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Research Article

Invariant Surfaces under Hyperbolic Translations in Hyperbolic Space

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We consider hyperbolic rotation (G_0), hyperbolic translation (G_1), and horocyclic rotation (G_2) groups in $\mathbb{H}^3$, which is called Minkowski model of hyperbolic space. Then, we investigate extrinsic differential geometry of invariant surfaces under subgroups of G_0 in $\mathbb{H}^3$. Also, we give explicit parametrization of these invariant surfaces with respect to constant hyperbolic curvature of profile curves. Finally, we obtain some corollaries for flat and minimal invariant surfaces which are associated with de Sitter and hyperbolic shape operator in $\mathbb{H}^3$.

1. Introduction

Hyperbolic space has five analytic models, which are isometrically equivalent to each other [1, 2]. In this study, we choose Minkowski model of hyperbolic space which is denoted by $\mathbb{H}^3$. In a different point of view, we may consider invariant surface as rotational surface. In this sense, rotational surfaces in different ambient spaces were studied by many authors. For instance, in [3], Carmo and Dajczer define rotational hypersurfaces with constant mean curvature (cmc) in hyperbolic n-space. They also give a local parametrization of this surface in terms of the cmc under some special conditions. In [4], Mori studied elliptic, spherical, and parabolic type rotational surfaces with cmc in $\mathbb{H}^3$. In [5], the total classification of the timelike and spacelike hyperbolic rotation surfaces is given in terms of cmc in 3-dimensional de Sitter space $\mathbb{S}^3_1$. As a general form, explicit parametrizations of rotational surfaces with cmc are given in Minkowski n-space by [6].

This paper is organized as follows. In Section 2, we give briefly the notions of H-point, H-line, H-plane, and H-distance in hyperbolic geometry of $\mathbb{H}^3$. Throughout this work, the prefix “H-” is used is the sense of belonging to hyperbolic space. It is well known that H-isometry is a map which is preserved H-distance in $\mathbb{H}^3$. The set of H-isometries is a group which is identified with restriction of isometries of Minkowski 4-space $\mathbb{R}^4_1$ to $\mathbb{H}^3$. Let the group of H-isometries be denoted by G in $\mathbb{H}^3$. We consider subgroups G_0, G_1, and G_2 of G with respect to leaving fixed timelike, spacelike, and lightlike planes of $\mathbb{R}^4_1$, respectively. Then, we give the notions of H-rotation, H-translation, and horocyclic rotation which are one-parameter actions of G in $\mathbb{H}^3$. Moreover, we obtain some properties of H-isometries. There exist three kinds of totally umbilical surfaces which are called H-sphere, equidistant surface, and horosphere in $\mathbb{H}^3$. We obtain a classification of H-isometries by the subgroups G_0, G_1, and G_2 with respect to leaving fixed equidistant surfaces, H-spheres, and horospheres in $\mathbb{H}^3$, respectively. In Section 3, we give the basic theory of extrinsic differential geometry of curves and surfaces in $\mathbb{H}^3$. In Section 4, we investigate surfaces which are invariant under a subgroup of H-translations in $\mathbb{H}^3$. Moreover, in the sense of de Sitter and hyperbolic shape operator in $\mathbb{H}^3$, we study extrinsic differential geometry of these invariant surfaces by using notations in [7, 8]. We give a relation between one of the principal curvatures of the invariant surface and hyperbolic curvature of profile curve of the invariant surface in $\mathbb{H}^3$. In a different viewpoint, we obtain explicit parametrization of some invariant surfaces in terms of constant hyperbolic curvature of profile curve. Moreover, we give some geometric results with respect to constant hyperbolic curvature of profile curve for flat and minimal invariant surfaces in $\mathbb{H}^3$. Finally, we give a classification theorem for the totally umbilical invariant surfaces in $\mathbb{H}^3$.

2. Isometries of $\mathbb{H}^3$

In [9], Reynold give a brief introduction to hyperbolic geometry of hyperbolic plane $\mathbb{H}^2$. Also, he described explicit descriptions of the hyperbolic metric and the isometries of the hyperbolic plane. In this section, we consider hyperbolic geometry in $\mathbb{H}^3$. We especially determine isometry groups of $\mathbb{H}^3$ with respect to causal character of hyperplanes of $\mathbb{R}^4_1$; then, these isometry groups are classified in terms of leaving those totally umbilic surfaces of $\mathbb{H}^3$ fixed.

Let $\mathbb{R}^4_1$ denote the 4-dimensional Minkowski space, that is, the real vector space $\mathbb{R}^4$ endowed with the scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_0 y_0 + \sum_{i=1}^{3} x_i y_i \tag{1}$$

for all $\mathbf{x} = (x_0, x_1, x_2, x_3)$, $\mathbf{y} = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$. Let $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be pseudo-orthonormal basis for $\mathbb{R}^4_1$. Then, $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}\varepsilon_j$ for signatures $\varepsilon_0 = -1$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$. The function

$$q : \mathbb{R}^4_1 \longrightarrow \mathbb{R}, \qquad q(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle \tag{2}$$

is called the associated *quadratic form* of $\langle \cdot, \cdot \rangle$.

A vector $\mathbf{v} \in \mathbb{R}^4_1$ is called *spacelike*, *timelike,* and *lightlike* if $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ (or $\mathbf{v} = 0$), $\langle \mathbf{v}, \mathbf{v} \rangle < 0$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$, respectively. The *Lorentzian norm* of a vector $\mathbf{v}$ is defined by $\|\mathbf{v}\| = \sqrt{|\langle \mathbf{v}, \mathbf{v} \rangle|}$.

The sets

$$\begin{aligned} \mathbb{H}^3 &= \left\{ \mathbf{x} \in \mathbb{R}^4_1 \ \langle \mathbf{x}, \mathbf{x} \rangle = -1, x_0 \geq 1 \right\}, \\ \mathbb{S}^3_1 &= \left\{ \mathbf{x} \in \mathbb{R}^4_1 \ \langle \mathbf{x}, \mathbf{x} \rangle = 1 \right\}, \\ \mathrm{LC}_+ &= \left\{ \mathbf{x} \in \mathbb{R}^4_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0, x_0 > 0 \right\} \end{aligned} \tag{3}$$

are called *Minkowski model of hyperbolic space, de Sitter space,* and *future light cone*, respectively.

Let P be a vector subspace of $\mathbb{R}^4_1$. Then, P is said to be timelike, spacelike, and lightlike if and only if P contains a timelike vector and every nonzero vector in P is spacelike otherwise, respectively.

Now, we give basic notions for hyperbolic geometry in $\mathbb{H}^3$. From now on, we use the prefix "H-" instead of "hyperbolic" for brevity.

An *H-point* is intersection $U_0 \cap \mathbb{H}^3$ such that U_0 is 1-dimensional timelike subspace of $\mathbb{R}^4_1$ and is called A_{U_0}. An *H-line* is intersection $U_1 \cap \mathbb{H}^3$ such that U_1 is 2-dimensional timelike subspace of $\mathbb{R}^4_1$ and is called l_{U_1}. An *H-plane* is intersection $U_2 \cap \mathbb{H}^3$ such that U_2 is 3-dimensional timelike subspace of $\mathbb{R}^4_1$ and is called D_{U_2}.

H-coordinate axes l_{0j} are denoted by intersections $l_{0j} = V_{0j} \cap \mathbb{H}^3$ such that $V_{0j} = \mathrm{Sp}\{\mathbf{e}_0, \mathbf{e}_j\}$ for $j = 1, 2, 3$. *H-coordinate planes* D_{ij} are denoted by intersections $D_{ij} = W_{0ij} \cap \mathbb{H}^3$ such that $W_{0ij} = \mathrm{Sp}\{\mathbf{e}_0, \mathbf{e}_i, \mathbf{e}_j\}$ for $i, j = 1, 2, 3$. *H-upper (H-lower) half-spaces* of D_{ij} are defined by intersections $\mathbb{H}^3$ and upper (lower) half-space of W_{0ij}.

A *hyperplane* in $\mathbb{R}^4_1$ is defined by $\mathrm{HP}(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}^4_1 \mid \langle \mathbf{x}, \mathbf{v} \rangle = c\}$ for a pseudo-normal $\mathbf{v} \in \mathbb{R}^4_1$ and a real number c. If $\mathbf{v}$ is spacelike, timelike, or lightlike, $\mathrm{HP}(\mathbf{v}, c)$ is called timelike, spacelike, or lightlike, respectively.

Three kinds of totally umbilic surfaces have $\mathbb{H}^3$ which are given by intersections of $\mathbb{H}^3$ and hyperplanes $\mathrm{HP}(\mathbf{v}, c)$ in $\mathbb{R}^4_1$. A surface $\mathrm{HP}(\mathbf{v}, c) \cap \mathbb{H}^3$ is called *H-sphere*, *equidistant surface,* and *horosphere* if $\mathrm{HP}(\mathbf{v}, c)$ is spacelike, timelike, and lightlike, respectively.

We now give the existence and uniqueness of any H-line or H-plane in $\mathbb{H}^3$. Any given two distinct points determine unique 2-plane through origin in $\mathbb{R}^4_1$ and three distinct points determine unique 3-plane through origin in $\mathbb{R}^4_1$. So the following propositions are clear.

Proposition 1. *Any given two distinct H-points lie on a unique H-line in* $\mathbb{H}^3$.

Proposition 2. *Any given three distinct H-points lie on a unique H-plane in* $\mathbb{H}^3$.

Also, we say that *H-line segments* $l_{\overline{AB}}$, *H-ray* $l_{\overrightarrow{AB}}$ are determined by two different H-points A and B in natural way.

Definition 3. Let $\gamma : [a, b] \subset \mathbb{R} \rightarrow l_{\overline{AB}} \subset \mathbb{H}^3$ be parametrization of $l_{\overline{AB}}$. Then, *H-length* of $l_{\overline{AB}}$ is given by

$$d_{\mathrm{H}}(A, B) = \int_a^b \left\| \gamma'(t) \right\| dt. \tag{4}$$

If we take any hyperbolic space curve instead of $l_{\overline{AB}}$ in Definition 3, then *H-arc length* of any hyperbolic space curve is calculated by formula (4) in the same way. Moreover, *H-distance* between H-points A and B is given by

$$d_{\mathrm{H}}(A, B) = -\operatorname{arccosh}(\langle A, B \rangle). \tag{5}$$

Let $T : \mathbb{R}^4_1 \rightarrow \mathbb{R}^4_1$ be a linear transformation. Then T is called *linear isometry* (with respect to q) if it satisfies the following equation:

$$q(T(x)) = q(x). \tag{6}$$

Let matrix form of linear transformation T be denoted by $\mathbf{T} = [t_{ij}]$, $0 \leqslant i, j \leqslant 3$ with respect to pseudo-orthonormal basis $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The set of all linear isometries of $\mathbb{R}^4_1$ is a group under matrix multiplication and it is denoted by

$$\mathrm{O}_1(4) = \left\{ \mathbf{T} \in \mathrm{GL}(4, \mathbb{R}) \mid \mathbf{T}^t \mathbf{J}_0 \mathbf{T} = \mathbf{J}_0 \right\}, \tag{7}$$

where signature matrix $\mathbf{J}_0 = \mathrm{diag}(-1, 1, 1, 1)$. It is also called *semiorthogonal* group of $\mathbb{R}^4_1$ and so

$$\det(\mathbf{T}) = \pm 1. \tag{8}$$

The subgroup $SO_1(4) = \{\mathbf{T} \in O_1(4) \mid \det(\mathbf{T}) = 1\}$ is called *special semiorthogonal* group. Let block matrix form of $\mathbf{T} \in O_1(4)$ be $\mathbf{T} = \left[\begin{smallmatrix} \mathbf{T}_t & b \\ c & \mathbf{T}_s \end{smallmatrix}\right]$. Then, $\mathbf{T}_t$ and $\mathbf{T}_s$ are called timelike and spacelike part of $\mathbf{T}$, respectively.

Definition 4. (i) If $\det(\mathbf{T}_t) > 0$ $(\det(\mathbf{T}_t) < 0)$, then $\mathbf{T}$ preserves (reverses) time orientation.

(ii) If $\det(\mathbf{T}_s) > 0$ $(\det(\mathbf{T}_s) < 0)$, then $\mathbf{T}$ preserves (reverses) space orientation.

Thus, $O_1(4)$ is decomposed into four disjoint sets indexed by the signs of $\det(\mathbf{T}_t)$ and $\det(\mathbf{T}_s)$ in that order. They are called $O_1^{++}(4), O_1^{+-}(4), O_1^{-+}(4)$, and $O_1^{--}(4)$. We define the group

$$G = \left\{T \in O_1(4) \mid T_{\mathbb{H}^3} : \mathbb{H}^3 \longrightarrow \mathbb{H}^3\right\}. \tag{9}$$

Elements of G preserve H-distance in $\mathbb{H}^3$. It is clear that

$$d_{\mathrm{H}}(T(A), T(B)) = d_{\mathrm{H}}(A, B) \tag{10}$$

for every $T \in G$ and $A, B \in \mathbb{H}^3$. Thus, we are ready to give the following definition.

Definition 5. Every element of G is an *H-isometry* in $\mathbb{H}^3$.

Thus, we say that G is union of subgroup which preserves time orientation of $O_1(4)$. That is, $G = O_1^{++}(4) \cup O_1^{+-}(4)$.

We consider G_0, G_1, and G_2 subgroups of G which leave fixed timelike, spacelike, and lightlike planes of $\mathbb{R}_1^4$, respectively. Let matrix representation of H-isometries be $\mathbf{T} = [t_{ij}]$, $0 \leqslant i, j \leqslant 3$ and let H-isometries $\mathbf{J}_1$, $\mathbf{J}_2$, and $\mathbf{J}_3$ be denoted by $\mathbf{J}_1 = \mathrm{diag}(1, -1, 1, 1)$, $\mathbf{J}_2 = \mathrm{diag}(1, 1, -1, 1)$, and $\mathbf{J}_3 = \mathrm{diag}(1, 1, 1, -1)$, respectively.

We suppose that $T \in G_0$. Then, T leaves fixed timelike planes $V_{0j} = \mathrm{Sp}\{\mathbf{e}_0, \mathbf{e}_j\}$ for $j = 1, 2, 3$ of $\mathbb{R}_1^4$. So that $T(V_{0j}) = V_{0j}$.

If $T(V_{01}) = V_{01}$ for $j = 1$, then entries of matrix $\mathbf{T}$ must be $t_{00} = 1$, $t_{10} = 0$, $t_{20} = 0$, and $t_{30} = 0$ and $t_{01} = 0$, $t_{11} = 1$, $t_{21} = 0$, and $t_{31} = 0$. By using (7) and (8), we have $t_{03} = 0$, $t_{02} = 0$, $t_{12} = 0$, and $t_{13} = 0$ and the following equation system:

$$\begin{gathered} t_{22}^2 + t_{32}^2 = 1, \qquad t_{23}^2 + t_{33}^2 = 1, \\ t_{22}t_{23} + t_{32}t_{33} = 0, \\ t_{22}t_{33} - t_{23}t_{32} = \pm 1. \end{gathered} \tag{11}$$

If the above system is solved under time orientation preserving and sign cases, then general form of H-isometries that leave fixed timelike plane V_{01} of $\mathbb{R}_1^4$ is given by

$$\mathbf{T}_{01} = \mathbf{R}_\theta^{01}\mathbf{J}_1^m\mathbf{J}_3^n, \quad m, n = 0, 1 \tag{12}$$

such that

$$\mathbf{R}_\theta^{01} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{13}$$

for all $\theta \in \mathbb{R}$. In other cases, if $T(V_{02}) = V_{02}$ and $T(V_{03}) = V_{03}$, then general forms of H-isometries that leave fixed timelike planes V_{02} and V_{03} of $\mathbb{R}_1^4$ are given by

$$\begin{gathered} \mathbf{T}_{02} = \mathbf{R}_\theta^{02}\mathbf{J}_1^m\mathbf{J}_2^n, \\ \mathbf{T}_{03} = \mathbf{R}_\theta^{03}\mathbf{J}_2^m\mathbf{J}_3^n, \\ m, n = 0, 1 \end{gathered} \tag{14}$$

such that

$$\begin{gathered} \mathbf{R}_\theta^{02} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix}, \\ \mathbf{R}_\theta^{03} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{gathered} \tag{15}$$

for all $\theta \in \mathbb{R}$, respectively. Thus, we say that the group G_0 is union of disjoint subgroups of G_0^+ and G_0^- such that

$$\begin{aligned} G_0^+ &= \left\{\mathbf{R}_{m_1\theta_1}^{01}\mathbf{R}_{m_2\theta_2}^{02}\mathbf{R}_{m_3\theta_3}^{03} \mid m_i \in \mathbb{Z}, \theta_j \in \mathbb{R}\right\}, \\ G_0^- &= \left\{\mathbf{R}_{m_1\theta_1}^{01}\mathbf{R}_{m_2\theta_2}^{02}\mathbf{R}_{m_3\theta_3}^{03}\mathbf{J}_1^{m_4}\mathbf{J}_2^{m_5}\mathbf{J}_3^{m_6} \mid \right. \\ &\qquad \left. m_i \in \mathbb{Z}, \theta_j \in \mathbb{R}, m_4 + m_5 + m_6 \equiv 1 \pmod 2\right\}; \end{aligned} \tag{16}$$

that is, $G_0 = G_0^+ \cup G_0^-$.

We suppose that $T \in G_1$. Then, T leaves fixed spacelike planes $V_{ij} = \mathrm{Sp}\{\mathbf{e}_i, \mathbf{e}_j\}$ for $i, j = 1, 2, 3$ of $\mathbb{R}_1^4$. That is, $T(V_{ij}) = V_{ij}$.

If $T(V_{23}) = V_{23}$ for $i = 2$, $j = 3$, then entries of $\mathbf{T}$ must be $t_{02} = 0$, $t_{12} = 0$, $t_{22} = 1$, and $t_{32} = 0$ and $t_{03} = 0$, $t_{13} = 0$, $t_{23} = 0$, and $t_{33} = 1$. By using (7) and (8), we have $t_{20} = 0$, $t_{30} = 0$, $t_{21} = 0$, and $t_{31} = 0$ and the following equation system:

$$\begin{gathered} -t_{00}^2 + t_{10}^2 = -1, \qquad -t_{01}^2 + t_{11}^2 = 1, \\ -t_{00}t_{01} + t_{10}t_{11} = 0, \qquad \left(t_{00}t_{11} - t_{01}t_{10}\right)t_{22}t_{33} = \pm 1. \end{gathered} \tag{17}$$

If the above system is solved under time orientation preserving and sign cases, then general form of H-isometries that leave fixed spacelike plane V_{23} of $\mathbb{R}_1^4$ is given by

$$\mathbf{T}_{23} = \mathbf{L}_s^{01}\mathbf{J}_1^m\mathbf{J}_2^n\mathbf{J}_3^k, \quad m, n, k = 0, 1 \tag{18}$$

such that

$$\mathbf{L}_s^{01} = \begin{bmatrix} \cosh s & \sinh s & 0 & 0 \\ \sinh s & \cosh s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

for all $s \in \mathbb{R}$. In other cases, if $T(V_{13}) = V_{13}$ and $T(V_{12}) = V_{12}$, then general forms of H-isometries that leave fixed spacelike planes V_{13} and V_{12} of $\mathbb{R}_1^4$ are given by

$$\begin{gathered} \mathbf{T}_{13} = \mathbf{L}_s^{02}\mathbf{J}_1^m\mathbf{J}_2^n\mathbf{J}_3^k, \\ \mathbf{T}_{12} = \mathbf{L}_s^{03}\mathbf{J}_1^m\mathbf{J}_2^n\mathbf{J}_3^k, \\ m, n, k = 0, 1 \end{gathered} \tag{20}$$

such that

$$\mathbf{L}_s^{02} = \begin{bmatrix} \cosh s & 0 & \sinh s & 0 \\ 0 & 1 & 0 & 0 \\ \sinh s & 0 & \cosh s & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{21}$$

$$\mathbf{L}_s^{03} = \begin{bmatrix} \cosh s & 0 & 0 & \sinh s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh s & 0 & 0 & \cosh s \end{bmatrix} \tag{22}$$

for all $s \in \mathbb{R}$, respectively. Thus, we say that the group G_1 is union of disjoint subgroups of G_1^+ and G_1^- such that

$$\begin{aligned} G_1^+ = \{ & \mathbf{L}_{m_1 s_1}^{01}\mathbf{L}_{m_2 s_2}^{02}\mathbf{L}_{m_3 s_3}^{03}\mathbf{J}_1^{m_4}\mathbf{J}_2^{m_5}\mathbf{J}_3^{m_6} \mid \\ & m_i \in \mathbb{Z}, s_j \in \mathbb{R}, m_4 + m_5 + m_6 \equiv 0 \pmod 2 \}, \end{aligned} \tag{23}$$

$$\begin{aligned} G_1^- = \{ & \mathbf{L}_{m_1 s_1}^{01}\mathbf{L}_{m_2 s_2}^{02}\mathbf{L}_{m_3 s_3}^{03}\mathbf{J}_1^{m_4}\mathbf{J}_2^{m_5}\mathbf{J}_3^{m_6} \mid \\ & m_i \in \mathbb{Z}, s_j \in \mathbb{R}, m_4 + m_5 + m_6 \equiv 1 \pmod 2 \}; \end{aligned} \tag{24}$$

that is, $G_1 = G_1^+ \cup G_1^-$.

We suppose that $T \in G_2$. Then, T leaves fixed lightlike planes $\mathbb{D}_{ij} = \mathrm{Sp}\{\mathbf{e}_0 + \mathbf{e}_i, \mathbf{e}_j\}$ for $j = 1, 2, 3$ of $\mathbb{R}_1^4$. So that $T(\mathbb{D}_{ij}) = \mathbb{D}_{ij}$.

If $T(\mathbb{D}_{12}) = \mathbb{D}_{12}$ for $i = 1$, $j = 2$, then entries of matrix $\mathbf{T}$ must be

$$\begin{gathered} t_{00} + t_{01} = 1, \qquad t_{10} + t_{11} = 1, \qquad t_{20} + t_{21} = 0, \\ t_{30} + t_{31} = 0, \qquad t_{02} = t_{12} = t_{32} = 0, \qquad t_{22} = 1. \end{gathered} \tag{25}$$

By using (7) and (8), we obtain the following equation system:

$$\begin{gathered} 1 - 2t_{00} + t_{03}^2 = -1, \qquad 1 - 2t_{10} + t_{13}^2 = 1, \\ t_{23}^2 = 0, \qquad t_{33}^2 = 1, \\ 1 - t_{00} - t_{10} + t_{03}t_{13} = 0, \qquad -t_{30} + t_{03}t_{33} = 0, \\ -t_{20} + t_{13}t_{23} = 0, \qquad -t_{30} + t_{13}t_{33} = 0, \\ t_{23}t_{33} = 0, \qquad -t_{03}t_{30} + t_{13}t_{30} + (t_{00} - t_{10})\, t_{33} = \pm 1. \end{gathered} \tag{26}$$

If the above system is solved under time orientation preserving and sign cases, then general form of H-isometries that leaves fixed lightlike plane $\mathbb{D}_{12}$ of $\mathbb{R}_1^4$ is given by

$$\mathbf{T}_{012} = \mathbf{H}_\lambda^{012}\mathbf{J}_2^m\mathbf{J}_3^n, \quad m, n = 0, 1 \tag{27}$$

such that

$$\mathbf{H}_\lambda^{012} = \begin{bmatrix} 1 + \dfrac{\lambda^2}{2} & -\dfrac{\lambda^2}{2} & 0 & \lambda \\ \dfrac{\lambda^2}{2} & 1 - \dfrac{\lambda^2}{2} & 0 & \lambda \\ 0 & 0 & 1 & 0 \\ \lambda & -\lambda & 0 & 1 \end{bmatrix} \tag{28}$$

for all $\lambda \in \mathbb{R}$. In other cases, we apply similar method. Hence, if H-isometry that leaves fixed lightlike plane $\mathbb{D}_{ij}$ of $\mathbb{R}_1^4$ is denoted by $\mathbf{H}_\lambda^{0ij}$, then we have the following H-isometries:

$$\begin{gathered} \mathbf{H}_\lambda^{013} = \begin{bmatrix} 1 + \dfrac{\lambda^2}{2} & -\dfrac{\lambda^2}{2} & \lambda & 0 \\ \dfrac{\lambda^2}{2} & 1 - \dfrac{\lambda^2}{2} & \lambda & 0 \\ \lambda & -\lambda & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ \mathbf{H}_\lambda^{021} = \begin{bmatrix} 1 + \dfrac{\lambda^2}{2} & 0 & -\dfrac{\lambda^2}{2} & \lambda \\ 0 & 1 & 0 & 0 \\ \dfrac{\lambda^2}{2} & 0 & 1 - \dfrac{\lambda^2}{2} & \lambda \\ \lambda & 0 & -\lambda & 1 \end{bmatrix}, \\ \mathbf{H}_\lambda^{023} = \begin{bmatrix} 1 + \dfrac{\lambda^2}{2} & \lambda & -\dfrac{\lambda^2}{2} & 0 \\ \lambda & 1 & -\lambda & 0 \\ \dfrac{\lambda^2}{2} & \lambda & 1 - \dfrac{\lambda^2}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ \mathbf{H}_\lambda^{031} = \begin{bmatrix} 1 + \dfrac{\lambda^2}{2} & 0 & \lambda & -\dfrac{\lambda^2}{2} \\ 0 & 1 & 0 & 0 \\ \lambda & 0 & 1 & -\lambda \\ \dfrac{\lambda^2}{2} & 0 & \lambda & 1 - \dfrac{\lambda^2}{2} \end{bmatrix}, \\ \mathbf{H}_\lambda^{032} = \begin{bmatrix} 1 + \dfrac{\lambda^2}{2} & \lambda & 0 & -\dfrac{\lambda^2}{2} \\ \lambda & 1 & 0 & -\lambda \\ 0 & 0 & 1 & 0 \\ \dfrac{\lambda^2}{2} & \lambda & 0 & 1 - \dfrac{\lambda^2}{2} \end{bmatrix}, \end{gathered} \tag{29}$$

and we obtained the following general forms:

$$\begin{gathered} \mathbf{T}_{013} = \mathbf{H}_\lambda^{013}\mathbf{J}_2^m\mathbf{J}_3^n \\ \mathbf{T}_{021} = \mathbf{H}_\lambda^{021}\mathbf{J}_1^m\mathbf{J}_3^n \\ \mathbf{T}_{023} = \mathbf{H}_\lambda^{023}\mathbf{J}_1^m\mathbf{J}_3^n \\ \mathbf{T}_{031} = \mathbf{H}_\lambda^{031}\mathbf{J}_1^m\mathbf{J}_2^n \\ \mathbf{T}_{032} = \mathbf{H}_\lambda^{032}\mathbf{J}_1^m\mathbf{J}_2^n, \\ m, n = 0, 1. \end{gathered} \tag{30}$$

So, we say that the group G_2 is union of disjoint subgroups of G_2^+ and G_2^- such that

$$G_2^+ = \{\mathbf{H}^{012}_{m_1\lambda_1}\mathbf{H}^{013}_{m_2\lambda_2}\mathbf{H}^{021}_{m_3\lambda_3}\mathbf{H}^{023}_{m_4\lambda_4}\mathbf{H}^{031}_{m_5\lambda_5}\mathbf{H}^{032}_{m_6\lambda_6}\mathbf{J}_1^{m_7}\mathbf{J}_2^{m_8}\mathbf{J}_3^{m_9} \mid m_i \in \mathbb{Z}, \lambda_j \in \mathbb{R}, m_7 + m_8 + m_9 \equiv 0 \pmod 2\},$$
$$G_2^- = \{\mathbf{H}^{012}_{m_1\lambda_1}\mathbf{H}^{013}_{m_2\lambda_2}\mathbf{H}^{021}_{m_3\lambda_3}\mathbf{H}^{023}_{m_4\lambda_4}\mathbf{H}^{031}_{m_5\lambda_5}\mathbf{H}^{032}_{m_6\lambda_6}\mathbf{J}_1^{m_7}\mathbf{J}_2^{m_8}\mathbf{J}_3^{m_9} \mid m_i \in \mathbb{Z}, \lambda_j \in \mathbb{R}, m_7 + m_8 + m_9 \equiv 1 \pmod 2\}; \quad (31)$$

that is, $G_2 = G_2^+ \cup G_2^-$.

Hence, it is clear that

$$G = G_0 \cup G_1 \cup G_2,$$
$$G_0 \cap G_1 \cap G_2 = \{\mathbf{I}_4, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3, \mathbf{J}_1\mathbf{J}_2, \mathbf{J}_1\mathbf{J}_3, \mathbf{J}_2\mathbf{J}_3, \mathbf{J}_1\mathbf{J}_2\mathbf{J}_3\}. \quad (32)$$

However, we see that easily from matrix multiplication

$$\mathbf{R}^{0j}_{\theta_1}\mathbf{R}^{0j}_{\theta_2} = \mathbf{R}^{0j}_{\theta_1+\theta_2},$$
$$\mathbf{L}^{0j}_{s_1}\mathbf{L}^{0j}_{s_2} = \mathbf{L}^{0j}_{s_1+s_2}, \quad (33)$$
$$\mathbf{H}^{0ij}_{\lambda_1}\mathbf{H}^{0ij}_{\lambda_2} = \mathbf{H}^{0ij}_{\lambda_1+\lambda_2},$$

for any $s_1, s_2, \theta_1, \theta_2, \lambda_1, \lambda_2 \in \mathbb{R}$.

Let parametrization of H-coordinate axes l_{01}, l_{02}, and l_{03} be

$$l_{01} : [0,\infty) \longrightarrow H^3, \quad l_{01}(r) = (\cosh r, \sinh r, 0, 0),$$
$$l_{02} : [0,\infty) \longrightarrow H^3, \quad l_{02}(r) = (\cosh r, 0, \sinh r, 0), \quad (34)$$
$$l_{03} : [0,\infty) \longrightarrow H^3, \quad l_{03}(r) = (\cosh r, 0, 0, \sinh r)$$

and let their matrix forms be

$$\mathbf{l}^{01}_r = \begin{bmatrix}\cosh r\\ \sinh r\\ 0\\ 0\end{bmatrix}, \qquad \mathbf{l}^{02}_r = \begin{bmatrix}\cosh r\\ 0\\ \sinh r\\ 0\end{bmatrix}, \qquad \mathbf{l}^{03}_r = \begin{bmatrix}\cosh r\\ 0\\ 0\\ \sinh r\end{bmatrix}, \quad (35)$$

respectively. After applying suitable H-isometry to H-coordinate axis $\mathbf{l}^{0j}_r$ for $j = 1, 2, 3$, we obtain the following different parametrizations:

$$\mathbf{H}_{(r,\theta_1,\theta_2)} = \mathbf{R}^{01}_{\theta_2}\mathbf{R}^{03}_{\theta_1}\mathbf{l}^{01}_r,$$
$$\widetilde{\mathbf{H}}_{(r,\theta_1,\theta_2)} = \mathbf{R}^{02}_{\theta_2}\mathbf{R}^{01}_{\theta_1}\mathbf{l}^{02}_r, \quad (36)$$
$$\widetilde{\widetilde{\mathbf{H}}}_{(r,\theta_1,\theta_2)} = \mathbf{R}^{03}_{\theta_2}\mathbf{R}^{02}_{\theta_1}\mathbf{l}^{03}_r$$

for hyperbolic polar coordinates $r > 0$, $\theta_1 \in [0,\pi]$ and $\theta_2 \in [0, 2\pi]$. Thus, we see roles of H-isometries $\mathbf{R}^{0j}_{\theta_i}$ and $\mathbf{L}^{0k}_s$ in $\mathbb{H}^3$ from equations

$$\mathbf{R}^{01}_{\varphi}\mathbf{H}_{(r,\theta_1,\theta_2)} = \mathbf{H}_{(r,\theta_1,\varphi+\theta_2)},$$
$$\mathbf{R}^{02}_{\varphi}\widetilde{\mathbf{H}}_{(r,\theta_1,\theta_2)} = \widetilde{\mathbf{H}}_{(r,\theta_1,\varphi+\theta_2)},$$
$$\mathbf{R}^{03}_{\varphi}\widetilde{\widetilde{\mathbf{H}}}_{(r,\theta_1,\theta_2)} = \widetilde{\widetilde{\mathbf{H}}}_{(r,\theta_1,\varphi+\theta_2)},$$
$$\mathbf{R}^{03}_{\theta}\mathbf{H}_{(r,\theta_1,0)} = \mathbf{H}_{(r,\theta+\theta_1,0)},$$
$$\mathbf{R}^{01}_{\phi}\widetilde{\mathbf{H}}_{(r,\theta_1,0)} = \widetilde{\mathbf{H}}_{(r,\phi+\theta_1,0)}, \quad (37)$$
$$\mathbf{R}^{02}_{\phi}\widetilde{\widetilde{\mathbf{H}}}_{(r,\theta_1,0)} = \widetilde{\widetilde{\mathbf{H}}}_{(r,\phi+\theta_1,0)},$$
$$\mathbf{L}^{01}_{s}\mathbf{H}_{(r,0,0)} = \mathbf{H}_{(s+r,0,0)},$$
$$\mathbf{L}^{02}_{s}\widetilde{\mathbf{H}}_{(r,0,0)} = \widetilde{\mathbf{H}}_{(s+r,0,0)},$$
$$\mathbf{L}^{03}_{s}\widetilde{\widetilde{\mathbf{H}}}_{(r,0,0)} = \widetilde{\widetilde{\mathbf{H}}}_{(s+r,0,0)}.$$

Moreover, $\mathbf{L}^{0j}_s$ and $\mathbf{R}^{0j}_\theta$ leave fixed $\mathbf{l}^{0j}_r$; that is,

$$\mathbf{L}^{0j}_s\mathbf{l}^{0j}_r = \mathbf{l}^{0j}_{s+r},$$
$$\mathbf{R}^{0j}_\theta\mathbf{l}^{0j}_r = \mathbf{l}^{0j}_r. \quad (38)$$

Thus, we are ready to give the following definitions by (33) and (38).

Definition 6. $\mathbf{L}^{0j}_s$ is *H-translation* by s along H-coordinate axis l_{0j} for $j = 1, 2, 3$ in $\mathbb{H}^3$.

Definition 7. $\mathbf{R}^{0j}_\theta$ is *H-rotation* by θ about H-coordinate axis l_{0j} for $j = 1, 2, 3$ in $\mathbb{H}^3$.

Definition 8. $\mathbf{H}^{0ij}_\lambda$ is *horocyclic rotation* by λ about lightlike plane $\mathbb{D}_{ij}$ for $i, j = 1, 2, 3$ $(i \neq j)$ in $\mathbb{H}^3$.

Now, we give corollaries about some properties of H-isometries and transition relation between H-coordinate axes l_{0j} with H-coordinate planes D_{ij}.

Corollary 9. *Any H-coordinate axis is converted to each other by suitable H-rotation. That is,*

$$\mathbf{R}^{0j}_\theta\mathbf{l}^{01}_r = \begin{cases}\mathbf{l}^{02}_r, & j = 3,\ \theta = \dfrac{\pi}{2}\\ \mathbf{l}^{03}_r, & j = 2,\ \theta = \dfrac{3\pi}{2}.\end{cases} \quad (39)$$

Corollary 10. *A H-plane consists in suitable H-coordinate axis and H-rotation. Namely,*

$$\mathbf{R}^{0j}_\phi\mathbf{l}^{0k}_r = \begin{cases}\mathbf{D}^{23}_{(r,\phi)}, & j = 1,\ k = 2\\ \mathbf{D}^{13}_{(r,\phi)}, & j = 2,\ k = 3\\ \mathbf{D}^{12}_{(r,\phi)}, & j = 3,\ k = 1.\end{cases} \quad (40)$$

Corollary 11. *Any H-coordinate plane is converted to each other by suitable H-rotation. That is,*

$$\mathbf{R}_{\theta}^{0j}\mathbf{D}_{(r,\phi)}^{12}=\begin{cases}\mathbf{D}_{(r,\phi)}^{23}, & j=3,\ \theta=\dfrac{3\pi}{2}\\ \mathbf{D}_{(r,\phi)}^{13}, & j=1,\ \theta=\dfrac{\pi}{2}.\end{cases} \tag{41}$$

Corollary 12. *Any horocyclic rotation is converted to each other by suitable H-rotations. Namely,*

$$\begin{aligned}&\mathbf{R}_{-\psi}^{0l}\mathbf{R}_{-\phi}^{0k}\mathbf{R}_{-\theta}^{0j}\mathbf{H}_{\lambda}^{012}\mathbf{R}_{\theta}^{0j}\mathbf{R}_{\phi}^{0k}\mathbf{R}_{\psi}^{0l}\\ &=\begin{cases}\mathbf{H}_{\lambda}^{013}, & j=1,\ \theta=\dfrac{\pi}{2},\ \phi=0,\ \psi=0\\ \mathbf{H}_{\lambda}^{031}, & j=1,\ k=2,\ \theta=\dfrac{\pi}{2},\ \phi=\dfrac{\pi}{2},\ \psi=0\\ \mathbf{H}_{\lambda}^{032}, & j=1,\ k=2,\ l=3,\ \theta=\dfrac{\pi}{2},\ \phi=\dfrac{\pi}{2},\ \psi=\dfrac{\pi}{2}\\ \mathbf{H}_{\lambda}^{023}, & j=3,\ k=2,\ \theta=-\dfrac{\pi}{2},\ \phi=-\dfrac{\pi}{2},\ \psi=0\\ \mathbf{H}_{\lambda}^{021}, & j=3,\ \theta=-\dfrac{\pi}{2},\ \phi=0,\ \psi=0.\end{cases}\end{aligned} \tag{42}$$

After the notion of congruent in $\mathbb{H}^3$, we will give a different classification theorem of H-isometries in terms of leaving those totally umbilic surfaces of $\mathbb{H}^3$ fixed.

Definition 13. Let S and S' be two subsets of $\mathbb{H}^3$. If $T(S)=S'$ for some $T\in G$, then S and S' are called *congruent* in $\mathbb{H}^3$.

Theorem 14. *An H-sphere is invariant under H-translation in $\mathbb{H}^3$.*

Proof. Suppose that M is an H-sphere. Then, there exists a spacelike hyperplane $\mathrm{HP}(\mathbf{v},-k)$ with timelike normal $\mathbf{v}$ such that $M=\mathbb{H}^3\cap\mathrm{HP}(\mathbf{v},-k)$ for $k>0$. So,

$$\begin{aligned}\mathrm{HP}(\mathbf{v},-k)=\Big\{\mathbf{x}\in\mathbb{R}_1^4\mid\ &\\ -v_0x_0+v_1x_1+v_2x_2+v_3x_3&=-k,k>0\Big\}.\end{aligned} \tag{43}$$

Moreover, for $\mathbf{w}=\mathbf{v}/\|\mathbf{v}\|\in\mathbb{H}^3$ and $c=k/\|\mathbf{v}\|$, we have

$$\mathrm{HP}(\mathbf{w},-c)=\left\{\mathbf{x}\in\mathbb{R}_1^4\mid\langle\mathbf{x},\mathbf{w}\rangle=-c\right\}. \tag{44}$$

Since $\mathbf{w}\in\mathbb{H}^3$,

$$\begin{aligned}\mathbf{w}=\big(&\cosh s_0,\sinh s_0\cos\phi_0,\sinh s_0\sin\phi_0\cos\theta_0,\\ &\sinh s_0\sin\phi_0\sin\theta_0\big)\end{aligned} \tag{45}$$

for any hyperbolic polar coordinates $s_0\in[0,\infty)$, $\phi_0\in[0,\pi]$ and $\theta_0\in[0,2\pi]$. If we apply H-isometry $\mathbf{T}=\mathbf{L}_{-s_0}^{01}\mathbf{R}_{-\phi_0}^{03}\mathbf{R}_{-\theta_0}^{01}\in G$, then we have unit timelike vector $\mathbf{e}_0$ such that

$$T(\mathbf{w})=\mathbf{e}_0. \tag{46}$$

However, unit timelike normal vector $\mathbf{e}_0$ is invariant under L_s^{0j}. That is,

$$L_s^{0j}(\mathbf{e}_0)=(\cosh s)\,\mathbf{e}_0,\quad j=1,2,3. \tag{47}$$

For this reason, if $\widetilde{M}$ is an H-sphere which is generated from spacelike hyperplane

$$\mathrm{HP}(\mathbf{e}_0,-c)=\left\{\mathbf{x}\in\mathbb{R}_1^4\mid x_0=c,\ c\geq1\right\}, \tag{48}$$

then we have

$$L_s^{0j}\big(\widetilde{M}\big)=\widetilde{M} \tag{49}$$

by (47). Therefore, $\widetilde{M}$ is invariant under H-translations. Finally, The proof is completed since M and $\widetilde{M}$ are congruent by (46). □

The following theorems also can be proved using similar method.

Theorem 15. *An equidistant surface is invariant under H-rotation in $\mathbb{H}^3$.*

Theorem 16. *A horosphere is invariant under horocyclic rotation in $\mathbb{H}^3$.*

Finally, we give the following corollary.

Corollary 17. *Equidistant surfaces, H-spheres, and horospheres are invariant under groups G_0, G_1, and G_2 in $\mathbb{H}^3$, respectively.*

3. Differential Geometry of Curves and Surfaces in $\mathbb{H}^3$

In this section, we give the basic theory of extrinsic differential geometry of curves and surfaces in $\mathbb{H}^3$. Unless otherwise stated, we use the notation in [7, 8].

The Lorentzian vector product of vectors $\mathbf{x}^1,\mathbf{x}^2,\mathbf{x}^3$ is given by

$$\mathbf{x}^1\wedge\mathbf{x}^2\wedge\mathbf{x}^3=\begin{vmatrix}-\mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ x_0^1 & x_1^1 & x_2^1 & x_3^1\\ x_0^2 & x_1^2 & x_2^2 & x_3^2\\ x_0^3 & x_1^3 & x_2^3 & x_3^3\end{vmatrix}, \tag{50}$$

where $\{\mathbf{e}_0,\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is the canonical basis of $\mathbb{R}_1^4$ and $\mathbf{x}^i=(x_0^i,x_1^i,x_2^i,x_3^i)$, $i=1,2,3$. Also, it is clear that

$$\left\langle\mathbf{x},\mathbf{x}^1\wedge\mathbf{x}^2\wedge\mathbf{x}^3\right\rangle=\det\left(\mathbf{x},\mathbf{x}^1,\mathbf{x}^2,\mathbf{x}^3\right) \tag{51}$$

for any $\mathbf{x}\in\mathbb{R}_1^4$. Therefore, $\mathbf{x}^1\wedge\mathbf{x}^2\wedge\mathbf{x}^3$ is pseudo-orthogonal to any $\mathbf{x}^i$, $i=1,2,3$.

We recall the basic theory of curves in $\mathbb{H}^3$. Let $\alpha: I\to\mathbb{H}^3$ be a unit speed regular curve for open subset $I\subset\mathbb{R}$. Since $\langle\alpha'(s),\alpha'(s)\rangle=1$, tangent vector of α is given by $\mathbf{t}(s)=\alpha'(s)$. The vector $\alpha''(s)-\alpha(s)$ is orthogonal to $\alpha(s)$ and $\mathbf{t}(s)$. We suppose that $\alpha''(s)-\alpha(s)\neq0$. Then, the normal vector of α is given by $\mathbf{n}(s)=(\alpha''(s)-\alpha(s))/\|\alpha''(s)-\alpha(s)\|$. However, the binormal vector of α is given by $\mathbf{e}(s)=\alpha(s)\wedge\mathbf{t}(s)\wedge\mathbf{n}(s)$.

Hence, we have a pseudo-orthonormal frame field $\{\alpha(s), \mathbf{t}(s), \mathbf{n}(s), \mathbf{e}(s)\}$ of $\mathbb{R}_1^4$ along α and the following Frenet-Serret formulas:

$$\begin{bmatrix} \alpha'(s) \\ \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{e}'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \kappa_h(s) & 0 \\ 0 & -\kappa_h(s) & 0 & \tau_h(s) \\ 0 & 0 & -\tau_h(s) & 0 \end{bmatrix} \begin{bmatrix} \alpha(s) \\ \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{e}(s) \end{bmatrix}, \quad (52)$$

where hyperbolic curvature and hyperbolic torsion of α are given by $\kappa_h(s) = \|\alpha''(s) - \alpha(s)\|$ and $\tau_h(s) = (-\det(\alpha(s), \alpha'(s), \alpha''(s), \alpha'''(s)))/(\kappa_h(s))^2$ under the assumption $\langle\alpha''(s), \alpha''(s)\rangle \neq -1$, respectively.

Remark 18. The condition $\langle\alpha''(s), \alpha''(s)\rangle \neq -1$ is equivalent to $\kappa_h(s) \neq 0$. Moreover, we see easily that $\kappa_h(s) = 0$ if and only if there exists a lightlike vector $\mathbf{c}$ such that $\alpha(s) - c$ a geodesic (H-line).

If $\kappa_h(s) = 1$ and $\tau_h(s) = 0$, then α in $\mathbb{H}^3$ is called a horocycle. We give a lemma about existence and uniqueness for horocycles (cf. [8, Proposition 4.3]).

Lemma 19. *For any $\mathbf{a}_0 \in \mathbb{H}^3$ and $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{S}_1^3$ such that $\langle\mathbf{a}_i, \mathbf{a}_j\rangle = 0$, the unique horocycle with the initial conditions $\gamma(0) = \mathbf{a}_0$, $\gamma'(0) = \mathbf{a}_1$, and $\gamma''(0) = \mathbf{a}_0 + \mathbf{a}_2$ is given by*

$$\gamma(s) = \mathbf{a}_0 + s\mathbf{a}_1 + \frac{s^2}{2}(\mathbf{a}_0 + \mathbf{a}_2). \quad (53)$$

Now, we recall the basic theory of surfaces in $\mathbb{H}^3$. Let $\mathbf{x} : U \to \mathbb{H}^3$ be embedding such that open subset $U \subset \mathbb{R}^2$. We denote that regular surface $M = \mathbf{x}(U)$ and identify M and U through the embedding $\mathbf{x}$, where $i : U \to M$ is a local chart. For $\mathbf{x}(u) = p \in M$ and $u = (u_1, u_2) \in U$, if we define spacelike unit normal vector

$$\eta(u) = \frac{\mathbf{x}(u) \wedge \mathbf{x}_{u_1}(u) \wedge \mathbf{x}_{u_2}(u)}{\left\|\mathbf{x}(u) \wedge \mathbf{x}_{u_1}(u) \wedge \mathbf{x}_{u_2}(u)\right\|}, \quad (54)$$

where $\mathbf{x}_{u_i} = \partial\mathbf{x}/\partial u_i$, $i = 1, 2$, then we have $\langle\mathbf{x}_{u_i}, \mathbf{x}\rangle = \langle\mathbf{x}_{u_i}, \eta\rangle = 0$, $i = 1, 2$. We also regard η as unit normal vector field along M in $\mathbb{H}^3$. Moreover, $\mathbf{x}(u) \pm \eta(u)$ is a lightlike vector since $\mathbf{x}(u) \in \mathbb{H}^3$, $\eta(u) \in \mathbb{S}_1^3$. Then the following maps $E : U \to S_1^3$, $E(u) = \eta(u)$ and $L^\pm : U \to LC_+$, $L^\pm(u) = \mathbf{x}(u) \pm \eta(u)$ are called *de Sitter Gauss map* and *light cone Gauss map* of $\mathbf{x}$, respectively [8]. Under the identification of U and M via the embedding $\mathbf{x}$, the derivative $d\mathbf{x}(u_0)$ can be identified with identity mapping I_{T_pM} on the tangent space T_pM at $\mathbf{x}(u_0) = p \in M$. We have that $-dL^\pm = -I_{T_pM} \pm (-dE)$.

For any given $\mathbf{x}(u_0) = p \in M$, the linear transforms $A_p = -dE(u_0) : T_pM \to T_pM$ and $S_p^\pm = -dL^\pm(u_0) : T_pM \to T_pM$ are called *de Sitter shape operator* and *hyperbolic shape operator* of $\mathbf{x}(U) = M$, respectively. The eigenvalues of A_p and $S_p^\pm$ are denoted by $k_i(p)$ and $k_i^\pm(p)$ for $i = 1, 2$, respectively. Obviously, A_p and $S_p^\pm$ have same eigenvectors. Also, the eigenvalues satisfy

$$k_i^\pm(p) = -1 \pm k_i(p), \quad i = 1, 2, \quad (55)$$

where $k_i(p)$ and $k_i^\pm(p)$ are called *de Sitter principal curvature* and *hyperbolic principal curvature* of M at $\mathbf{x}(u_0) = p \in M$, respectively.

The *de Sitter Gauss curvature* and the *de Sitter mean curvature* of M are given by

$$K_d(u_0) = \det A_p = k_1(p)\,k_2(p),$$
$$H_d(u_0) = \frac{1}{2}\operatorname{Tr} A_p = \frac{k_1(p) + k_2(p)}{2} \quad (56)$$

at $\mathbf{x}(u_0) = p$, respectively. Similarly, The *hyperbolic Gauss curvature* and the *hyperbolic mean curvature* of M are given by

$$K_h^\pm(u_0) = \det S_p^\pm = k_1^\pm(p)\,k_2^\pm(p),$$
$$H_h^\pm(u_0) = \frac{1}{2}\operatorname{Tr} S_p^\pm = \frac{k_1^\pm(p) + k_2^\pm(p)}{2} \quad (57)$$

at $\mathbf{x}(u_0) = p$, respectively. Evidently, we have the following relations:

$$K_h^\pm = 1 \mp 2H_d + K_d,$$
$$H_h^\pm = -1 \pm H_d. \quad (58)$$

We say that a point $\mathbf{x}(u_0) = p \in M$ is an *umbilical point* if $k_1(p) = k_2(p)$. Also, M is *totally umbilical* if all points on M are umbilical. Now, we give the following classification theorem of totally umbilical surfaces in $\mathbb{H}^3$ (cf. [8, Proposition 2.1]).

Lemma 20. *Suppose that $M = \mathbf{x}(U)$ is totally umbilical. Then, $k(p)$ is a constant k. Under this condition, one has the following classification.*

(1) *Supposing that $k^2 \neq 1$,*

(a) *if $k \neq 0$ and $k^2 < 1$, then M is a part of an equidistant surface;*

(b) *if $k \neq 0$ and $k^2 > 1$, then M is a part of a sphere;*

(c) *if $k = 0$, then M is a part of a plane (H-plane).*

(2) *If $k^2 = 1$, then M is a part of horosphere.*

4. G_1-Invariant Surfaces in $\mathbb{H}^3$

In this section, we investigate surfaces which are invariant under some one parameter subgroup of H-translations in $\mathbb{H}^3$. Moreover, we study extrinsic differential geometry of these invariant surfaces.

Let $M = \mathbf{x}(U)$ be a regular surface via embedding $\mathbf{x} : U \to \mathbb{H}^3$ such that open subset $U \subset \mathbb{R}^2$. We denote by A the shape operator of M with respect to unit normal vector field η in $\mathbb{H}^3$. Let us represent by $\overline{\overline{D}}$, $\overline{D}$, and D the Levi-Civita connections of $\mathbb{R}_1^4$, $\mathbb{H}^3$, and M, respectively. Then the Gauss and Weingarten explicit formulas for M in $\mathbb{H}^3$ are given by

$$\overline{\overline{D}}_X Y = D_X Y + \langle A(X), Y\rangle\,\eta + \langle X, Y\rangle\,\mathbf{x}, \quad (59)$$

$$A(X) = -\overline{D}_X\eta = -\overline{\overline{D}}_X\eta \quad (60)$$

for all tangent vector fields $X, Y \in \mathcal{X}(M)$, respectively.

Let I be an open interval of $\mathbb{R}$ and let

$$\alpha : I \longrightarrow D_{23} \subset H^3, \qquad \alpha(t) = (\alpha_1(t), 0, \alpha_3(t), \alpha_4(t)) \tag{61}$$

be a unit speed regular curve which is lying on H-plane $D_{23} = \{(x_0, x_1, x_2, x_3) \in \mathbb{H}^3 \mid x_1 = 0\}$. Without loss of generality, we consider the subgroup

$$\overline{G_1} = \left\{ \mathbf{L}_s = \begin{bmatrix} \cosh s & \sinh s & 0 & 0 \\ \sinh s & \cosh s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : s \in \mathbb{R} \right\} \subset G_1 \tag{62}$$

which is H-translation group along H-coordinate axis $l_{01}(s) = (\cosh s, \sinh s, 0, 0)$ in $\mathbb{H}^3$. Let $M = \mathbf{x}(U)$ be a regular surface which is given by the embedding

$$\mathbf{x} : U \longrightarrow \mathbb{H}^3, \qquad \mathbf{x}(s,t) = L_s(\alpha(t)), \tag{63}$$

where $U = \mathbb{R} \times I$ is open subset of $\mathbb{R}^2$. Since $L_s(M) = M$ for all $s \in \mathbb{R}$, M is invariant under H-translation group $\overline{G_1}$. Hence we say that M is a $\overline{G_1}$*-invariant surface* and α is the *profile curve* of M in $\mathbb{H}^3$.

Remark 21. From now on, we will not use the parameter "t" in case of necessity for brevity.

By (61) and (62), the parametrization of M is

$$\mathbf{x}(s,t) = (\alpha_1(t)\cosh s, \alpha_1(t)\sinh s, \alpha_3(t), \alpha_4(t)) \tag{64}$$

for all $(s,t) \in U$. By (59),

$$\begin{aligned} \partial_s &= \overline{\overline{D}}_{\mathbf{x}_s}\mathbf{x} - \langle \mathbf{x}_s, \mathbf{x}\rangle \mathbf{x} = \overline{\overline{D}}_{\mathbf{x}_s}\mathbf{x} = \mathbf{x}_s, \\ \partial_t &= \overline{\overline{D}}_{\mathbf{x}_t}\mathbf{x} - \langle \mathbf{x}_t, \mathbf{x}\rangle \mathbf{x} = \overline{\overline{D}}_{\mathbf{x}_t}\mathbf{x} = \mathbf{x}_t, \end{aligned} \tag{65}$$

where $\partial_s = \overline{D}_{\mathbf{x}_s}\mathbf{x}$, $\partial_t = \overline{D}_{\mathbf{x}_t}\mathbf{x}$, $\mathbf{x}_s = \partial\mathbf{x}/\partial s$, and $\mathbf{x}_t = \partial\mathbf{x}/\partial t$. So, we have

$$\begin{aligned} \partial_s(s,t) &= (\alpha_1(t)\sinh s, \alpha_1(t)\cosh s, 0, 0), \\ \partial_t(s,t) &= \left(\alpha_1'(t)\cosh s, \alpha_1'(t)\sinh s, \alpha_3'(t), \alpha_4'(t)\right), \\ \langle \partial_s(s,t), \partial_t(s,t)\rangle &= 0. \end{aligned} \tag{66}$$

Hence, $\psi = \{\partial_s, \partial_t\}$ is orthogonal tangent frame of $\mathcal{X}(M)$. If $\omega(s,t) = \mathbf{x}(s,t) \wedge \partial_s(s,t) \wedge \partial_t(s,t)$, then we have that $\langle \omega(s,t), \omega(s,t)\rangle = \alpha_1(t)^2$ and also $\alpha_1(t) > 0$ for all $t \in I$ since $\alpha(I) \in \mathbb{H}^3$. If the unit normal vector of M in $\mathbb{H}^3$ is denoted by $\eta(s,t) = \omega(s,t)/\|\omega(s,t)\|$, then we have that

$$\begin{aligned} \eta(s,t) = \Big(&\left(\alpha_3\alpha_4' - \alpha_3'\alpha_4\right)\cosh s, \left(\alpha_3\alpha_4' - \alpha_3'\alpha_4\right)\sinh s, \\ &\alpha_1\alpha_4' - \alpha_1'\alpha_4, \alpha_1'\alpha_3 - \alpha_1\alpha_3' \Big) \end{aligned} \tag{67}$$

and it is clear that

$$\langle \eta, \partial_s\rangle \equiv \langle \eta, \partial_t\rangle \equiv 0 \tag{68}$$

for all $(s,t) \in U$. From (59) and (60), the matrix of de Sitter shape operator of M with respect to orthogonal tangent frame ψ of $\mathcal{X}(M)$ is $\mathbf{A}_p = \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$ at any $\mathbf{x}(s,t) = p \in M$, where

$$\begin{aligned} a &= \frac{\langle -\overline{D}_{\partial_s}\eta, \partial_s\rangle}{\langle \partial_s, \partial_s\rangle} = \frac{\langle \overline{\overline{D}}_{x_s}x_s, \eta\rangle}{\langle x_s, x_s\rangle}, \\ b &= \frac{\langle -\overline{D}_{\partial_s}\eta, \partial_t\rangle}{\langle \partial_t, \partial_t\rangle} = \frac{\langle \overline{\overline{D}}_{x_s}x_t, \eta\rangle}{\langle x_t, x_t\rangle}; \qquad c = b, \\ d &= \frac{\langle -\overline{D}_{\partial_t}\eta, \partial_t\rangle}{\langle \partial_t, \partial_t\rangle} = \frac{\langle \overline{\overline{D}}_{x_t}x_t, \eta\rangle}{\langle x_t, x_t\rangle}. \end{aligned} \tag{69}$$

After basic calculations, the de Sitter principal curvatures of M are

$$k_1 = \frac{\alpha_3'\alpha_4 - \alpha_3\alpha_4'}{\alpha_1}, \tag{70}$$

$$\begin{aligned} k_2 = {} &\alpha_1''\left(\alpha_3'\alpha_4 - \alpha_3\alpha_4'\right) + \alpha_3''\left(\alpha_1\alpha_4' - \alpha_1'\alpha_4\right) \\ &+ \alpha_4''\left(\alpha_1'\alpha_3 - \alpha_1\alpha_3'\right). \end{aligned} \tag{71}$$

Let Frenet-Serret apparatus of M be denoted by $\{\mathbf{t}, \mathbf{n}, \mathbf{e}, \kappa_h, \tau_h\}$ in $\mathbb{H}^3$.

Proposition 22. *The binormal vector of the profile curve of $\overline{G_1}$-invariant surface M is constant in $\mathbb{H}^3$.*

Proof. Let α be the profile curve of M. By (61), we know that α is a hyperbolic plane curve; that is, $\tau_h = 0$. Moreover, by (52) and (59), we have that $\overline{D}_{\mathbf{t}}\mathbf{e} = -\tau_h\mathbf{n} = 0$. Hence, by (52), $\overline{\overline{D}}_{\mathbf{t}}\mathbf{e} = 0$. This completes the proof. □

From now on, let the binormal vector of the profile curve of M be given by $\mathbf{e} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ such that λ_i is scalar for $i = 0, 1, 2, 3$. Now, we will give the important relation between the one of de Sitter principal curvatures and hyperbolic curvature of the profile curve of M.

Theorem 23. *Let M be $\overline{G_1}$-invariant surface in $\mathbb{H}^3$. Then, $k_2 = \lambda_1\kappa_h$.*

Proof. Let the binormal vector of the profile curve of M be denoted by $\mathbf{e}$. In Section 3, from the definition of Serret-Frenet vectors, we have that $\kappa_h\mathbf{e} = \alpha \wedge \alpha' \wedge \alpha''$. Also, by (50) and (71), we obtain that $\alpha \wedge \alpha' \wedge \alpha'' = (0, k_2, 0, 0)$. Thus, it follows that $\kappa_h\mathbf{e} = (0, k_2, 0, 0)$. For this reason, we have that $k_2 = \lambda_1\kappa_h$. □

As a result of Theorem 23, the de Sitter Gauss curvature and the de Sitter mean curvature of $M = \mathbf{x}(U)$ are

$$K_d(p) = \frac{\alpha_3'\alpha_4 - \alpha_3\alpha_4'}{\alpha_1}\lambda_1\kappa_h, \tag{72}$$

$$H_d(p) = \frac{\left(\alpha_3'\alpha_4 - \alpha_3\alpha_4'\right) + \lambda_1\kappa_h\alpha_1}{2\alpha_1}, \tag{73}$$

at any $\mathbf{x}(s,t) = p$, respectively. Moreover, if we apply (58), then the hyperbolic Gauss curvature and the hyperbolic mean curvature of M are

$$K_h^{\pm}(p) = \frac{\left(\alpha_1 \mp \left(\alpha_3'\alpha_4 - \alpha_3\alpha_4'\right)\right)\left(1 \mp \lambda_1\kappa_h\right)}{\alpha_1}, \quad (74)$$

$$H_h^{\pm}(p) = \frac{\alpha_1\left(-2 \pm \lambda_1\kappa_h\right) \pm \left(\alpha_3'\alpha_4 - \alpha_3\alpha_4'\right)}{2\alpha_1}, \quad (75)$$

at any $\mathbf{x}(s,t) = p$, respectively.

Proposition 24. *Let $\alpha : I \to D_{23} \subset H^3$, $\alpha(t) = (\alpha_1(t), 0, \alpha_3(t), \alpha_4(t))$, be unit speed regular profile curve of $\overline{G_1}$-invariant surface M. Then its components are given by*

$$\begin{aligned} \alpha_3(t) &= \sqrt{\alpha_1(t)^2 - 1}\cos\varphi(t), \\ \alpha_4(t) &= \sqrt{\alpha_1(t)^2 - 1}\sin\varphi(t), \\ \varphi(t) &= \int_0^t \frac{\sqrt{\alpha_1(u)^2 - \alpha_1'(u)^2 - 1}}{\alpha_1(u)^2 - 1}du. \end{aligned} \quad (76)$$

Proof. Suppose that the profile curve of M is unit speed and regular. So that, it satisfies the following equations:

$$-\alpha_1(t)^2 + \alpha_3(t)^2 + \alpha_4(t)^2 = -1, \quad (77)$$

$$-\alpha_1'(t)^2 + \alpha_3'(t)^2 + \alpha_4'(t)^2 = 1, \quad (78)$$

for all $t \in I$. By (77) and $\alpha_1(t) \geq 1$, we have that

$$\begin{aligned} \alpha_3(t) &= \sqrt{\alpha_1(t)^2 - 1}\cos\varphi(t), \\ \alpha_4(t) &= \sqrt{\alpha_1(t)^2 - 1}\sin\varphi(t), \end{aligned} \quad (79)$$

such that φ is a differentiable function. Moreover, by (78) and (79), we obtain that

$$\varphi'(t)^2 = \frac{\alpha_1(t)^2 - \alpha_1'(t)^2 - 1}{\left(\alpha_1(t)^2 - 1\right)^2}. \quad (80)$$

Finally, by (80), we have that

$$\varphi(t) = \pm\int_0^t \frac{\sqrt{\alpha_1(u)^2 - \alpha_1'(u)^2 - 1}}{\alpha_1(u)^2 - 1}du \quad (81)$$

such that $\alpha_1(t)^2 - \alpha_1'(t)^2 - 1 > 0$ for all $t \in I$. Without loss of generality, when we choose positive of signature of φ, this completes the proof. □

Remark 25. If M is a de Sitter flat surface in $\mathbb{H}^3$, then we say that M is an H-plane in $\mathbb{H}^3$.

Now, we will give some results which are obtained by (72) and (74).

Corollary 26. *Let α be the profile curve of $\overline{G_1}$-invariant surface M in $\mathbb{H}^3$. Then,*

(i) *if $\alpha_3 = 0$ or $\alpha_4 = 0$, then M is a part of de Sitter flat surface;*

(ii) *if $\kappa_h = 0$, then M is a part of de Sitter flat surface;*

(iii) *if $\lambda_1 = 0$, then M is a part of de Sitter flat surface.*

Corollary 27. *Let α be the profile curve of $\overline{G_1}$-invariant surface M in $\mathbb{H}^3$. If $\alpha_3 = \mu\alpha_4$ such that $\mu \in \mathbb{R}$, then M is a de Sitter flat surface.*

Theorem 28. *Let α be the profile curve of $\overline{G_1}$-invariant surface M in $\mathbb{H}^3$. Then, M is hyperbolic flat surface if and only if $\kappa_h = \pm 1/\lambda_1$.*

Proof. Suppose that M is hyperbolic flat surface; that is, $K_h^{\pm} = 0$. By (74), it follows that

$$\alpha_1(t) \mp \left(\alpha_3'(t)\alpha_4(t) - \alpha_3(t)\alpha_4'(t)\right) = 0 \quad (82)$$

or

$$1 \mp \lambda_1\kappa_h(t) = 0 \quad (83)$$

for all $t \in I$. Firstly, let us find solution of (82). If Proposition 24 is applied to (82), we have that $\alpha_1(t) \mp (-\sqrt{\alpha_1(t)^2 - \alpha_1'(t)^2 - 1}) = 0$. Hence, it follows that

$$\alpha_1'(t)^2 + 1 = 0. \quad (84)$$

There is no real solution of (84). This means that the only one solution is $\kappa_h = \pm 1/\lambda_1$ by (83). On the other hand, if we assume that $\kappa_h = \pm 1/\lambda_1$, then the proof is clear. □

Corollary 29. *Let α be the profile curve of $\overline{G_1}$-invariant surface M in $\mathbb{H}^3$. Then,*

(i) *if $\lambda_1 = 1$ and $\kappa_h = 1$, then M is K_h^+-flat surface which is generated from horocyle;*

(ii) *if $\lambda_1 = -1$ and $\kappa_h = 1$, then M is K_h^--flat surface which is generated from horocyle.*

Now, we will give theorem and corollaries for $\overline{G_1}$-invariant surface which satisfy minimal condition in $\mathbb{H}^3$ by (73) and (75).

Theorem 30. *Let α be the profile curve with constant hyperbolic curvature of $\overline{G_1}$-invariant surface M in $\mathbb{H}^3$. Then, M is de*

Sitter minimal surface if and only if the parametrization of α is given by

$$\begin{aligned}\alpha_1(t) = &\left(\left(-2+\lambda_1^2\kappa_h^2\right)\cosh\left(\left(t+c_1\right)\sqrt{1-\lambda_1^2\kappa_h^2}\right)\right.\\ &\left.-\lambda_1^2\kappa_h^2\sinh\left(\left(t+c_1\right)\sqrt{1-\lambda_1^2\kappa_h^2}\right)\right)\\ &\times\left(-2\left(1-\lambda_1^2\kappa_h^2\right)\right)^{-1},\\ \alpha_3(t) = &\sqrt{\alpha_1(t)^2-1}\cos\varphi(t),\\ \alpha_4(t) = &\sqrt{\alpha_1(t)^2-1}\sin\varphi(t),\\ \varphi(t) = &\int_0^t \frac{\sqrt{\alpha_1(u)^2-\alpha_1'(u)^2-1}}{\alpha_1(u)^2-1}du,\end{aligned} \tag{85}$$

with the condition $\kappa_h \in (-1/\lambda_1, 1/\lambda_1)$ such that c_1 is an arbitrary constant.

Proof. Suppose that M is de Sitter minimal surface; that is, $H_d \equiv 0$. By (73), it follows that

$$\left(\alpha_3'(t)\alpha_4(t)-\alpha_3(t)\alpha_4'(t)\right)+\lambda_1\kappa_h\alpha_1(t)=0, \tag{86}$$

for all $t \in I$. By using Proposition 24, we have the following differential equation:

$$\alpha_1'(t)^2-\left(1-\lambda_1^2\kappa_h^2\right)\alpha_1(t)^2+1=0. \tag{87}$$

There exists only one real solution of (87) under the condition $1-\lambda_1^2\kappa_h^2>0$. Moreover, λ_1 must not be zero by Corollary 26. So that, we obtain

$$\kappa_h \in \left(-\frac{1}{\lambda_1}, \frac{1}{\lambda_1}\right). \tag{88}$$

Hence, the solution is

$$\begin{aligned}\alpha_1(t) = &\left(\left(-2+\lambda_1^2\kappa_h^2\right)\cosh\left(\left(t+c_1\right)\sqrt{1-\lambda_1^2\kappa_h^2}\right)\right.\\ &\left.-\lambda_1^2\kappa_h^2\sinh\left(\left(t+c_1\right)\sqrt{1-\lambda_1^2\kappa_h^2}\right)\right)\\ &\times\left(-2\left(1-\lambda_1^2\kappa_h^2\right)\right)^{-1},\end{aligned} \tag{89}$$

where c_1 is an arbitrary constant under the condition (88). Finally, the parametrization of α is given explicitly by Proposition 24.

On the other hand, let the parametrization of profile curve α of M be given by (85) under the condition (87). Then it satisfies (86). It means that $H_d \equiv 0$. □

Theorem 31. *Let α be the profile curve with constant hyperbolic curvature of $\overline{G_1}$-invariant surface M in $\mathbb{H}^3$. Then, M is hyperbolic minimal surface if and only if the parametrization of α is given by*

$$\begin{aligned}\alpha_1(t) &\\ = &\left(\left(\left(2\mp\lambda_1\kappa_h\right)^2-2\right)\cosh\left(\left(t+c_1\right)\sqrt{1-\left(2\mp\lambda_1\kappa_h\right)^2}\right)\right.\\ &\left.+\left(2-\lambda_1\kappa_h\right)^2\sinh\left(\left(t+c_1\right)\sqrt{1-\left(2\mp\lambda_1\kappa_h\right)^2}\right)\right)\\ &\times\left(2\left(-1+\left(2\mp\lambda_1\kappa_h\right)^2\right)\right)^{-1},\\ \alpha_3(t) = &\sqrt{\alpha_1(t)^2-1}\cos\varphi(t),\\ \alpha_4(t) = &\sqrt{\alpha_1(t)^2-1}\sin\varphi(t),\\ \varphi(t) = &\int_0^t \frac{\sqrt{\alpha_1(t)^2-\alpha_1'(t)^2-1}}{\alpha_1(t)^2-1}du,\end{aligned} \tag{90}$$

with the condition $1-(2\mp\lambda_1\kappa_h)^2>0$ such that c_1 is an arbitrary constant.

Proof. Suppose that M is hyperbolic minimal surface; that is, $H_h^{\pm} \equiv 0$. By (75), it follows that

$$\left(-2\pm\lambda_1\kappa_h\right)\alpha_1(t)\pm\left(\alpha_3'(t)\alpha_4(t)-\alpha_3(t)\alpha_4'(t)\right)=0. \tag{91}$$

If Proposition 24 is applied to (91), we have that

$$\alpha_1'(t)^2+\left(\left(-2\pm\lambda_1\kappa_h\right)^2-1\right)\alpha_1(t)^2+1=0. \tag{92}$$

There exists only one real solution of (92) under the condition

$$\left(-2\pm\lambda_1\kappa_h\right)^2-1<0. \tag{93}$$

Thus, the solution is

$$\begin{aligned}\alpha_1(t) &\\ = &\left(\left(\left(2\mp\lambda_1\kappa_h\right)^2-2\right)\cosh\left(\left(t+c_1\right)\sqrt{1-\left(2\mp\lambda_1\kappa_h\right)^2}\right)\right.\\ &\left.+\left(2-\lambda_1\kappa_h\right)^2\sinh\left(\left(t+c_1\right)\sqrt{1-\left(2\mp\lambda_1\kappa_h\right)^2}\right)\right)\\ &\times\left(2\left(-1+\left(2\mp\lambda_1\kappa_h\right)^2\right)\right)^{-1},\end{aligned} \tag{94}$$

where c_1 is an arbitrary constant under the condition (93). However, λ_1 must not be zero by Corollary 26. So that, if M is H_h^+-minimal surface (H_h^--minimal surface), then we obtain $\kappa_h \in (1/\lambda_1, 3/\lambda_1)$ ($\kappa_h \in (-3/\lambda_1, -1/\lambda_1)$) by (93). Finally, the parametrization of α is given explicitly by Proposition 24.

Conversely, let the parametrization of profile curve α of M be given by (90) under the condition (93). Then, it satisfies (91). It means that $H_h^{\pm} \equiv 0$. □

Now, we will give classification theorem for totally umbilical $\overline{G_1}$-invariant surfaces.

Theorem 32. *Let α be the profile curve of totally umbilical $\overline{G_1}$-invariant surface M in $\mathbb{H}^3$. Then, the hyperbolic curvature of α is constant.*

Proof. Let M be totally umbilical $\overline{G_1}$-invariant surface. By Theorem 23 and (70), we may assume that $k_1(p) = k_2(p) = \lambda_1\kappa_h(t)$ for all $\mathbf{x}(s,t) = p \in M$. Then, we have the following equations:

$$\begin{gathered} A(\partial_s) = -\overline{D}_{\partial_s}\eta = -\overline{\overline{D}}_{\mathbf{x}_s}\eta = -\lambda_1\kappa_h(t)\,\mathbf{x}_s, \\ A(\partial_t) = -\overline{D}_{\partial_t}\eta = -\overline{\overline{D}}_{\mathbf{x}_t}\eta = -\lambda_1\kappa_h(t)\,\mathbf{x}_t, \\ \overline{D}_{\partial_t}\left(\overline{D}_{\partial_s}\eta\right) = \overline{\overline{D}}_{\mathbf{x}_t}\left(\overline{\overline{D}}_{\mathbf{x}_s}\eta\right) = \overline{\overline{D}}_{\mathbf{x}_t}\eta_s, \\ \overline{D}_{\partial_s}\left(\overline{D}_{\partial_t}\eta\right) = \overline{\overline{D}}_{\mathbf{x}_s}\left(\overline{\overline{D}}_{\mathbf{x}_t}\eta\right) = \overline{\overline{D}}_{\mathbf{x}_s}\eta_t. \end{gathered} \tag{95}$$

By (95), we obtain that

$$\begin{gathered} -\overline{\overline{D}}_{\mathbf{x}_t}\eta_s = -\lambda_1\left(\kappa_h'(t)\,\mathbf{x}_s + \kappa_h(t)\,\mathbf{x}_{st}\right), \\ -\overline{\overline{D}}_{\mathbf{x}_s}\eta_t = -\lambda_1\kappa_h(t)\,\mathbf{x}_{ts}. \end{gathered} \tag{96}$$

Also, if we use the equations $\overline{\overline{D}}_{\mathbf{x}_t}\eta_s = \overline{\overline{D}}_{\mathbf{x}_s}\eta_t$ and $\mathbf{x}_{st} = \mathbf{x}_{ts}$ in (96), then it follows that $\lambda_1\kappa_h'(t) = 0$ for all $t \in I$. Moreover, λ_1 must not be zero by Corollary 26. Thus, κ_h is constant. □

Corollary 33. *Let α be the profile curve of totally umbilical $\overline{G_1}$-invariant surface M in $\mathbb{H}^3$. Then we have the following classification.*

(1) *Supposing that $\xi^2 \neq 1$,*

  (a) *if $\xi \neq 0$ and $\xi^2 < 1$, then M is a part of an equidistant surface;*
  (b) *if $\xi \neq 0$ and $\xi^2 > 1$, then M is a part of a sphere;*
  (c) *if $\xi = 0$, then M is a part of a H-plane.*

(2) *If $\xi^2 = 1$, then M is a part of horosphere,*

where $\xi = \lambda_1\kappa_h$ is a constant.

Proof. We suppose that $\xi = \lambda_1\kappa_h$. By Proposition 22 and Theorem 32, we have that ξ is constant. Moreover, ξ is de Sitter principal curvature of M by Theorem 23. Since M is totally umbilical surface, de Sitter shape operator of M is $A_p = \xi I_2$ where I_2 is identity matrix. Finally, the proof is complete by Lemma 20. □

Now, we will give some examples of $\overline{G_1}$-invariant surface in $\mathbb{H}^3$. Let the Poincaré ball model of hyperbolic space be given by

$$\mathbb{B}^3 = \left\{\left(x_1, x_2, x_3\right) \in \mathbb{R}^3 \mid \sum_{i=1}^{3} x_i^2 < 1\right\} \tag{97}$$

with the hyperbolic metric $ds^2 = 4(dx_1^2 + dx_2^2 + dx_3^2)/(1 - x_1^2 - x_2^2 - x_3^2)$. Then, it is well known that stereographic projection of $\mathbb{H}^3$ is given by

$$\begin{gathered} \Phi : \mathbb{H}^3 \longrightarrow \mathbb{B}^3, \\ \Phi\left(x_0, x_1, x_2, x_3\right) = \left(\frac{x_1}{1 + x_0}, \frac{x_2}{1 + x_0}, \frac{x_3}{1 + x_0}\right). \end{gathered} \tag{98}$$

We can draw the pictures of surface $\mathbf{x}(U) = M$ by using stereographic projection Φ. That is, $\Phi(M) \subset \mathbb{B}^3$ such that $\mathbf{x}(U) = M \subset \mathbb{H}^3$.

Example 34. The $\overline{G_1}$-invariant surface which is generated from $\alpha(t) = (\sqrt{2}, 0, \cos t, \sin t)$ with hyperbolic curvature $\kappa_h = \sqrt{2}$ is drawn in Figure 1(a).

Example 35. Let the profile curve of M be given by

$$\alpha(t) = \left(\sqrt{2} + \frac{1}{2}\left(-1 + \sqrt{2}\right)t^2, 0, 1 - \frac{1}{2}\left(-1 + \sqrt{2}\right)t^2, t\right) \tag{99}$$

such that hyperbolic curvature $\kappa_h = 1$. Then, M is hyperbolic flat $\overline{G_1}$-invariant surface which is generated from horocycle in $\mathbb{H}^3$ (see Figure 1(b)).

Example 36. The $\overline{G_1}$-invariant surface which is generated from

$$\begin{aligned} \alpha(t) = \Bigg(&\frac{1}{3}\left(-1 + 4\cosh\frac{\sqrt{3}t}{2}\right), 0, \\ &\frac{2}{3}\left(-1 + \cosh\frac{\sqrt{3}t}{2}\right), \frac{2}{\sqrt{3}}\sinh\frac{\sqrt{3}t}{2}\Bigg) \end{aligned} \tag{100}$$

with hyperbolic curvature $\kappa_h = 1/2$ is drawn in Figure 1(c).

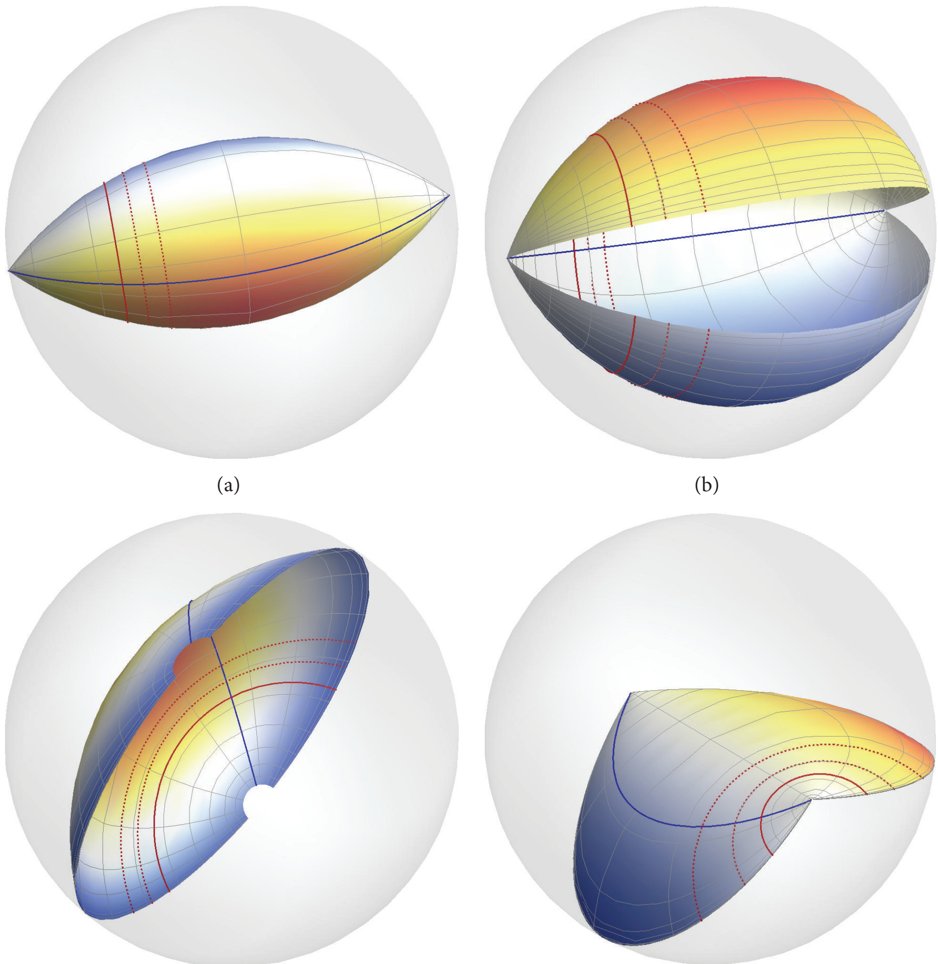
Example 37. Let the profile curve of M be given by

$$\begin{aligned} &\alpha(t) \\ &= \left(2\cosh\frac{t}{\sqrt{3}} - \sinh\frac{t}{\sqrt{3}}, 0, \cosh\frac{t}{\sqrt{3}} - 2\sinh\frac{t}{\sqrt{3}}, \sqrt{2}\right) \end{aligned} \tag{101}$$

such that hyperbolic curvature $\kappa_h = \sqrt{2}/\sqrt{3}$. Then, M is totally umbilical $\overline{G_1}$-invariant surface with $\xi^2 = 2/3$ in $\mathbb{H}^3$ (see Figure 1(d)).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.



(a) (b) (c) (d)

Figure 1: Examples of some $\overline{G_1}$-invariant surfaces.

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