gamma\left(\sum_{i=1}^{l+1}\beta_{k_i}+l+1\right)} \times \left(t-f_{k_{l+1}}(\tau)\right)^{\sum_{i=1}^{l+1}\beta_{k_i}+l}, \tag{42}$$

where $z := \{s - f_{k_{l+1}}(\tau)\}/\{t - f_{k_{l+1}}(\tau)\}$ and $z_1 := \{b_{k_{l+1}}^{-1}(\tau) - f_{k_{l+1}}(\tau)\}/\{t - f_{k_{l+1}}(\tau)\} \geq 0$ by (H_4). Thus, from (37) we see that inequality (35) holds for $n = l + 1$ and the claimed (35) is proved.

For every $\phi(t) \in L^1_{\mathrm{loc},w}[t_0, T)$ and each $n = 0, 1, \ldots, \infty$, define a sequence of linear operators $A_n : L^1_{\mathrm{loc},w}[t_0, T) \to C[t_0, T)$ by

$$A_n\phi(t) := \sum_{1\leq k_1,\ldots,k_n\leq m} \frac{\prod_{i=1}^{n}\Gamma\left(\beta_{k_i}+1\right)\widetilde{h}_{k_i}(t)}{\Gamma\left(\sum_{i=1}^{n}\beta_{k_i}+n\right)} \cdot \int_{b_{k_n}(t_0)}^{b_{k_n}(t)} \left(t-f_{k_n}(s)\right)^{\sum_{i=1}^{n}\beta_{k_i}+n-1} \times f'_{k_n}(s)\phi(s)\,ds, \tag{43}$$

where $\int_{b_{k_n}(t_0)}^{b_{k_n}(t)} (t - f_{k_n}(s))^{\sum_{i=1}^{n}\beta_{k_i}+n-1} f'_{k_n}(s)\phi(s)ds < \infty$ since $\int_{b_{k_n}(t_0)}^{b_{k_n}(t)} (t - f_{k_n}(s))^{\beta_{k_n}}\phi(s)ds < \infty$ from the supposition of Theorem 1 and $(t - f_{k_n}(s))^{\sum_{i=1}^{n-1}(\beta_{k_i}+1)} f'_{k_n}(s)$ are continuous on $[b_{k_n}(t_0), b_{k_n}(t))$. It implies from (35) that

$$B^n\phi(t) \leq A_n\phi(t). \tag{44}$$

In what follows we will prove that for all $\phi \in L^1_{\mathrm{loc},w}[t_0, T)$ and arbitrarily given $t \in [t_0, T)$, $\sum_{n=1}^{\infty} A_n\phi(t)$ converges. Define

$$\Psi_a(x) := \frac{\Gamma(x)}{\Gamma(x+a)}, \tag{45}$$

where $x \in (0, \infty)$ and $a > 0$ is a constant. By Euler's definition on Gamma function (see [24]), $\Gamma(x) = (1/x)\prod_{n=1}^{\infty}(1 + (1/n))^x/(1 + (x/n))$, we know that

$$\Psi_a(x) = \left(1+\frac{a}{x}\right)\prod_{n=1}^{\infty}\frac{1}{(1+(1/n))^a}\left(1+\frac{a}{n+x}\right), \tag{46}$$

and therefore Ψ_a is strictly decreasing on $(0, \infty)$. Thus, for all integers $1 \leq k_i, k_j \leq m$, $1 \leq i, j \leq n$ and $\delta \geq 0$ satisfying that $-1 < \beta_{k_j} - \delta < 0$, we have $\Psi_a(\beta_{k_j} + 1) \leq \Psi_a(\beta_{k_j} - \delta + 1)$, where $a := (\sum_{i=1}^{n}\beta_{k_i}) + n - \beta_{k_j} - 1$, implying that

$$\frac{\Gamma\left(\beta_{k_j}+1\right)}{\Gamma\left(\sum_{i=1}^{n}\beta_{k_i}+n\right)} \leq \frac{\Gamma\left(\beta_{k_j}-\delta+1\right)}{\Gamma\left(\sum_{i=1}^{n}\beta_{k_i}-\delta+n\right)}. \tag{47}$$

Multiplying the above inequality by $(\prod_{i=1}^{n}\Gamma(\beta_{k_i} + 1))/\Gamma(\beta_{k_j} + 1)$, we get that

$$\frac{\prod_{i=1}^{n}\Gamma\left(\beta_{k_i}+1\right)}{\Gamma\left(\sum_{i=1}^{n}\beta_{k_i}+n\right)} \leq \frac{\prod_{i=1}^{j-1}\Gamma\left(\beta_{k_i}+1\right)\Gamma\left(\beta_{k_j}-\delta+1\right)\prod_{i=j+1}^{n}\Gamma\left(\beta_{k_i}+1\right)}{\Gamma\left(\sum_{i=1}^{n}\beta_{k_i}-\delta+n\right)}. \tag{48}$$

Let $\beta_* := \min_{1\leq i\leq m}\beta_i$, $\beta^* := \max_{1\leq i\leq m}\beta_i$ and $c := (\beta^* + 1)/(\beta_* + 1) \geq 1$. Then (48) implies that

$$\frac{\prod_{i=1}^{n}\Gamma\left(\beta_{k_i}+1\right)}{\Gamma\left(\sum_{i=1}^{n}\beta_{k_i}+n\right)} \leq \frac{\left(\Gamma\left(\beta_*+1\right)\right)^n}{\Gamma\left(n\left(\beta_*+1\right)\right)}. \tag{49}$$

When $t \in (t_0 + 1, T)$, we have

$$A_n\phi(t) = \sum_{1\leq k_1,\ldots,k_n\leq m} \frac{\prod_{i=1}^{n}\Gamma\left(\beta_{k_i}+1\right)\widetilde{h}_{k_i}(t)}{\Gamma\left(\sum_{i=1}^{n}\beta_{k_i}+n\right)} \times \int_{b_{k_n}(t_0)}^{b_{k_n}(t)} \left(t-f_{k_n}(s)\right)^{\sum_{i=1}^{n}\beta_{k_i}+n-1} f'_{k_n}(s)\phi(s)\,ds$$
$$= \sum_{1\leq k_1,\ldots,k_n\leq m} \frac{\prod_{i=1}^{n}\Gamma\left(\beta_{k_i}+1\right)\widetilde{h}_{k_i}(t)}{\Gamma\left(\sum_{i=1}^{n}\beta_{k_i}+n\right)} \times \left\{\int_{b_{k_n}(t_0)}^{f_{k_n}^{-1}(t-1)} \left(t-f_{k_n}(s)\right)^{\beta_{k_n}}\phi(s) \times \left(t-f_{k_n}(s)\right)^{\sum_{i=1}^{n-1}\beta_{k_i}+n-1} f'_{k_n}(s)\,ds \right.$$
$$\left. + \int_{f_{k_n}^{-1}(t-1)}^{b_{k_n}(t)} \left(t-f_{k_n}(s)\right)^{\beta_{k_n}}\phi(s) \times \left(t-f_{k_n}(s)\right)^{\sum_{i=1}^{n-1}\beta_{k_i}+n-1} f'_{k_n}(s)\,ds\right\}$$

$$\le m^{n-1}\left(\max_{i=1,\dots,m}\{\widetilde{h}_i(t)\}\right)^n \frac{(\Gamma(\beta_*+1))^n}{\Gamma(n(\beta_*+1))}$$
$$\times\sum_{i=1}^{m}\left\{\int_{b_i(t_0)}^{f_i^{-1}(t-1)}(t-f_i(s))^{\beta_i}\phi(s)\times(t-f_i(s))^{(n-1)c(\beta_*+1)}f_i'(s)\,ds\right.$$
$$\left.+\int_{f_i^{-1}(t-1)}^{b_i(t)}(t-f_i(s))^{\beta_i}\phi(s)\times(t-f_i(s))^{(n-1)(\beta_*+1)}f_i'(s)\,ds\right\}$$
$$\le\left(m\max_{i=1,\dots,m}\{\widetilde{h}_i(t)\}\right)^n\frac{(\Gamma(\beta_*+1))^n}{\Gamma(n(\beta_*+1))}$$
$$\times\left\{(t-t_0)^{(n-1)c(\beta_*+1)}\times\int_{b_j(t_0)}^{f_j^{-1}(t-1)}\left(t-f_j(s)\right)^{\beta_j}\phi(s)f_j'(s)\,ds+\int_{f_j^{-1}(t-1)}^{b_j(t)}\left(t-f_j(s)\right)^{\beta_j}\phi(s)f_j'(s)\,ds\right\}$$
$$\le\left(m\max_{i=1,\dots,m}\left\{\widetilde{h}_i(t)\right\}\right)^n\frac{(\Gamma(\beta_*+1))^n}{\Gamma(n(\beta_*+1))}\times(t-t_0)^{(n-1)c(\beta_*+1)}\times\int_{b_j(t_0)}^{b_j(t)}\left(t-f_j(s)\right)^{\beta_j}\phi(s)f_j'(s)\,ds, \tag{50}$$

where

$$\int_{b_j(t_0)}^{b_j(t)}\left(t-f_j(s)\right)^{\beta_j}\phi(s)f_j'(s)\,ds=\max_{i=1,\dots,m}\left\{\int_{b_i(t_0)}^{b_i(t)}(t-f_i(s))^{\beta_i}\phi(s)f_i'(s)\,ds\right\}<\infty. \tag{51}$$

Let

$$c_n(t):=\left(m\max_{i=1,\dots,m}\{\widetilde{h}_i(t)\}\right)^n\frac{(\Gamma(\beta_*+1))^n}{\Gamma(n(\beta_*+1))}\times(t-t_0)^{(n-1)c(\beta_*+1)}\times\int_{b_j(t_0)}^{b_j(t)}\left(t-f_j(s)\right)^{\beta_j}\phi(s)f_j'(s)\,ds. \tag{52}$$

Then $\sum_{n=1}^{\infty}c_n(t)$ is convergent on (t_0+1,T) by the ratio test ([25, pages 66-67]) because

$$\begin{aligned}\lim_{n\to+\infty}\frac{c_{n+1}(t)}{c_n(t)}&=\lim_{n\to+\infty}m\max_{i=1,\dots,m}\left\{\widetilde{h}_i(t)\right\}\times\frac{\Gamma(\beta_*+1)\Gamma(n(\beta_*+1))}{\Gamma((n+1)(\beta_*+1))}(t-t_0)^{c(\beta_*+1)}\\&=\lim_{n\to+\infty}B(\beta_*+1,n(\beta_*+1))\,m\times\max_{i=1,\dots,m}\left\{\widetilde{h}_i(t)\right\}(t-t_0)^{c(\beta_*+1)}\\&=\lim_{n\to+\infty}\frac{\Gamma(\beta_*+1)}{(n(\beta_*+1))^{\beta_*+1}}m\times\max_{i=1,\dots,m}\left\{\widetilde{h}_i(t)\right\}(t-t_0)^{c(\beta_*+1)}=0,\end{aligned} \tag{53}$$

where we note that $m\max_{i=1,\dots,m}\{\widetilde{h}_i(t)\}(t-t_0)^{c(\beta_*+1)}$ is a continuous function on $[t_0,T)$ and $\Gamma(\beta_*+1)/(n(\beta_*+1))^{\beta_*+1}$ is a Stirling's approximation of $B(\beta_*+1,n(\beta_*+1))$ as $n\to+\infty$, as known in [26, pages 59]. It implies that $\sum_{n=1}^{\infty}A_n\phi(t)$ is convergent on (t_0+1,T). The case of $t\in[t_0,t_0+1]$ can be proved similarly. Then, $\sum_{n=1}^{\infty}A_n\phi(t)$ is convergent for $\phi\in L^1_{\mathrm{loc},w}[t_0,T)$ and arbitrarily given $t\in[t_0,T)$. Passing to the limit as $n\to+\infty$ in (34), by (44) we get

$$\begin{aligned}u(t)&\le\sum_{n=0}^{\infty}A_na(t)\\&=a(t)+\sum_{n=1}^{\infty}\left\{\sum_{1\le k_1,\dots,k_n\le m}\frac{\prod_{i=1}^{n}\Gamma\left(\beta_{k_i}+1\right)\widetilde{h}_{k_i}(t)}{\Gamma\left(\sum_{i=1}^{n}\beta_{k_i}+n\right)}\times\int_{b_{k_n}(t_0)}^{b_{k_n}(t)}\left(t-f_{k_n}(s)\right)^{\sum_{i=1}^{n}\beta_{k_i}+n-1}\times f_{k_n}'(s)\,a(s)\,ds\right\}.\end{aligned} \tag{54}$$

Since k_i's are chosen arbitrarily, by interchanging k_n with k_1 in (54), we obtain the estimate (17) and complete the proof of the theorem.

4. Application to Dependence

Recently, increasing interest was given to fractional differential equations (see monographs [11, 23]). In this section we consider the Cauchy problem of the general fractional differential equation

$$D_{t_0}^{1}x(t)=\sum_{i=1}^{m}D_{t_0}^{\alpha_i}f_i\left(t,x\left(b_i(t)\right)\right), \tag{55}$$

$$x(t_0)=a, \tag{56}$$

where $0 \le \alpha_i < 1$, $t_0 \le t \le T < +\infty$, $D_{t_0}^{\alpha_i}$ is defined as in the Introduction, particularly, $D_{t_0}^1$ denotes d/dt and $D_{t_0}^0$ denotes identity map, $b_i : [t_0, T] \to [t_0, T]$ and $f_i : [t_0, T] \times \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, m$. This system includes the system

$$D_0^1 x(t) + p D_0^{\alpha} x(t) + D_0^0 x(t) = q(t), \quad x(0^+) = a, \tag{57}$$

as a special case, which was considered in [27] and corresponds to the Basset problem when $\alpha = 1/2$, a classical problem in fluid dynamics concerning the unsteady motion of a particle accelerating in a viscous fluid under the action of gravity (see [28]). We will give continuous dependence for solutions of (55) associated with the initial condition (56) on the derivative orders α_i's, the initial data a, and the functions f_i and b_i.

Proposition 3. *Suppose that each $b_i \in C^1([t_0, T], [t_0, T])$ is strictly increasing such that $b_i(t) \le t$ and each $f_i \in C([t_0, T] \times [a-\rho, a+\rho], \mathbb{R})$ satisfies that $|f_i(t, x_1) - f_i(t, x_2)| \le L_i |x_1 - x_2|$, $i = 1, \ldots, m$, where $T > 0$, $\rho > 0$, and $L_i > 0$ are constant. Then, the Cauchy problem* (55)-(56) *has a unique solution on $[t_0, t_0 + h]$ for a certain constant $0 < h < T$ and the solution depends continuously on a, α_i's, f_i's, and b_i's for all $t \in [t_0, t_0 + h]$.*

Proof. The Cauchy problem (55)-(56) is equivalent to the integral equation

$$\begin{aligned} x(t) &= a + \sum_{i=1}^{m} \int_{t_0}^{t} \frac{1}{\Gamma(1-\alpha_i)} d\left(\int_{t_0}^{s} (s-\tau)^{-\alpha_i} f_i(\tau, x(b_i(\tau))) \, d\tau \right) \\ &= a + \sum_{i=1}^{m} \frac{1}{\Gamma(1-\alpha_i)} \int_{t_0}^{t} (t-s)^{-\alpha_i} f_i(s, x(b_i(s))) \, ds. \end{aligned} \tag{58}$$

Define a sequence $\{\varphi_n(t)\}$ such that $\varphi_0(t) \equiv a$ and

$$\varphi_n(t) = a + \sum_{i=1}^{m} \frac{1}{\Gamma(1-\alpha_i)} \times \int_{t_0}^{t} (t-s)^{-\alpha_i} f_i(s, \varphi_{n-1}(b_i(s))) \, ds, \quad n = 1, 2, \ldots, \tag{59}$$

where $t \in [t_0, t_0 + h]$. We first claim that all $\varphi_n(t)$ are well-defined and continuous in $[t_0, t_0 + h]$, such that $|\varphi_n(t) - a| \le \rho$. In fact, it is true for $n = 0$. Suppose that it is true for some n. Then, φ_{n+1} is also well defined by (59) and continuous in $[t_0, t_0 + h]$. Moreover,

$$\begin{aligned} |\varphi_{n+1}(t) - a| &\le \sum_{i=1}^{m} \frac{1}{\Gamma(1-\alpha_i)} \int_{t_0}^{t} (t-s)^{-\alpha_i} |f_i(s, \varphi_n(b_i(s)))| \, ds \\ &\le \sum_{i=1}^{m} \frac{M h^{1-\alpha_i}}{\Gamma(2-\alpha_i)} \le \rho, \end{aligned} \tag{60}$$

where

$$M := \max_{\substack{(t,x)\in[t_0,T]\times[a-\rho,a+\rho] \\ i=1,\ldots,m}} |f_i(t,x)|, \quad h := \min\{T - t_0, \sigma\}, \tag{61}$$

and σ is a positive constant such that

$$\sum_{i=1}^{m} \frac{M \sigma^{1-\alpha_i}}{\Gamma(2-\alpha_i)} = \rho. \tag{62}$$

The existence of σ is guaranteed by the intermediate theorem for continuous functions. Thus, the claim is proved by induction for all n.

The convergence of the sequence $\{\varphi_n(t)\}$ is equivalent to the convergence of the series $\varphi_0(t) + \sum_{l=1}^{\infty} (\varphi_l(t) - \varphi_{l-1}(t))$. We claim that

$$|\varphi_l(t) - \varphi_{l-1}(t)| \le \sum_{1 \le k_1, \ldots, k_l \le m} \frac{M \prod_{i=2}^{l} L_{k_i}}{\Gamma\left(1 + \sum_{i=1}^{l} (1 - \alpha_{k_i})\right)} (t - t_0)^{\sum_{i=1}^{l} (1-\alpha_{k_i})}, \tag{63}$$

for all $t \in [t_0, t_0 + h]$ and all integers l. It can be checked easily for $l = 1$. Suppose that it is true for some l. By changing t into $b_{k_{l+1}}(s)$ in (63), we get from $b_{k_{l+1}}(s) \le s$ that

$$\begin{aligned} \left|\varphi_l(b_{k_{l+1}}(s)) - \varphi_{l-1}(b_{k_{l+1}}(s))\right| &\le \sum_{1 \le k_1, \ldots, k_l \le m} \frac{M \prod_{i=2}^{l} L_{k_i}}{\Gamma\left(1 + \sum_{i=1}^{l} (1 - \alpha_{k_i})\right)} \times \left(b_{k_{l+1}}(s) - t_0\right)^{\sum_{i=1}^{l} (1-\alpha_{k_i})} \\ &\le \sum_{1 \le k_1, \ldots, k_l \le m} \frac{M \prod_{i=2}^{l} L_{k_i}}{\Gamma\left(1 + \sum_{i=1}^{l} (1 - \alpha_{k_i})\right)} \times (s - t_0)^{\sum_{i=1}^{l} (1-\alpha_{k_i})}. \end{aligned} \tag{64}$$

Combining with (59), we have

$$\begin{aligned} |\varphi_{l+1}(t) - \varphi_l(t)| &\le \sum_{k_{l+1}=1}^{m} \frac{1}{\Gamma(1 - \alpha_{k_{l+1}})} \times \int_{t_0}^{t} (t-s)^{-\alpha_{k_{l+1}}} \\ &\quad \times \left| f_{k_{l+1}}\left(s, \varphi_l(b_{k_{l+1}}(s))\right) - f_{k_{l+1}}\left(s, \varphi_{l-1}(b_{k_{l+1}}(s))\right) \right| ds \end{aligned}$$

$$\leq \sum_{k_{l+1}=1}^{m} \frac{L_{k_{l+1}}}{\Gamma\left(1-\alpha_{k_{l+1}}\right)} \times \int_{t_0}^{t} (t-s)^{-\alpha_{k_{l+1}}} \left|\varphi_l\left(b_{k_{l+1}}(s)\right)-\varphi_{l-1}\left(b_{k_{l+1}}(s)\right)\right| ds$$

$$\leq \sum_{1\leq k_1,\dots,k_{l+1}\leq m} \frac{M\prod_{i=2}^{l+1}L_{k_i}}{\Gamma\left(1+\sum_{i=1}^{l}\left(1-\alpha_{k_i}\right)\right)\Gamma\left(1-\alpha_{k_{l+1}}\right)} \times \int_{t_0}^{t} (t-s)^{-\alpha_{k_{l+1}}}\left(s-t_0\right)^{\sum_{i=1}^{l}(1-\alpha_{k_i})} ds$$

$$= \sum_{1\leq k_1,\dots,k_{l+1}\leq m} \left(M\left(\prod_{i=2}^{l+1}L_{k_i}\right) \times B\left(1+\sum_{i=1}^{l}\left(1-\alpha_{k_i}\right),1-\alpha_{k_{l+1}}\right) \times\left(\Gamma\left(1+\sum_{i=1}^{l}\left(1-\alpha_{k_i}\right)\right) \times\Gamma\left(1-\alpha_{k_{l+1}}\right)\right)^{-1}\right) \times\left(t-t_0\right)^{\sum_{i=1}^{l+1}(1-\alpha_{k_i})}$$

$$= \sum_{1\leq k_1,\dots,k_{l+1}\leq m} \frac{M\prod_{i=2}^{l+1}L_{k_i}}{\Gamma\left(1+\sum_{i=1}^{l+1}\left(1-\alpha_{k_i}\right)\right)}\left(t-t_0\right)^{\sum_{i=1}^{l+1}(1-\alpha_{k_i})}. \tag{65}$$

Thus, (63) holds for $l+1$ and the claim is proved for all l by induction. From (63), for $t\leq t_0+h$ we have

$$\begin{aligned}\left|\varphi_l(t)-\varphi_{l-1}(t)\right| &\leq \sum_{1\leq k_1,\dots,k_l\leq m} \frac{Mh^{\sum_{i=1}^{l}(1-\alpha_{k_i})}\prod_{i=2}^{l}L_{k_i}}{\Gamma\left(1+\sum_{i=1}^{l}\left(1-\alpha_{k_i}\right)\right)} \\ &\leq \frac{m^l M h^{l(1-\alpha_*)}L^{l-1}}{\Gamma\left(1+l\left(1-\alpha^*\right)\right)},\end{aligned} \tag{66}$$

where $L := \max_{1\leq i\leq m}L_i$, $\alpha^* := \max_{1\leq i\leq m}\alpha_i$, and $\alpha_* := \min_{1\leq i\leq m}\alpha_i$. By the ratio test ([25, pp. 66-67]), we get that $\sum_{l=1}^{\infty} m^l M h^{l(1-\alpha_*)}L^{l-1}/\Gamma(1+l(1-\alpha^*))$ is convergent, implying that the series $\varphi_0(t)+\sum_{l=1}^{\infty}(\varphi_l(t)-\varphi_{l-1}(t))$ and therefore the sequence $\{\varphi_n(t)\}$ are uniformly convergent in $[t_0,t_0+h]$. Let $\varphi(t) := \lim_{n\to\infty}\varphi_n(t)$, which is well-defined and continuous in $[t_0,t_0+h]$ such that $|\varphi(t)-a|\leq\rho$. One can check that φ is a continuous solution of the Cauchy problem (55)-(56) in $[t_0,t_0+h]$.

Next, we prove the continuous dependence of the solution. Consider the Cauchy problem

$$D_{t_0}^{1}x(t)=\sum_{i=1}^{m}D_{t_0}^{\tilde{\alpha}_i}\tilde{f}_i\left(t,x\left(\tilde{b}_i(t)\right)\right), \tag{67}$$

$$x\left(t_0\right)=\tilde{a}, \tag{68}$$

where $0\leq\tilde{\alpha}_i<1$, $t_0\leq t\leq T<+\infty$, $\tilde{b}_i\in C^1([t_0,T],[t_0,T])$ is strictly increasing such that $b_i(t)\leq t$, and $\tilde{f}_i\in C([t_0,T]\times[\tilde{a}-\rho,\tilde{a}+\rho],\mathbb{R})$ is Lipschitzian in the second variable with the Lipschitz constant $\tilde{L}_i$, $i=1,\dots,m$. As above, we similarly obtain a solution ψ of the Cauchy problem (67)-(68). Similar to (58), we have

$$\begin{aligned}\varphi(t)&=a+\sum_{i=1}^{m}\frac{1}{\Gamma\left(1-\alpha_i\right)}\int_{t_0}^{t}(t-s)^{-\alpha_i}f_i\left(s,\varphi\left(b_i(s)\right)\right)ds,\\ \psi(t)&=\tilde{a}+\sum_{i=1}^{m}\frac{1}{\Gamma\left(1-\tilde{\alpha}_i\right)}\int_{t_0}^{t}(t-s)^{-\tilde{\alpha}_i}\tilde{f}_i\left(s,\psi\left(\tilde{b}_i(s)\right)\right)ds.\end{aligned} \tag{69}$$

It follows that

$$\begin{aligned}&\left|\varphi(t)-\psi(t)\right|\\ &\quad\leq|a-\tilde{a}|+\sum_{i=1}^{m}\left|\frac{1}{\Gamma\left(1-\alpha_i\right)}\times\int_{t_0}^{t}(t-s)^{-\alpha_i}f_i\left(s,\varphi\left(b_i(s)\right)\right)ds-\frac{1}{\Gamma\left(1-\tilde{\alpha}_i\right)}\times\int_{t_0}^{t}(t-s)^{-\tilde{\alpha}_i}\tilde{f}_i\left(s,\psi\left(\tilde{b}_i(s)\right)\right)ds\right|\\ &\quad\leq\Xi(t)+\sum_{i=1}^{m}\left\{\left|\frac{1}{\Gamma\left(1-\alpha_i\right)}\times\int_{t_0}^{t}(t-s)^{-\alpha_i}f_i\left(s,\varphi\left(b_i(s)\right)\right)ds-\frac{1}{\Gamma\left(1-\alpha_i\right)}\times\int_{t_0}^{t}(t-s)^{-\alpha_i}f_i\left(s,\psi\left(b_i(s)\right)\right)ds\right|\right\}\\ &\quad\leq\Xi(t)+\sum_{i=1}^{m}\frac{L_i}{\Gamma\left(1-\alpha_i\right)}\times\int_{b_i(t_0)}^{b_i(t)}\left(t-b_i^{-1}(s)\right)^{-\alpha_i}\left|\varphi(s)-\psi(s)\right|d\left(b_i^{-1}(s)\right),\end{aligned} \tag{70}$$

where

$$\begin{aligned}
\Xi(t) &:= |a-\widetilde{a}| \\
&+ \sum_{i=1}^{m} \left\{ \left| \frac{1}{\Gamma(1-\alpha_i)} \times \int_{t_0}^{t} (t-s)^{-\alpha_i} f_i(s,\psi(b_i(s)))\,ds - \frac{1}{\Gamma(1-\widetilde{\alpha}_i)} \times \int_{t_0}^{t} (t-s)^{-\widetilde{\alpha}_i} f_i(s,\psi(b_i(s)))\,ds \right| \right. \\
&+ \left| \frac{1}{\Gamma(1-\widetilde{\alpha}_i)} \int_{t_0}^{t} (t-s)^{-\widetilde{\alpha}_i} f_i(s,\psi(b_i(s)))\,ds - \frac{1}{\Gamma(1-\widetilde{\alpha}_i)} \int_{t_0}^{t} (t-s)^{-\widetilde{\alpha}_i} \widetilde{f}_i(s,\psi(b_i(s)))\,ds \right| \\
&+ \left. \left| \frac{1}{\Gamma(1-\widetilde{\alpha}_i)} \int_{t_0}^{t} (t-s)^{-\widetilde{\alpha}_i} \widetilde{f}_i(s,\psi(b_i(s)))\,ds - \frac{1}{\Gamma(1-\widetilde{\alpha}_i)} \int_{t_0}^{t} (t-s)^{-\widetilde{\alpha}_i} \widetilde{f}_i(s,\psi(\widetilde{b}_i(s)))\,ds \right| \right\} \\
&\le |a-\widetilde{a}| \\
&+ \sum_{i=1}^{m} \left\{ \left| \frac{(t-t_0)^{1-\alpha_i}}{\Gamma(2-\alpha_i)} - \frac{(t-t_0)^{1-\widetilde{\alpha}_i}}{\Gamma(2-\widetilde{\alpha}_i)} \right| \|f_i\| + \frac{(t-t_0)^{1-\widetilde{\alpha}_i}}{\Gamma(2-\widetilde{\alpha}_i)} \|f_i-\widetilde{f}_i\| \right. \\
&+ \left. \frac{\widetilde{L}_i}{\Gamma(1-\widetilde{\alpha}_i)} \int_{t_0}^{t} (t-s)^{-\widetilde{\alpha}_i} \times \left|\psi(b_i(s)) - \psi(\widetilde{b}_i(s))\right| ds \right\},
\end{aligned} \tag{71}$$

$\|f_i\| := \max_{(t,x)\in[t_0,T]\times[a-\rho,a+\rho]}|f_i(t,x)|$, and $\|f_i-\widetilde{f}_i\| := \max_{(t,x)\in[t_0,T]\times[a-\rho,a+\rho]}|f_i(t,x)-\widetilde{f}_i(t,x)|$. Thus, applying the inequality given in Corollary 2 to (70), we obtain that

$$\left|\varphi(t)-\psi(t)\right| \le \widetilde{\Xi}(t) \prod_{i=1}^{m} E_{1-\alpha_i}\left(L_i(t-t_0)^{1-\alpha_i}\right), \quad \forall t \in [t_0, t_0+h], \tag{72}$$

where $\widetilde{\Xi}(t) := \max_{t_0\le\tau\le t}\Xi(\tau)$.

Note from (71) that $\widetilde{\Xi}(t)$ contains $|(t-t_0)^{1-\alpha_i}/\Gamma(2-\alpha_i) - (t-t_0)^{1-\widetilde{\alpha}_i}/\Gamma(2-\widetilde{\alpha}_i)|$ and $|\psi(b_i(s)) - \psi(\widetilde{b}_i(s))|$, which will be estimated with $\|b_i-\widetilde{b}_i\| := \max_{t_0\le\tau\le t_0+h}|b_i(\tau)-\widetilde{b}_i(\tau)|$ and $|\alpha_i-\widetilde{\alpha}_i|$, respectively. In (71) we have $\|f_i\| \le M$ in (61). Since $(t-t_0)^{1-\alpha_i}$ and $\Gamma(2-\alpha_i)$ are both C^1 functions in $\alpha_i \in [0,1]$, and $\Gamma(2-\alpha_i) > 0$ for all $\alpha_i \in [0,1]$, we see that $(t-t_0)^{1-\alpha_i}/\Gamma(2-\alpha_i)$ is C^1 in $\alpha_i \in [0,1]$. By the Lagrange mean value theorem (see [25, page 108]), function $(t-t_0)^{1-\alpha_i}/\Gamma(2-\alpha_i)$ satisfies the Lipschitz condition uniformly with respect to $t \in [t_0, t_0+h]$; that is, there exists a constant $N > 0$ such that

$$\left|\frac{(t-t_0)^{1-\alpha_i}}{\Gamma(2-\alpha_i)} - \frac{(t-t_0)^{1-\widetilde{\alpha}_i}}{\Gamma(2-\widetilde{\alpha}_i)}\right| \le N\left|\alpha_i-\widetilde{\alpha}_i\right|, \quad \forall \alpha_i, \widetilde{\alpha}_i \in [0,1]. \tag{73}$$

On the other hand, from the equivalent integral equation of (67)

$$\psi(t) = \psi(t_0) + \sum_{j=1}^{m} \frac{1}{\Gamma(1-\widetilde{\alpha}_j)} \int_{t_0}^{t} (t-s)^{-\widetilde{\alpha}_j} \widetilde{f}_j\left(s, \psi\left(\widetilde{b}_j(s)\right)\right) ds, \tag{74}$$

we get

$$\left|\psi(t)-\psi(t_0)\right| \le \sum_{j=1}^{m} \frac{\widetilde{M}}{\Gamma\left(2-\widetilde{\alpha}_j\right)} \left|t-t_0\right|^{1-\widetilde{\alpha}_j}, \tag{75}$$

where

$$\widetilde{M} := \max_{\substack{(t,x)\in[t_0,T]\times[a-\rho,a+\rho]\\ j=1,\ldots,m}} \left|\widetilde{f}_j(t,x)\right|. \tag{76}$$

Choosing $t = b_i(s)$ and $t_0 = \widetilde{b}_i(s)$ in (74), we obtain

$$\begin{aligned}
\left|\psi(b_i(s)) - \psi\left(\widetilde{b}_i(s)\right)\right| &\le \sum_{j=1}^{m} \frac{\widetilde{M}}{\Gamma\left(2-\widetilde{\alpha}_j\right)} \left|b_i(s)-\widetilde{b}_i(s)\right|^{1-\widetilde{\alpha}_j} \\
&\le \sum_{j=1}^{m} \frac{\widetilde{M}}{\Gamma\left(2-\widetilde{\alpha}_j\right)} \left\|b_i-\widetilde{b}_i\right\|^{1-\widetilde{\alpha}_j}.
\end{aligned} \tag{77}$$

Summarily, from (71), (73) and (75) we get

$$\begin{aligned}
\Xi(t) \le\ & |a-\widetilde{a}| \\
&+ \sum_{i=1}^{m} \left\{ MN\left|\alpha_i-\widetilde{\alpha}_i\right| + \frac{(t-t_0)^{1-\widetilde{\alpha}_i}}{\Gamma(2-\widetilde{\alpha}_i)} \left\|f_i-\widetilde{f}_i\right\| \right. \\
&+ \left. \frac{\widetilde{L}_i}{\Gamma(1-\widetilde{\alpha}_i)} \int_{t_0}^{t} (t-s)^{-\widetilde{\alpha}_i} \sum_{j=1}^{m} \frac{\widetilde{M}}{\Gamma\left(2-\widetilde{\alpha}_j\right)} \times \left\|b_i-\widetilde{b}_i\right\|^{1-\widetilde{\alpha}_j} ds \right\}
\end{aligned}$$

$$\leq |a-\tilde{a}| + \sum_{i=1}^{m} \left\{ MN \left|\alpha_i - \tilde{\alpha}_i\right| + \frac{h^{1-\tilde{\alpha}_i}}{\Gamma(2-\tilde{\alpha}_i)} \left\| f_i - \tilde{f}_i \right\| + \frac{\widetilde{M}\widetilde{L}_i h^{1-\tilde{\alpha}_i}}{\Gamma(2-\tilde{\alpha}_i)} \sum_{j=1}^{m} \frac{\left\| b_i - \tilde{b}_i \right\|^{1-\tilde{\alpha}_j}}{\Gamma\left(2-\tilde{\alpha}_j\right)} \right\}. \tag{78}$$

It follows from (72) that

$$\begin{aligned} \|\varphi-\psi\| \leq & \prod_{i=1}^{m} E_{1-\alpha_i}\left(L_i h^{1-\alpha_i}\right) |a-\tilde{a}| \\ & + MN \prod_{i=1}^{m} E_{1-\alpha_i}\left(L_i h^{1-\alpha_i}\right) \sum_{i=1}^{m} \left|\alpha_i - \tilde{\alpha}_i\right| \\ & + \prod_{i=1}^{m} E_{1-\alpha_i}\left(L_i h^{1-\alpha_i}\right) \sum_{i=1}^{m} \frac{h^{1-\tilde{\alpha}_i}}{\Gamma(2-\tilde{\alpha}_i)} \left\| f_i - \tilde{f}_i \right\| \\ & + \widetilde{M} \prod_{i=1}^{m} E_{1-\alpha_i}\left(L_i h^{1-\alpha_i}\right) \\ & \times \sum_{1\leq i,j\leq m} \frac{\widetilde{L}_i h^{1-\tilde{\alpha}_i}}{\Gamma(2-\tilde{\alpha}_i)\,\Gamma\left(2-\tilde{\alpha}_j\right)} \left\| b_i - \tilde{b}_i \right\|^{1-\tilde{\alpha}_j}; \end{aligned} \tag{79}$$

that is, the solution φ of the Cauchy problem (55)-(56) depends continuously on a, α_i, f_i and b_i.

As a complement, we note from (79) that $\varphi \equiv \psi$ in $[t_0, t_0 + h]$ if $a = \tilde{a}$, $\alpha_i = \tilde{\alpha}_i$, $f_i \equiv \tilde{f}_i$, and $b_i \equiv \tilde{b}_i$. This implies the uniqueness of the solution.

□

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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