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*Research Article*

# Stability of a Class of Coupled Systems

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We consider a class of coupled systems with damping terms. By using multiplier method and the estimation techniques of the energy, we show that even if the kernel function is nonincreasing and integrable without additional conditions, the energy of the system decays also to zero in a good rate.

## 1. Introduction

This work is motivated by the recent researches on the Cauchy problem for the coupled evolution equations with memory (e.g., Alabau-Boussouira et al. [1], Cannarsa and Sforza [2], Wan and Xiao [3], and Xiao and Liang [4]).

We study the following abstract Cauchy problem for coupled systems with damping terms:

$$u''(t) + Au(t) - \int_0^t g_1(t-s)\, Au(s)\, ds + bv(t) = 0, \quad (1)$$

$$v''(t) + Av(t) - \int_0^t g_2(t-s)\, Av(s)\, ds + bu(t) = 0, \quad (2)$$

$$u(0) = u_0, \qquad u'(0) = u_1, \quad (3)$$

$$v(0) = v_0, \qquad v'(0) = v_1, \quad (4)$$

where $A$ is a positive self-adjoint linear operator in a Hilbert space $H$; $g_1(t)$ and $g_2(t)$ are two nonnegative functions on $[0, +\infty)$ and denote the memory kernel, which will be specified later. The problem arises in the theory of viscoelasticity.

We are concerned with the delay behavior of the energy of the systems. In the real world, for the viscoelastic material, the kernel function is almost all nonincreasing and nonnegative. Therefore, we are more interested in decay behavior when the kernel is nonnegative and nonincreasing. In this case, $\int_t^{+\infty} g(s)ds$ is a strongly positive definite kernel (as in [2, 5]). By using multiplier method and the estimation techniques of the energy, we show that even if the kernel function is nonincreasing and integrable without additional conditions, the energy of the system decays also to zero in a good rate.

Let us recall the following assumptions which were used in related literature:

($I_1$) $A$ is a positive self-adjoint linear operator in $H$, satisfying

$$a\langle Au, u\rangle \geq \|u\|^2, \quad u \in \mathscr{D}(A), \quad (5)$$

for a constant $a > 0$.

($I_2$) $g_i(t) : [0, \infty) \to [0, \infty)$ is a nonincreasing and integrable function such that

$$0 < \int_0^\infty g_i(t)\, dt < 1, \qquad 1 - \int_0^\infty g_i(t)\, dt + ab^2 > 0, \quad (6)$$

where $i = 1, 2$.

A pair $(u, v)$ of functions is called a (classical) solution of (1)–(4) on $[0, T)$, $T > 0$ if

$$u, v \in C^2([0,T); H) \cap C^1\left([0,T); \left[\mathscr{D}\left(\sqrt{A}\right)\right]\right) \cap C([0,T); [\mathscr{D}(A)]) \quad (7)$$

satisfying (1)–(4) for $t \in [0, T)$.

We define the energy of a solution $(u,v)$ of (1)–(4) as

$$\begin{aligned} E(t) &= E_{u,v}(t) \\ &= \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1-\int_0^t g_1(s)\,ds}{2}\left\|\sqrt{A}u(t)\right\|^2 \\ &\quad + \frac{1}{2}\int_0^t g_1(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^2 ds \\ &\quad + \frac{1}{2}\left\|v'(t)\right\|^2 + \frac{1-\int_0^t g_2(s)\,ds}{2}\left\|\sqrt{A}v(t)\right\|^2 \\ &\quad + \frac{1}{2}\int_0^t g_2(t-s)\left\|\sqrt{A}v(s)-\sqrt{A}v(t)\right\|^2 ds \\ &\quad + \frac{1}{2}b\left(\|v(t)+u(t)\|^2 - \|u(t)\|^2 - \|v(t)\|^2\right). \end{aligned} \tag{8}$$

About the information on $\sqrt{A}$, see Xiao and Liang's monograph [6].

**Theorem 1.** *Let $(I_1)$-$(I_2)$ hold. Then, for $u_0, v_0 \in \mathscr{D}(A)$ and $u_1, v_1 \in \mathscr{D}(\sqrt{A})$, (1)–(4) have a unique solution $(u(t), v(t))$ on $[0,\infty)$ and*

$$\begin{aligned} \frac{d}{dt}E(t) &= -\frac{1}{2}g_1(t)\left\|\sqrt{A}u(t)\right\|^2 - \frac{1}{2}g_2(t)\left\|\sqrt{A}v(t)\right\|^2 \\ &\quad + \frac{1}{2}\int_0^t g_1'(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^2 ds \\ &\quad + \frac{1}{2}\int_0^t g_2'(t-s)\left\|\sqrt{A}v(s)-\sqrt{A}v(t)\right\|^2 ds, \quad t \ge 0. \end{aligned} \tag{9}$$

*Proof.* The existence and uniqueness of solution can be obtained by the standard operator theory. Here, we omit it.

Multiplying (1) by $u'(t)$ and (2) by $v'(t)$, respectively, and summing-up, we obtained the equality (21). □

*Remark 2.* From assumption $(I_2)$ and (21), we have

$$E'(t) \le 0, \quad 0 \le E(t) \le E(0), \quad \forall t \ge 0. \tag{10}$$

For any $h \in L^1_{\text{loc}}(0,\infty)$ and any $\varphi \in L^1_{\text{loc}}(0,\infty;H)$, we define

$$h * \varphi(t) = \int_0^t h(t-s)\varphi(s)\,ds, \quad t \ge 0. \tag{11}$$

Next, let us recall the concept of strongly positive definite kernel. It can be found in [2, 5].

*Definition 3.* Set $h(t) \in L^\infty(0,\infty)$; $h(t)$ is called positive definite kernel if, for any $\varphi(s) \in L^2_{\text{loc}}(0,\infty;H)$,

$$\int_0^t \langle h*\varphi(s), \varphi(s)\rangle\, ds \ge 0, \quad \forall t \ge 0. \tag{12}$$

Also, $h(t)$ is said to be a strongly positive definite kernel if there exists a constant $\delta > 0$ such that $h(t) - \delta e^{-t}$ is positive definite, for any $\varphi(s) \in L^2_{\text{loc}}(0,\infty;H)$.

See more properties of the strongly positive definite kernel in [2, 5].

## 2. Result and Proof

**Theorem 4.** *Let $(I_1)$-$(I_2)$ hold, and let $u_0, v_0, u_1$, and $v_1$ be as in Theorem 1. Then, the energy $E(t)$ satisfies*

$$E(t) \le C(t+1)^{-1}, \quad \forall t \ge 0, \tag{13}$$

*where $C > 0$ is a positive constant and depends on the initial data. Moreover,*

$$tE(t) \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty. \tag{14}$$

To prove Theorem 4, we need the following lemmas. From now on, we write

$$G_i(t) := \int_t^\infty g_i(s)\,ds. \tag{15}$$

Then, $G_i(t)$ is a strongly positive definite kernel; see [2, Theorem 2.1].

**Lemma 5.** *Let $(I_1)$-$(I_2)$ hold, $u_0, v_0 \in \mathscr{D}(A)$, and $u_1, v_1 \in \mathscr{D}(\sqrt{A})$. Then, for any $t \ge 0$,*

$$\begin{aligned} &\int_0^t \left\langle G_1 * \sqrt{A}u'(s), \sqrt{A}u'(s)\right\rangle ds \\ &\qquad + \int_0^t \left\langle G_2 * \sqrt{A}v'(s), \sqrt{A}v'(s)\right\rangle ds \le C_1, \end{aligned} \tag{16}$$

*where $C_1 > 0$ depends only on the initial data.*

*Proof.* It follows from (1) that

$$\begin{aligned} &\frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\left\|\sqrt{A}u(t)\right\|^2 - \int_0^t \left\langle g_1 * \sqrt{A}u(s), \sqrt{A}u'(s)\right\rangle ds \\ &\quad = \frac{1}{2}\left\|u'(0)\right\|^2 + \frac{1}{2}\left\|\sqrt{A}u(0)\right\|^2 - \int_0^t \left\langle bv(s), u'(s)\right\rangle ds. \end{aligned} \tag{17}$$

Moreover, taking the inner product of (2) with $v'(t)$ and integrating over $[0,t]$, we obtain

$$\frac{1}{2}\left\|v'(t)\right\|^2+\frac{1}{2}\left\|\sqrt{A}v(t)\right\|^2-\int_0^t\left\langle g_2*\sqrt{A}v(s),\sqrt{A}v'(s)\right\rangle ds=\frac{1}{2}\left\|v'(0)\right\|^2+\frac{1}{2}\left\|\sqrt{A}v(0)\right\|^2-\int_0^t\left\langle bu(s),v'(s)\right\rangle ds. \tag{18}$$

Combining the above two equations and using integration by parts, we get

$$\begin{aligned}&\frac{1}{2}\left\|u'(t)\right\|^2+\frac{1}{2}\left\|\sqrt{A}u(t)\right\|^2-\int_0^t\left\langle g_1*\sqrt{A}u(s),\sqrt{A}u'(s)\right\rangle ds\\&\quad+\frac{1}{2}\left\|v'(t)\right\|^2+\frac{1}{2}\left\|\sqrt{A}v(t)\right\|^2\\&\quad-\int_0^t\left\langle g_2*\sqrt{A}v(s),\sqrt{A}v'(s)\right\rangle ds\\&\quad+b\langle u(t),v(t)\rangle\le\widetilde{C}_1(u_0,u_1,v_0,v_1).\end{aligned}\tag{19}$$

Applying Lemma 3.4 and (3.13) in [2] to the two integral terms on the left-hand side, we have

$$\begin{aligned}&\frac{1}{2}\left\|u'(t)\right\|^2+\frac{1-G_1(0)}{2}\left\|\sqrt{A}u(t)\right\|^2\\&\quad+\int_0^t\left\langle G_1*\sqrt{A}u'(s),\sqrt{A}u'(s)\right\rangle ds\\&\quad+\frac{1}{2}\left\|v'(t)\right\|^2+\frac{1-G_2(0)}{2}\left\|\sqrt{A}v(t)\right\|^2\\&\quad+\int_0^t\left\langle G_2*\sqrt{A}v'(s),\sqrt{A}v'(s)\right\rangle ds\\&\quad+\frac{1}{2}b\left(\|v(t)+u(t)\|^2-\|u(t)\|^2-\|v(t)\|^2\right)\\&\le\frac{G(0)}{2}\left\|\sqrt{A}u(0)\right\|^2-G_1(t)\left\langle\sqrt{A}u(0),\sqrt{A}u(t)\right\rangle\\&\quad-\int_0^t g_1(s)\left\langle\sqrt{A}u(0),\sqrt{A}u(s)\right\rangle ds\\&\quad+\frac{G(0)}{2}\left\|\sqrt{A}v(0)\right\|^2-G_2(t)\left\langle\sqrt{A}v(0),\sqrt{A}v(t)\right\rangle\\&\quad-\int_0^t g_2(s)\left\langle\sqrt{A}v(0),\sqrt{A}v(s)\right\rangle ds\\&\quad+\widetilde{C}_1(u_0,u_1,v_0,v_1).\end{aligned}\tag{20}$$

Noticing $(I_2)$ and Remark 2, we obtain (16). □

**Lemma 6.** *Let $(I_1)$-$(I_2)$ hold, $u_0,v_0\in\mathscr{D}(A)$, and $u_1,v_1\in\mathscr{D}(\sqrt{A})$. Then, for any $t\ge0$,*

$$\begin{aligned}&\int_0^t\left\langle G_1*\sqrt{A}u''(s),\sqrt{A}u''(s)\right\rangle ds\\&\quad+\int_0^t\left\langle G_2*\sqrt{A}v''(s),\sqrt{A}v''(s)\right\rangle ds\le C_2,\end{aligned}\tag{21}$$

*where $C_2>0$ depends only on the initial data.*

*Proof.* Differentiating the systems (1)-(2) with respect to $t$, we get

$$\begin{aligned}&u'''(t)+Au'(t)-g_1(t)Au(0)\\&\quad-\int_0^t g_1(t-s)Au'(s)ds+bv'(t)=0,\\&v'''(t)+Av'(t)-g_2(t)Av(0)\\&\quad-\int_0^t g_2(t-s)Av'(s)ds+bu'(t)=0.\end{aligned}\tag{22}$$

Thus, similar to the proof of the Lemma 5 for the above (22), we deduce (21). □

In view of Lemma 2.9 and (2.14) of [2], (16), and (21), we have

$$\begin{aligned}&\int_0^t\left\|\sqrt{A}u'(s)\right\|^2ds\le C_3,\\&\int_0^t\left\|\sqrt{A}v'(s)\right\|^2ds\le C_3,\end{aligned}\tag{23}$$

where $C_3>0$ depends only on the initial data.

Moreover, in view of (23) and $(I_1)$, we have

$$\int_0^t\left\|u'(s)\right\|^2ds\le C_4,\tag{24}$$

$$\int_0^t\left\|v'(s)\right\|^2ds\le C_4,\tag{25}$$

where $C_4>0$ depends only on the initial data.

**Lemma 7.** *Let $(I_1)$-$(I_2)$ hold, $u_0,v_0\in\mathscr{D}(A)$, and $u_1,v_1\in\mathscr{D}(\sqrt{A})$. Then, for any $t\ge0$,*

$$\int_0^t\left\|\sqrt{A}u(s)\right\|^2ds\le C_5,\tag{26}$$

$$\int_0^t\left\|\sqrt{A}v(s)\right\|^2ds\le C_5,\tag{27}$$

*where $C_5>0$ depends only on the initial data.*

*Proof.* It follow from (1) and (2) that

$$\begin{aligned}
&\int_0^t \left( \left\| \sqrt{A}u(s) \right\|^2 + \left\| \sqrt{A}v(s) \right\|^2 + b\langle v(s), u(s) \rangle \right. \\
&\qquad \left. + b\langle u(s), v(s) \rangle \right) ds \\
&= -\left\langle u'(t), u(t) \right\rangle \Big|_0^t - \left\langle v'(t), v(t) \right\rangle \Big|_0^t \\
&\quad + \int_0^t \left( \left\| u'(s) \right\|^2 + \left\| v'(s) \right\|^2 \right) ds \\
&\quad + \int_0^t \left\langle g_1 * \sqrt{A}u(s), \sqrt{A}u(s) \right\rangle ds \\
&\quad + \int_0^t \left\langle g_2 * \sqrt{A}v(s), \sqrt{A}v(s) \right\rangle ds \\
&\le C + \int_0^t \left\langle g_1 * \sqrt{A}u(s), \sqrt{A}u(s) \right\rangle ds \\
&\quad + \int_0^t \left\langle g_2 * \sqrt{A}v(s), \sqrt{A}v(s) \right\rangle ds \\
&\le C + \frac{G_1(0)}{2} \int_0^t \left\| \sqrt{A}u(s) \right\|^2 ds \\
&\quad + \frac{1}{2G_1(0)} \int_0^t \left\| g_1 * \sqrt{A}u(s) \right\|^2 ds \\
&\quad + \frac{G_2(0)}{2} \int_0^t \left\| \sqrt{A}v(s) \right\|^2 ds \\
&\quad + \frac{1}{2G_2(0)} \int_0^t \left\| g_2 * \sqrt{A}v(s) \right\|^2 ds.
\end{aligned} \tag{28}$$

Note that we have used (24)-(25) in the above calculation. Hence, we have

$$\begin{aligned}
&\int_0^t \left( \left\| \sqrt{A}u(s) \right\|^2 + \left\| \sqrt{A}v(s) \right\|^2 + b\|v(t) + u(t)\|^2 \right. \\
&\qquad \left. - b\|u(t)\|^2 - b\|v(t)\|^2 \right) ds \\
&\le C + \frac{G_1(0)}{2} \int_0^t \left\| \sqrt{A}u(s) \right\|^2 ds \\
&\quad + \frac{1}{2G_1(0)} \int_0^t \left\| g_1 * \sqrt{A}u(s) \right\|^2 ds \\
&\quad + \frac{G_2(0)}{2} \int_0^t \left\| \sqrt{A}v(s) \right\|^2 ds \\
&\quad + \frac{1}{2G_2(0)} \int_0^t \left\| g_2 * \sqrt{A}v(s) \right\|^2 ds.
\end{aligned} \tag{29}$$

On the other hand, we see that

$$\begin{aligned}
&\left\| g_1 * \sqrt{A}u(s) \right\|^2 \le G_1(0)\, g_1 * \left\| \sqrt{A}u(s) \right\|^2, \\
&\left\| g_2 * \sqrt{A}v(s) \right\|^2 \le G_2(0)\, g_2 * \left\| \sqrt{A}v(s) \right\|^2.
\end{aligned} \tag{30}$$

By Young's inequality, we obtain

$$\int_0^t \left\| g_1 * \sqrt{A}u(s) \right\|^2 \le G_1^2(0) \int_0^t \left\| \sqrt{A}u(s) \right\|^2 ds, \tag{31}$$

$$\int_0^t \left\| g_2 * \sqrt{A}u(s) \right\|^2 \le G_2^2(0) \int_0^t \left\| \sqrt{A}u(s) \right\|^2 ds. \tag{32}$$

Putting (31)-(32) into (29), we obtain

$$\begin{aligned}
&\int_0^t \left( \left\| \sqrt{A}u(s) \right\|^2 + \left\| \sqrt{A}v(s) \right\|^2 + b\|v(t) + u(t)\|^2 \right. \\
&\qquad \left. - b\|u(t)\|^2 - b\|v(t)\|^2 \right) ds \\
&\qquad \le C + G_1(0) \int_0^t \left\| \sqrt{A}u(s) \right\|^2 ds \\
&\qquad\quad + G_2(0) \int_0^t \left\| \sqrt{A}v(s) \right\|^2 ds.
\end{aligned} \tag{33}$$

Noticing assumption $(I_2)$, we obtain the desired estimates (26)-(27). □

*Proof of Theorem 4.* First, we estimate the two memory energy terms.

By a direct calculation, we have

$$\begin{aligned}
&\int_0^t \left( \int_0^t g_1(t-s) \left\| \sqrt{A}u(t) - \sqrt{A}u(s) \right\|^2 ds \right) dt \\
&\quad \le C \int_0^t \left( \int_0^t g_1(t-s) \left( \left\| \sqrt{A}u(t) \right\|^2 + \left\| \sqrt{A}u(s) \right\|^2 \right) ds \right) dt \\
&\quad \le C \int_0^t \left( \left\| \sqrt{A}u(t) \right\|^2 \int_0^t g_1(t-s)\, ds \right) dt \\
&\qquad + C \int_0^t \left( \int_0^t g_1(t-s) \left\| \sqrt{A}u(s) \right\|^2 ds \right) dt \\
&\quad \le C \int_0^t \left( \left\| \sqrt{A}u(t) \right\|^2 \int_0^{+\infty} g_1(s)\, ds \right) dt \\
&\qquad + C \int_0^t \left( \int_s^t g_1(t-s) \left\| \sqrt{A}u(s) \right\|^2 dt \right) ds \\
&\quad \le C \int_0^t \left\| \sqrt{A}u(t) \right\|^2 dt \\
&\qquad + C \int_0^t \left( \left\| \sqrt{A}u(s) \right\|^2 \int_0^{+\infty} g_1(t)\, dt \right) ds \\
&\quad \le C + C \int_0^t \left\| \sqrt{A}u(s) \right\|^2 ds.
\end{aligned} \tag{34}$$

Hence, by (26), we obtain

$$\int_0^t \left( \int_0^t g_1(t-s) \left\| \sqrt{A}u(t) - \sqrt{A}u(s) \right\|^2 ds \right) dt \le C_6. \tag{35}$$

Similarly, we have

$$\int_0^t \left( \int_0^t g_2 (t-s) \left\| \sqrt{A} v(t) - \sqrt{A} v(s) \right\|^2 ds \right) dt \le C_6. \quad (36)$$

Thus, (24)–(27) and (35)-(36) yield

$$\int_0^\infty E(t)\, dt \le C, \quad (37)$$

for a positive constant $C$. As $E'(s) \le 0$, we have

$$\frac{d}{dt}(tE(t)) \le E(t), \quad t \ge 0. \quad (38)$$

Accordingly, (37) means that

$$tE(t) \le \int_0^t E(s)\, ds \le C, \quad t \ge 0. \quad (39)$$

Hence, the estimate (13) follows. Furthermore, since the integral $\int_0^{+\infty} E(t)dt$ is convergent, it follows that

$$tE(t) \le 2 \int_{t/2}^t E(s)\, ds \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty, \quad (40)$$

via the Cauchy convergence principle. Then, the proof of Theorem 4 is completed. □

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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