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Research Article

A Variational Principle for Three-Point Boundary Value Problems with Impulse

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We construct a variational functional of a class of three-point boundary value problems with impulse. Using the critical points theory, we study the existence of solutions to second-order three-point boundary value problems with impulse.

1. Introduction

In this paper, we study the following three-point boundary value problems with impulse:

$$\begin{aligned} x'' &= f(t,x), \quad t \neq t_1,\ t \in [0,1], \\ x'(0) &= a_{11}x(0) + a_{12}x(t_1) + a_{13}x(1), \\ \Delta x'(t_1) &= a_{12}x(0) + a_{22}x(t_1) + a_{23}x(1), \\ x'(1) &= -a_{13}x(0) - a_{23}x(t_1) - a_{33}x(1), \end{aligned} \tag{1}$$

where $0 < t_1 < 1$, $f : [0,1] \times R \to R$, $\Delta x'(t_1) = x'(t_1^+) - x'(t_1^-)$, and $x'(t_1^+)$ (respectively, $x'(t_1^-)$) denote the right limit (respectively, left limit) of $x'(t)$ at t_1.

The existence of solutions for three-point boundary value problems has been investigated by many authors. See, for example, [1–11] and references cited therein. In [1], Bao et al. studied a class of three-point boundary value problems

$$\begin{gathered} y''(t) + f\left(t, y(t), y'(t)\right) = 0, \quad 0 < t < 1, \\ y(0) = 0, \quad y(1) = \gamma y(\eta). \end{gathered} \tag{2}$$

Using the method upper and lower solutions, some existence results for positive solutions of problems (2) had been obtained. By applying the fixed point theory, many authors have studied the existence of positive solutions for three-point boundary value problems (see [7–11]). In [12], the authors studied Sturm-Liouville boundary value problem of a class of second-order impulsive differential equations:

$$\begin{gathered} -u''(t) + u(t) = f(t, u(t)), \quad t \in [0,T] \setminus \left\{t_1, t_2, \ldots, t_p\right\}, \\ -\Delta u\left(t_j\right) = A_j u\left(t_j^-\right), \quad j = 1,2,\ldots,p, \\ -\Delta u'\left(t_j\right) = I_j\left(u\left(t_j^-\right)\right) - A_j u'\left(t_j^+\right), \quad j = 1,2,\ldots,p, \\ \alpha u(0) - \beta u'(0) = 0, \qquad \gamma u(T) + \sigma u'(T) = 0. \end{gathered} \tag{3}$$

By establishing the corresponding variational principle of problem (3), the existence for solutions of the problems had been obtained. In papers [12–16], the variational methods were applied to impulsive differential equations too. In this paper, we will use the critical point theorem to study three-point boundary value problems (1) with impulse.

The paper is organized as follows: in Section 2, we will construct a variational functional of the boundary value problem (1). In Section 3, using the critical point theory, we will give some sufficient conditions in which problem (1) has solutions. Throughout this paper, we always assume that the following conditions hold.

(A_1) $f(t,x)$ is measurable in t for each $x \in R$, continuous in x for almost every $t \in [0,1]$.

(A_2) for any $k > 0$, there exists $h_k \in L^1(0,1)$ such that

$$\left|f(t,x)\right| \le h_k(t) \tag{4}$$

for almost every $t \in [0,1]$ and for all $|x| \le k$.

2. Variational Structure

Let W be the space of absolutely continuous functions $x : [0,1] \to R$ with a weak derivative $x' \in L^2(0,1;R)$. We define the operator P on W by

$$(Px)(t) = \begin{cases} \dfrac{x(t_1) - x(0)}{t_1}(t - t_1) + x(t_1), & 0 \le t \le t_1, \\ \dfrac{x(1) - x(t_1)}{1 - t_1}(t - t_1) + x(t_1), & t_1 < t \le 1, \end{cases} \tag{5}$$

and set $W_1 = PW$ and $W_2 = (I - P)W$. The following properties are easy consequence of the definition:

(B_1) $\dim W_1 = 3$.

(B_2) $W = W_1 \oplus W_2$.

(B_3) For each $x \in W_2$, $x(0) = x(t_1) = x(1) = 0$.

Now, we define the norm $\|\cdot\|$ over W by

$$\|x\|^2 = \int_0^1 \left[((I-P)x)'(t)\right]^2 dt + x^2(0) + x^2(t_1) + x^2(1). \tag{6}$$

Then W is a Hilbert space and the corresponding inner product (x, y) is

$$\begin{aligned}(x, y) = &\int_0^1 \left[((I-P)x)'(t) \cdot ((I-P)y)'(t)\right] dt \\ &+ x(0)y(0) + x(t_1)y(t_1) + x(1)y(1).\end{aligned} \tag{7}$$

For each $x \in W$, it follows from (6) that

$$\begin{aligned} x^2(t) &= \left[x(0) + \int_0^t x'(s)\,ds\right]^2 \\ &= \left[x(1) + \int_0^t ((I-P)x)'(s)\,ds\right]^2 \\ &\le 2x^2(1) + 2\int_0^1 \left[((I-P)x)'(t)\right]^2 dt \le 2\|x\|^2, \end{aligned} \tag{8}$$

and so $|x(t)| \le \sqrt{2}\|x\|$ on $[0,1]$.

In order to study problem (1), we define the functional ϕ on W by

$$\begin{aligned}\phi(x) = &\frac{1}{2}\|(I-P)x\|^2 + \int_0^1 F(t, x(t))\,dt \\ &+ \frac{1}{2}(x(0), x(t_1), x(1))\,A(x(0), x(t_1), x(1))^T,\end{aligned} \tag{9}$$

where $F(t,x) = \int_0^x f(t,u)du$ and

$$A = \begin{pmatrix} a_{11} + \dfrac{1}{t_1} & a_{12} - \dfrac{1}{t_1} & a_{13} \\ a_{12} - \dfrac{1}{t_1} & a_{22} + \dfrac{1}{t_1} + \dfrac{1}{1-t_1} & a_{23} - \dfrac{1}{1-t_1} \\ a_{13} & a_{23} - \dfrac{1}{1-t_1} & a_{33} + \dfrac{1}{1-t_1} \end{pmatrix}. \tag{10}$$

Under the conditions (A_1) and (A_2), ϕ is continuously differentiable, weakly lower semicontinuous on W and

$$\begin{aligned}(\phi'(x), y) = &((I-P)x, (I-P)y) \\ &+ \int_0^1 f(t, x(t))\,y(t)\,dt \\ &+ (x(0), x(t_1), x(1))\,A\,(y(0), y(t_1), y(1))^T\end{aligned} \tag{11}$$

for all $y \in W$; see [17].

The following theorem is the main conclusion of this paper.

Theorem 1. *Assume that f satisfies the conditions (A_1) and (A_2). If x is a critical point of the functional ϕ defined by* (9), *then $x(t)$ is a solution of problem* (1).

Proof. Let x be a critical point of the functional ϕ defined by (9). We prove this theorem in three steps.

Step 1. In this step, we prove that $x(t)$ satisfies the equation $x''(t) = f(t, x(t))$ except at t_1.

We define $\omega \in C(0,1;R)$ by

$$\omega(t) = \int_{t_1}^t f(s, x(s))\,ds. \tag{12}$$

It follows from (11) that for all $y \in W$,

$$\begin{aligned}&((I-P)x, (I-P)y) + \int_0^1 f(t, x(t))\,y(t)\,dt \\ &\quad + (x(0), x(t_1), x(1))\,A(y(0), y(t_1), y(1))^T = 0.\end{aligned} \tag{13}$$

By the Fubini theorem and (13), we obtain

$$\begin{aligned} &\int_0^1 \left[((I-P)x)'(t) - \omega(t)\right] y'(t)\,dt \\ &= \int_0^1 ((I-P)x)'(t)\left[((I-P)y)'(t) + (Py)'(t)\right] dt \\ &\quad - \int_0^1 \omega(t)\,y'(t)\,dt \\ &= -\int_0^1 f(t, x(t))\,y(t)\,dt \\ &\quad - (x(0), x(t_1), x(1))\,A(y(0), y(t_1), y(1))^T \\ &\quad - \int_0^1 y'(t) \int_{t_1}^t f(s, x(s))\,ds\,dt \\ &= -y(0)\int_0^{t_1} f(t, x(t))\,dt - y(1)\int_{t_1}^1 f(t, x(t))\,dt \\ &\quad - (x(0), x(t_1), x(1))\,A(y(0), y(t_1), y(1))^T. \end{aligned} \tag{14}$$

In particular, we can choose

$$y(t)=\begin{cases}\sin\dfrac{2n\pi t}{t_1} & 0\le t\le t_1,\\ 0 & t_1<t\le 1,\end{cases}\quad n=1,2,\ldots,$$
$$y(t)=\begin{cases}1-\cos\dfrac{2n\pi t}{t_1} & 0\le t\le t_1,\\ 0 & t_1<t\le 1,\end{cases}\quad n=1,2,\ldots, \tag{15}$$

so that

$$\int_0^{t_1}\left[((I-P)x)'(t)-\omega(t)\right]\sin\frac{2n\pi t}{t_1}dt = \int_0^{t_1}\left[((I-P)x)'(t)-\omega(t)\right]\cos\frac{2n\pi t}{t_1}dt=0,\quad n=1,2,\ldots. \tag{16}$$

The theorem of Fourier series implies that

$$((I-P)x)'(t)-\omega(t)=C \tag{17}$$

on $[0,t_1]$ for some $C\in R$. Integrating (17) over $[0,t_1]$, we obtain

$$Ct_1=-\int_0^{t_1}\omega(t)\,dt=\int_0^{t_1}tf(t,x(t))\,dt, \tag{18}$$

and hence

$$((I-P)x)'(t)-\omega(t)=\int_0^{t_1}\frac{t}{t_1}f(t,x(t))\,dt. \tag{19}$$

Similarly, setting

$$y(t)=\begin{cases}0 & 0\le t\le t_1,\\ \sin\dfrac{2n\pi(t-t_1)}{1-t_1} & t_1<t\le 1,\end{cases}\quad n=1,2,\ldots,$$
$$y(t)=\begin{cases}0 & 0\le t\le t_1,\\ 1-\cos\dfrac{2n\pi(t-t_1)}{1-t_1} & t_1<t\le 1,\end{cases}\quad n=1,2,\ldots, \tag{20}$$

in (14), we have

$$((I-P)x)'(t)-\omega(t)=-\int_{t_1}^{1}\frac{1-t}{1-t_1}f(t,x(t))\,dt \tag{21}$$

on $[t_1,1]$. Thus, (19) and (21) imply that $x(t)$ satisfies the equation $x''(t)=f(t,x(t))$ except at t_1.

Step 2. In this step, we prove that $x(t)$ satisfies the boundary value conditions $x'(0)=a_{11}x(0)+a_{12}x(t_1)+a_{13}x(1)$ and $x'(1)=-a_{13}x(0)-a_{23}x(t_1)-a_{33}x(1)$.

Set

$$y(t)=\begin{cases}\dfrac{t}{t_1}-1 & 0\le t\le t_1,\\ 0 & t_1<t\le 1.\end{cases} \tag{22}$$

Inserting (22) into (13), we have

$$\int_0^{t_1}\left(\frac{t}{t_1}-1\right)f(t,x(t))\,dt = \left(a_{11}+\frac{1}{t_1}\right)x(0)+\left(a_{12}-\frac{1}{t_1}\right)x(t_1)+a_{13}x(1). \tag{23}$$

It follows form (19) and (23) that

$$\begin{aligned}x'(0)&=((I-P)x)'(0)+(Px)'(0)\\&=\int_0^{t_1}\left(\frac{t}{t_1}-1\right)f(t,x(t))\,dt+\frac{x(t_1)-x(0)}{t_1}\\&=a_{11}x(0)+a_{12}x(t_1)+a_{13}x(1).\end{aligned} \tag{24}$$

Similarly, setting

$$y(t)=\begin{cases}0 & 0\le t\le t_1,\\ \dfrac{t-t_1}{1-t_1} & t_1<t\le 1,\end{cases} \tag{25}$$

equality (13) becomes

$$\int_{t_1}^{1}\frac{t-t_1}{1-t_1}f(t,x(t))\,dt+a_{13}x(0)+\left(a_{23}-\frac{1}{1-t_1}\right)x(t_1)+\left(a_{33}+\frac{1}{1-t_1}\right)x(1)=0, \tag{26}$$

and hence

$$\begin{aligned}x'(1)&=((I-P)x)'(1)+(Px)'(1)\\&=\int_{t_1}^{1}\frac{t-t_1}{1-t_1}f(t,x(t))\,dt+\frac{x(1)-x(t_1)}{1-t_1}\\&=-a_{13}x(0)-a_{23}x(t_1)-a_{33}x(1).\end{aligned} \tag{27}$$

Step 3. In this step, we prove that $x(t)$ satisfies the conditions $\Delta x'(t_1)=a_{12}x(0)+a_{22}x(t_1)+a_{23}x(1)$.

Inserting

$$y(t)=\begin{cases}\dfrac{t}{t_1} & 0\le t\le t_1,\\ \dfrac{1-t}{1-t_1} & t_1<t\le 1\end{cases} \tag{28}$$

into (13), we have

$$-\int_0^{t_1}\frac{t}{t_1}f(t,x(t))\,dt-\int_{t_1}^{1}\frac{1-t}{1-t_1}f(t,x(t))\,dt = \left(a_{12}-\frac{1}{t_1}\right)x(0)+\left(a_{22}+\frac{1}{t_1}+\frac{1}{1-t_1}\right)x(t_1)+\left(a_{23}-\frac{1}{1-t_1}\right)x(1). \tag{29}$$

It follows from (19) and (21) that

$$\begin{aligned}\Delta x'(t_1) &= \Delta((I-P)x)'(t_1) + \Delta(Px)'(t_1)\\ &= -\int_0^{t_1}\frac{t}{t_1}f(t,x(t))\,dt - \int_{t_1}^1 \frac{1-t}{1-t_1}f(t,x(t))\,dt\\ &\quad + \frac{x(1)-x(t_1)}{1-t_1} - \frac{x(t_1)-x(0)}{t_1}\\ &= a_{12}x(0) + a_{22}x(t_1) + a_{23}x(1).\end{aligned} \tag{30}$$

This completes the proof of Theorem 1. □

3. Solutions of Problem (1)

As applications of Theorem 1, we consider solutions of problem (1). Let k_1 and k_2 denote the minimum and maximum eigenvalue of the matrix A in (9). We have the following theorems.

Theorem 2. *Assume that f satisfies (A_1) and (A_2). Assume also that the following conditions hold:*

(A_3) $k_1 > 0$.

(A_4) *there is a positive constant l, with $l < 2$, and a positive function $c \in L^1(0,1)$ such that*

$$F(t,x) \ge -c(t)\left(1+|x|^l\right) \tag{31}$$

for almost every $t \in [0,1]$ and for all $x \in R$.

Then, problem (1) has a solution.

Proof. (A_3) implies that for each $x \in W$

$$\begin{aligned}&(x(0),x(t_1),x(1))A(x(0),x(t_1),x(1))^T\\ &\quad \ge k_1\left(x^2(0)+x^2(t_1)+x^2(1)\right).\end{aligned} \tag{32}$$

By (9), (32), and (A_4),

$$\begin{aligned}\phi(x) &\ge \frac{1}{2}\|(I-P)x\|^2 - \int_0^1 c(t)\left(|x(t)|^l+1\right)dt\\ &\quad + \frac{1}{2}k_1\left(x^2(0)+x^2(t_1)+x^2(1)\right)\\ &\ge \frac{1}{2}k_3\|x\|^2 - \int_0^1 c(t)\,dt\left(2\|x\|^l+1\right),\end{aligned} \tag{33}$$

where $k_3 = \min\{k_1, 1\}$. It follows that $\phi(x) \to +\infty$ as $\|x\| \to \infty$. Thus, ϕ has a critical point and problem (1) has a solution. □

Theorem 3. *Assume that (A_1), (A_2), and (A_3) are satisfied. Assume also that the following conditions hold:*

(A_4') *there is a positive function $c \in L^1(0,1)$ such that*

$$F(t,x) \ge -c(t)\left(1+x^2\right) \tag{34}$$

for almost every $t \in [0,1]$ and for all $x \in R$.

(A_5) $4\int_0^1 c(t)dt < k_3$, *where k_3 is in the proof of Theorem 2.*

Then, problem (1) has a solution.

Proof. This proof is similar to the proof of Theorem 2. □

Theorem 4. *Assume that (A_1)–(A_4) are satisfied. Assume also that the following condition holds:*

(A_6) *there are two positive constants d_1 and d_2 with $d_1 < \sqrt{6}d_2$ such that*

$$\begin{aligned}&\int_0^1 F(t,d_2)\,dt < -\frac{3k_2}{2}d_2^2,\\ &\int_0^1 \max_{|x|\le\sqrt{k_4}d_1}[-F(t,x)]\,dt < \frac{k_1}{4}d_1^2,\end{aligned} \tag{35}$$

where $k_4 = \max\{k_1, 1\}$.

Then, problem (1) has at least three solutions.

In order to prove this theorem, we need the following theorem (see Theorem 2.1 of [18]).

Theorem A. *Let X be a reflexive real Banach space; let $\Phi : X \to R$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*; and let $\Psi : X \to R$ be a sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exist $r \in R$ and $x_0, \overline{x} \in X$, with $\Phi(x_0) < r < \Phi(\overline{x})$ and $\Psi(x_0) = 0$, such that*

(i) $\sup_{\Phi(x)\le r}\Psi(x) < (r-\Phi(x_0))(\Psi(\overline{x})/(\Phi(\overline{x})-\Phi(x_0)))$

(ii) *for each* $\lambda \in \Lambda_r :=](\Phi(\overline{x})-\Phi(x_0))/\Psi(\overline{x}), (r-\Phi(x_0))/\sup_{\Phi(x)\le r}\Psi(x)[$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Proof of Theorem 4. Let $X = W$ and define

$$\begin{aligned}\Phi(x) &= \frac{1}{2}\|(I-P)x\|^2\\ &\quad + \frac{1}{2}(x(0),x(t_1),x(1))A(x(0),x(t_1),x(1))^T\end{aligned} \tag{36}$$

for each $x \in X$. Then, Φ is a sequentially weakly lower semicontinuous and continuously differentiable functional whose derivative is given by

$$\begin{aligned}\left(\Phi'(x),y\right) &= ((I-P)x,(I-P)y)\\ &\quad + (x(0),x(t_1),x(1))A\left(y(0),y(t_1),y(1)\right)^T\end{aligned} \tag{37}$$

for all $y \in X$. It follows from (36) that for each $x \in W$,

$$\Phi(x) \geq \frac{1}{2}\|(I-P)x\|^2 + \frac{1}{2}k_1\left[x^2(0) + x^2(t_1) + x^2(1)\right] \geq \frac{1}{2}k_3\|x\|^2, \tag{38}$$

so that Φ is coercive. Because $I - \Phi'$ is a compact operator, Φ' has the continuous inverse if and only if 0 is not the eigenvalues of Φ'. If 0 is the eigenvalues of Φ' and $\beta(t)$ is a eigenvector of Φ' associated with the eigenvalue 0, then (37) implies that

$$\begin{aligned} 0 &= \left(\Phi'(\beta), \beta\right) \\ &= \|(I-P)\beta\|^2 \\ &\quad + (\beta(0), \beta(t_1), \beta(1)) A(\beta(0), \beta(t_1), \beta(1))^T \\ &\geq k_3\|\beta\|^2 > 0. \end{aligned} \tag{39}$$

This is a contradiction, and hence Φ' has the continuous inverse. Set $\Psi(x) = -\int_0^1 F(t, x(t))dt$ for $x \in X$. Then, Ψ is a sequentially weakly upper semicontinuous and continuously differentiable functional whose derivative is compact.

Setting $x_0 = 0$ and $\overline{x} = d_2$ for all $t \in [0,1]$, then $\Phi(x_0) = \Psi(x_0) = 0$ and

$$\frac{3}{2}k_2d_2^2 \geq \Phi(\overline{x}) = \frac{1}{2}(d_2, d_2, d_2) A(d_2, d_2, d_2)^T \geq \frac{3}{2}k_1d_2^2, \tag{40}$$

since $(I-P)\overline{x} = 0$. Setting $r = (1/4)k_1d_1^2$, by $d_1 < \sqrt{6}d_2$ and (40), we obtain

$$\Phi(x_0) = 0 < r < \frac{3}{2}k_1d_2^2 \leq \Phi(\overline{x}). \tag{41}$$

By (38), $\Phi(x) \leq r$ implies that $|x(t)| \leq \sqrt{k_4}d_1$ for every $t \in [0, 2\tau]$ since $k_1 = k_3k_4$. From (A_6) and (40), we have

$$\begin{aligned} \sup_{\Phi(x)\leq r} \Psi(x) &\leq \int_0^1 \max_{|x|\leq\sqrt{k_4}d_1} [-F(t,x)]\,dt \\ &< \frac{k_1}{4}d_1^2 < \frac{k_1d_1^2\int_0^1[-F(t,d_2)]\,dt}{6k_2d_2^2} \\ &\leq (r - \Phi(0))\frac{\Psi(\overline{x})}{\Phi(\overline{x}) - \Phi(0)}, \end{aligned} \tag{42}$$

and (i) in Theorem A holds.

Since

$$\begin{aligned} \frac{\Phi(\overline{x}) - \Phi(x_0)}{\Psi(\overline{x})} &\leq \frac{(3/2)k_2d_2^2}{\int_0^1[-F(t,d_2)]\,dt} < 1 \\ &< \frac{(1/4)k_1d_1^2}{\int_0^1 \max_{|x|\leq\sqrt{k_4}d_1}[-F(t,x)]\,dt} \\ &\leq \frac{r - \Phi(x_0)}{\sup_{\Phi(x)\leq r}\Psi(x)}, \end{aligned} \tag{43}$$

we can take $\lambda = 1$ in Theorem A. Therefore, it is easy to show that

$$(\Phi - \Psi)(x) \geq \frac{1}{2}k_3\|x\|^2 - \int_0^1 c(t)\,dt\left(2\|x\|^l + 1\right), \tag{44}$$

so that $\Phi - \Psi$ is coercive. Using Theorem A, problem (1) has at least three solutions. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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