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Research Article

New Travelling Wave Solutions for Sine-Gordon Equation

Yunchuan Sun

Business School, Beijing Normal University, Beijing 100875, China

Correspondence should be addressed to Yunchuan Sun; yunch@bnu.edu.cn

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We propose a method to deal with the general sine-Gordon equation. Several new exact travelling wave solutions with the form of *JacobiAmplitude* function are derived for the general sine-Gordon equation by using some reasonable transformation. Compared with previous solutions, our solutions are more general than some of the previous.

1. Introduction

The sine-Gordon equation

$$\omega_{tt} = \omega_{xx} + \sin(\omega) \tag{1}$$

appears in differential geometry and relativistic field theory. It is denominated following its similar form to the Klein-Gordon equation. The equation, as well as several solution techniques, was known in the 19th century, but the equation grew greatly in importance when it was realized that it led to solutions ("kink" and "antikink") with the collisional properties of solitons [1, 2]. The sine-Gordon equation is widely applied in physical and engineering applications, including the propagation of fluxons in Josephson junctions (a junction between two superconductors), the motion of rigid pendular attached to a stretched wire, and dislocations in crystals [3–6]. It also arises in nonlinear optics ^{3}He spin waves and other fields. The single sine-Gordon equation and double sine-Gordon equation are usually applied for the propagation and creation of ultrashort optical pulses in the resonant fivefold degenerate medium [7].

Indeed, there are two equivalent forms of the single sine-Gordon equation (SSG for short). Equation (1) is the form of the (real) space-time, where t and x represent the space and time coordinates, respectively [8]. If we use the light cone coordinates (u, v), akin to asymptotic coordinates, where $u = (x + t)/2$ and $v = (x - t)/2$, then we get the equivalent form as

$$\omega_{uv} = \sin(\omega). \tag{2}$$

Many mathematicians have put forward a number of approaches to solve the sine-Gordon equation based on different conditions due to its wide applications. Most of these approaches are followed by a travelling wave transformation which is a powerful means to unite a partial differential equation (PDE for short) into an ordinary differential equation (ODE for short). A number of solutions with $\tan^{-1}\coth(\xi)$, $\tan^{-1}\tanh(\xi)$, $\tan^{-1}\text{sech}(\xi)$, $\tan^{-1}\text{sn}(\xi)$, and so on have been provided in different functional forms by different methods [9–15]. References [16, 17] present several cases of solutions for the single sine-Gordon equation (2) by using the knowledge of elliptic equation and Jacobian elliptic functions through different transformations. The tanh method assumes that the travelling wave solutions can be expressed in terms of the tanh function [18–20]. And [21] also presents some exact travelling wave solutions for a more general sine-Gordon equation:

$$\omega_{tt} = a\omega_{xx} + b\sin(\lambda\omega). \tag{3}$$

In this paper, a method will be employed to derive a set of exact travelling wave solutions with a JacobiAmplitude function form which has been employed to the Dodd-Bullough equation and some new travelling wave solutions have been derived [22]. Compared with other solutions, we find that some previous solutions presented in [18, 21] are special cases of our solutions.

2. The Proposed Method

Our method is based on two assumptions.

(1) After the travelling wave transformation, the involved equation can be represented as the form

$$F\left(\omega''\right) = G\left(g\left(\omega\right)\right), \tag{4}$$

where F, G, and g might be any functions.

(2) The traveling wave solutions for the above equation meets that ω' can be expressed in a specific function form of ω.

Then, the main steps are as follows.

(1) Unite the independent variables x and t into one wave variable $\xi = x + ct$. Consider that

$$P\left(\omega_{xt}, \omega'_{xt}, \omega''_{xt}, \ldots,\right) = 0 \tag{5}$$

can change into an ODE

$$O\left(\omega\left(\xi\right), \omega'\left(\xi\right), \omega''\left(\xi\right), \ldots,\right) = 0. \tag{6}$$

The precondition for our method is that (6) meets a form of (4) after some transformation.

(2) Find solutions for (4). Firstly, we assume that

$$\omega' = f\left(g\left(\omega\right)\right). \tag{7}$$

So,

$$\omega'' = f'\left(g\left(\omega\right)\right) g'\left(\omega\right) \omega'. \tag{8}$$

It is trivial to obtain that

$$\omega'' = f\left(g\left(\omega\right)\right) f'\left(g\left(\omega\right)\right) g'\left(\omega\right). \tag{9}$$

Then, by substituting (9) into (4), we have

$$F\left(f\left(g\left(\omega\right)\right) f'\left(g\left(\omega\right)\right) g'\left(\omega\right)\right) = G\left(g\left(\omega\right)\right). \tag{10}$$

Let $\psi = g(\omega)$. We have

$$F\left(f\left(\psi\right) f'\left(\psi\right) \psi'\right) = G\left(\psi\right). \tag{11}$$

(3) Find the solutions for f from (11). In some cases, (11) is a variable separated ODE.

(4) Get ξ by integration. Equation (7) is also a variable separated ODE, so the solution for ξ can be retrieved. We have

$$\frac{d\omega}{f\left(g\left(\omega\right)\right)} = d\xi, \tag{12}$$

$$\int \frac{d\omega}{f\left(g\left(\omega\right)\right)} = \int d\xi = \xi + P, \tag{13}$$

where P is an integration constant.

In some cases, the integration of the left side of (13) is so complex that only the implicit solutions can be derived. However, we can get the explicit solutions by seeking the inverse function otherwise.

3. The Solution for Sine-Gordon Equation

Firstly, we unite the independent variables t and x into one wave variable $\xi = t + cx$ to carry out a PDE in two variables into an ODE. Thus, we get that

$$\left(1 - ac^2\right) \omega'' = b \sin\left(\lambda\omega\right). \tag{14}$$

Next, we assume that $\omega'(\xi)$ meets an ODE form of some function of $\sin(\lambda\omega)$ given by the following equation for $\omega''(\xi)$ satisfying a simple function form of $\sin(\lambda\omega)$:

$$\omega' = f\left(\sin\left(\lambda\omega\right)\right). \tag{15}$$

From (15), we can get that

$$\omega'' = \lambda f\left(\sin\left(\lambda\omega\right)\right) f'\left(\sin\left(\lambda\omega\right)\right) \cos\left(\lambda\omega\right). \tag{16}$$

Consequently, we substitute (16) into (14); we obtain

$$\left(1 - ac^2\right) \lambda f\left(\sin\left(\lambda\omega\right)\right) f'\left(\sin\left(\lambda\omega\right)\right) \cos\left(\lambda\omega\right) = b \sin\left(\lambda\omega\right). \tag{17}$$

Remember that ω is a function of $\sin(\lambda\omega)$, so we can view $\sin(\lambda\omega)$ as a new variable y; that is, $y = \sin(\lambda\omega)$, and then f is a function of y. We have

$$\left(1 - ac^2\right) \lambda f\left(y\right) f'\left(y\right) \left(\pm\sqrt{1 - y^2}\right) = by. \tag{18}$$

Now, we try to get the form of the function f.

Equation (18) is a variable separated ODE; using a symbol computation software program, such as Mathematica, we obtain

$$f\left(y\right) = \pm\sqrt{\frac{-2b}{\lambda\left(1 - ac^2\right)} \sqrt{1 - y^2} + P}, \tag{19}$$

where P is a constant of integration.

By substituting (19) into (15), we get that

$$\frac{d\omega}{d\xi} = \omega'\left(\xi\right) = \pm\sqrt{\frac{-2b}{\lambda\left(1 - ac^2\right)} \cos\left(\lambda\omega\right) + P}. \tag{20}$$

Equation (20) is also a variable separated ODE; using symbol computation tool, we obtain two sets of solutions for two different cases.

Case 1. For $b\lambda(1 - ac^2) < 0$, we can get that

$$\xi + Q = \pm\frac{2}{\lambda} \sqrt{\frac{\lambda\left(1 - ac^2\right)}{-2b + \left(1 - ac^2\right) P\lambda}} \times \text{Elliptic}F\left(\frac{\lambda\omega}{2}, \frac{4b}{2b - \left(1 - ac^2\right) P\lambda}\right), \tag{21}$$

where Q is a constant of integration. JacobiAmplitude is the inverse of the elliptic integral of the first kind. We have

$$\omega = \frac{2}{\lambda}\text{JacobiAmplitude}\left(\pm\frac{\lambda}{2}(\xi+Q)\sqrt{\frac{2b+\left(-1+ac^2\right)P\lambda}{\left(-1+ac^2\right)\lambda}}, \frac{4b}{2b+\left(-1+ac^2\right)P\lambda}\right). \quad (22)$$

Then, by substituting ξ with $t+cx$ we get the exact solution of the sine-Gordon equation (2) as follows:

$$\omega = \frac{2}{\lambda}\text{JacobiAmplitude}\left(\pm\frac{\lambda}{2}(t+cx+Q)\sqrt{\frac{2b}{\left(-1+ac^2\right)\lambda}+P}, \frac{4b}{2b+\left(-1+ac^2\right)P\lambda}\right). \quad (23)$$

Case 2. On the contrary, for $b\lambda(1-ac^2) > 0$, we can get that

$$\xi + Q = \pm\frac{2}{\lambda}\sqrt{\frac{\lambda\left(-1+ac^2\right)}{-2b+\left(-1+ac^2\right)P\lambda}} \times \text{Elliptic}F\left(\frac{-\pi+\lambda\omega}{2}, \frac{4b}{2b-\left(-1+ac^2\right)P\lambda}\right), \quad (24)$$

where Q is a constant of integration. Similar to Case 1, we obtain

$$\omega = \frac{\pi}{\lambda}+\frac{2}{\lambda} \times \text{JacobiAmplitude}\left(\pm\frac{\lambda}{2}(\xi+Q)\sqrt{\frac{2b+\left(1-ac^2\right)P\lambda}{\left(1-ac^2\right)\lambda}}, \frac{4b}{2b+\left(1-ac^2\right)P\lambda}\right). \quad (25)$$

Substituting ξ with $t+cx$, we get another exact solution of the sine-Gordon equation (3) as follows:

$$\omega = \frac{\pi}{\lambda}+\frac{2}{\lambda} \times \text{JacobiAmplitude}\left(\pm\frac{\lambda}{2}(t+cx+Q)\sqrt{\frac{2b}{\left(1-ac^2\right)\lambda}+P}, \frac{4b}{2b+\left(1-ac^2\right)P\lambda}\right). \quad (26)$$

4. Comparing to Previous Solutions

Many researchers have proposed different solutions for sine-Gordon equation. Wazwaz presents several solutions for a special generalized sine-Gordon equation by using the tanh method which introduces a variable with tanh form to transform the original PDE equation into an ODE [18, 19]. Fu et al. solve the single sine-Gordon equation by taking three kinds of different transformations and gain a list of solutions for 22 cases in [17]. Furthermore, [16] provides three kinds of solutions by transforming the equation in three different ways and summarizes some results in [17]. Reference [21] presents two exact travelling waving solutions. In this section, we compare our results with the solutions in [18, 21].

4.1. Comparing to Solutions in [21]. Two exact travelling solutions are provided in [21]. In fact these two solutions are special cases of our solutions (23) and (26). For P is a constant of integration, we can fix it to satisfy the fact that $4b/(2b+(-1+ac^2)P\lambda) = 1$; that is, $P = 2b/\lambda(-1+ac^2)$.

Then, the JacobiAmplitude function in (23) and (26) degenerates to Arctan function. Thus, (23) and (26) change into

$$\omega = \frac{2}{\lambda}\left(-\frac{\pi}{2}+2\arctan\left(\exp\left(\pm\lambda\sqrt{\frac{b}{\left(-1+ac^2\right)\lambda}} \times (t+cx+Q)\right)\right)\right), \quad b\lambda\left(1-ac^2\right) < 0; \quad (27)$$

$$\omega = \frac{4}{\lambda}\arctan\left(\exp\left(\pm\lambda\sqrt{\frac{b}{\left(1-ac^2\right)\lambda}}(t+cx+Q)\right)\right), \quad b\lambda\left(1-ac^2\right) > 0. \quad (28)$$

It is clear that the two sets of travelling waving solutions proposed in [21] are equal to (27) and (28), respectively. Indeed, these two solutions are only special cases of our solutions by fixing the constant of integration to a special value.

4.2. Comparing to Solutions in [18]. The tanh method is usually used to solve the nonlinear equations by transforming a PDE equation into an ODE. Four kinds of solutions are presented in [18] for a special generalized sine-Gordon equation by fixing $a = 1$, $b = 1$, and $\lambda = 2$:

$$\omega_{tt} - \omega_{xx} + \sin(2\omega). \quad (29)$$

Indeed, the proposed solutions in [18] are also special cases of our results. In (23) and (26), we also set $P = 2b/\lambda(-1+ac^2)$

and then $4b/(2b+(-1+ac^2)P\lambda)=1$. Meanwhile, let $a=1$, $b=1$, and $\lambda=2$. Then, we get that

$$\omega=-\frac{\pi}{2}+2\arctan\left(\exp\left(\pm 2\sqrt{\frac{1}{-2\left(1-c^2\right)}}\left(t+cx+Q\right)\right)\right),\quad 1-c^2<0;$$
$$\omega=2\arctan\left(\exp\left(\pm 2\sqrt{\frac{2}{2\left(1-c^2\right)}}\left(t+cx+Q\right)\right)\right),\quad 1-c^2>0. \tag{30}$$

It is easy to verify that (30) are equivalent to the solutions that occurred in [18] for the generalized sine-Gordon equations by reasonable transformations.

5. Conclusions

This paper develops a method to deal with the general sine-Gordon equation $\omega_{tt}=a\omega_{xx}+b\sin(\lambda\omega)$. The proposed method is based on the assumption that the travelling wave solution meets a specific form $\omega'=f(\sin(\lambda\omega))$. Two sets of new exact solutions for the general sine-Gordon equation have been retrieved. And, by comparison, we find that some previous solutions presented in [18, 21] are special cases of our solutions.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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