

Research Article

Multiplicity of Positive Solutions for a p - q -Laplacian Type Equation with Critical Nonlinearities

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We study the effect of the coefficient $f(x)$ of the critical nonlinearity on the number of positive solutions for a p - q -Laplacian equation. Under suitable assumptions for $f(x)$ and $g(x)$, we should prove that for sufficiently small $\lambda > 0$, there exist at least k positive solutions of the following p - q -Laplacian equation, $-\Delta_p u - \Delta_q u = f(x)|u|^{p^*-2}u + \lambda g(x)|u|^{r-2}u$ in Ω , $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain, $N > p$, $1 < q < N(p-1)/(N-1) < p \leq \max\{p, p^* - q/(p-1)\} < r < p^*$, $p^* = Np/(N-p)$ is the critical Sobolev exponent, and $\Delta_s u = \operatorname{div}(|\nabla u|^{s-2}\nabla u)$ is the s -Laplacian of u .

1. Introduction

This paper is concerned with the multiplicity of positive solutions to the following p - q -Laplacian equation with critical nonlinearities:

$$\begin{aligned} -\Delta_p u - \Delta_q u &= f(x)|u|^{p^*-2}u + \lambda g(x)|u|^{r-2}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (E_\lambda)$$

where $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain with smooth boundary $\partial\Omega$, $1 < q < p < N$, $\lambda > 0$, and $\Delta_s u = \operatorname{div}(|\nabla u|^{s-2}\nabla u)$ is the s -Laplacian of u , and assume that

(H1) $1 < q < N(p-1)/(N-1) < p \leq \max\{p, p^* - q/(p-1)\} < r < p^* = Np/(N-p)$;

(H2) f and g are positive continuous functions in $\overline{\Omega}$;

(H3) There exist k points a^1, a^2, \dots, a^k in Ω such that $f(a^i)$ are strict local maxima satisfying

$$f(a^i) = \max_{x \in \overline{\Omega}} f(x) = 1 \quad \text{for } 1 \leq i \leq k, \quad (1)$$

and for some $\beta > N/(p-1)$, $f(x) = f(a^i) + O(|x-a^i|^\beta)$ as $x \rightarrow a^i$ uniformly in i .

Problem (E_λ) comes, for example, from a general reaction-diffusion system

$$u_t = \operatorname{div}[H(u)\nabla u] + c(x, u), \quad (2)$$

where $H(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. This system has a wide range of applications in physics and related science such as biophysics, plasma physics, and chemical reaction design. In such applications, the function u describes a concentration, the first term on the right-hand side of (2) corresponds to the diffusion with a diffusion coefficient $H(u)$, whereas the second one is the reaction and relates to sources and loss processes. Typically, in chemical and biological applications, the reaction term $c(x, u)$ has a polynomial form with respect to the concentration u .

The stationary solution of (2) was studied by many authors; that is, many works are considered the solutions of the following problem:

$$-\operatorname{div}[H(u)\nabla u] = c(x, u). \quad (3)$$

See [1–5] for different $c(x, u)$. In the present paper we are concerned with problem (E_λ) in a bounded domain with $c(x, u) = f(x)|u|^{p^*-2}u + \lambda g(x)|u|^{r-2}u$ in (3). Recently, in [6], the authors obtain the existence of $\operatorname{cat}_\Omega(\Omega)$ positive solutions

of problem (E_λ) for $N > p^2$ and $f(x) \equiv g(x) \equiv 1$ when condition (H1) holds, where $\text{cat}_\Omega(\Omega)$ denotes the Lusternik-Schnirelmann category of Ω in itself.

Specially, if $p = q$, (E_λ) can be reduced to the following elliptic problems:

$$-\Delta_p u = f(x) |u|^{p^*-2} u + \lambda g(x) |u|^{r-2} u \quad \text{in } \Omega, \tag{4}$$

$$u = 0 \quad \text{on } \partial\Omega.$$

After the well-known results of Brézis and Nirenberg [7], who studied (4) in the case of $p = r = 2$ and $f(x) \equiv g(x) \equiv 1$, a lot of problems involving the critical growth in bounded and unbounded domains have been considered; see, for example, [8–10] and reference therein. In particular, the first multiplicity result for (4) has been achieved by Rey in [11] in the semilinear case. Precisely Rey proved that if $N \geq 5$, $p = r = 2$, and $f(x) \equiv g(x) \equiv 1$, for λ small enough, problem (4) has at least $\text{cat}_\Omega(\Omega)$ solutions. Furthermore, Alves and Ding in [12] obtained the existence of $\text{cat}_\Omega(\Omega)$ positive solutions to problem (4) with $p \geq 2$, $r \in [p, p^*)$, and $f(x) \equiv g(x) \equiv 1$. Finally, we mention that [13] studied (4) when $1 < r < p < N$ and f, g are sign-changing and verified the existence of two positive solutions for $\lambda \in (0, \lambda_0)$ for some positive constant λ_0 .

The main purpose of this paper is to analyze the effect of the coefficient $f(x)$ of the critical nonlinearity to prove the multiplicity of positive solutions of problem (E_λ) for small $\lambda > 0$. By the similar argument in [14], we can construct the k compact Palais-Smale sequences that are suitably localized in correspondence of k maximum points of f . Under some assumptions (H1)–(H3), we could show that there are at least k positive solutions of problem (E_λ) for sufficiently small $\lambda > 0$.

This paper is organized as follows. First of all, we study the argument of the Nehari manifold \mathcal{M}_λ . Next, we prove the existence of a positive solution $u_0 \in \mathcal{M}_\lambda$. Finally, we show that the condition (H3) affects the number of positive solution of (E_λ) ; that is, there are at least k critical points $u_i \in \mathcal{M}_\lambda$ of J_λ such that $J_\lambda(u_i) = \alpha_\lambda^i$ ((PS)-value) for $1 \leq i \leq k$.

The main results of this paper are given as follows.

Theorem 1. *Suppose that (H1)–(H3) hold; then problem (E_λ) has a positive solution u_0 in $W_0^{1,p}(\Omega)$ for all $\lambda > 0$.*

Theorem 2. *Suppose that (H1)–(H3) hold; then there exists a $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, problem (E_λ) admits at least k positive solutions in $W_0^{1,p}(\Omega)$.*

2. Preliminaries

In what follows, we denote by $\|\cdot\|_p, |\cdot|_p$ the norms on $W_0^{1,p}(\Omega)$ and $L^p(\Omega)$, respectively; that is,

$$\|u\|_p = \left(\int_\Omega |\nabla u|^p dx \right)^{1/p}, \quad |u|_p = \left(\int_\Omega |u|^p dx \right)^{1/p}. \tag{5}$$

We denote the dual space of $W_0^{1,p}(\Omega)$ by $W'(\Omega)$. Set also

$$D^{1,p}(\mathbf{R}^N) := \left\{ u \in L^{p^*}(\mathbf{R}^N) : \frac{\partial u}{\partial x_i} \in L^p(\mathbf{R}^N) \text{ for } i = 1, 2, \dots, N \right\} \tag{6}$$

equipped with the norm

$$\|u\|_* = \left(\int_{\mathbf{R}^N} |\nabla u|^p dx \right)^{1/p}. \tag{7}$$

We will denote by S the best Sobolev constant as follows:

$$S = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_p^p}{|u|_{p^*}^p}. \tag{8}$$

It is well known that S is independent of Ω and is never achieved except when $\Omega = \mathbf{R}^N$ (see [15]). Throughout this paper, we denote the Lebesgue measure of Ω by $|\Omega|$ and denote a ball centered at $a \in \mathbf{R}^N$ with radius r by $B_r(a)$ and also denote positive constants (possibly different) by C, C_i . $O(\varepsilon^t)$ denotes $|O(\varepsilon^t)|/\varepsilon^t \leq C$, $o(\varepsilon^t)$ denotes $|o(\varepsilon^t)|/\varepsilon^t \rightarrow 0$ as $\varepsilon \rightarrow 0$, and $o_n(1)$ denotes $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$.

Associated with (E_λ) , we consider the energy functional J_λ in $W_0^{1,p}(\Omega)$, for each $u \in W_0^{1,p}(\Omega)$,

$$J_\lambda(u) = \frac{1}{p} \|u\|_p^p + \frac{1}{q} \|u\|_q^q - \frac{1}{p^*} \int_\Omega f(x) |u|^{p^*} dx - \frac{1}{r} \int_\Omega \lambda g(x) |u|^r dx. \tag{9}$$

It is well known that J_λ is of C^1 in $W_0^{1,p}(\Omega)$ and the solutions of (E_λ) are the critical points of the energy functional J_λ (see [16]).

We define the Nehari manifold

$$\mathcal{M}_\lambda := \{u \in W_0^{1,p}(\Omega) \setminus \{0\} : \langle J'_\lambda(u), u \rangle = 0\}, \tag{10}$$

where

$$\langle J'_\lambda(u), u \rangle = \|u\|_p^p + \|u\|_q^q - \int_\Omega f(x) |u|^{p^*} dx - \int_\Omega \lambda g(x) |u|^r dx = 0. \tag{11}$$

The Nehari manifold \mathcal{M}_λ contains all nontrivial solutions of (E_λ) .

Note that J_λ is not bounded from below in $W_0^{1,p}(\Omega)$. From the following lemma, we have that J_λ is bounded from below on the Nehari manifold \mathcal{M}_λ .

Lemma 3. *Suppose that $1 < q < p < r < p^*$ and (H2) hold. Then for any $\lambda > 0$, one has that J_λ is bounded from below on \mathcal{M}_λ . Moreover, $J_\lambda(u) > 0$ for all $u \in \mathcal{M}_\lambda$.*

Proof. For $u \in \mathcal{M}_\lambda$, (10) leads to

$$\begin{aligned}
 J_\lambda(u) &= \left(\frac{1}{p} - \frac{1}{r}\right) \|u\|_p^p + \left(\frac{1}{q} - \frac{1}{r}\right) \|u\|_q^q \\
 &\quad + \left(\frac{1}{r} - \frac{1}{p^*}\right) \int_\Omega f(x) |u|^{p^*} dx > 0.
 \end{aligned}
 \tag{12}$$

Define

$$\alpha_\lambda := \inf_{u \in \mathcal{M}_\lambda} J_\lambda(u). \tag{13}$$

Now we show that J_λ possesses the mountain-pass (MP, in short) geometry.

Lemma 4. *Suppose $1 < q < p < r < p^*$ and (H2) holds. Then for any $\lambda > 0$, one has that*

- (i) *there exist positive numbers R and d_0 such that $J_\lambda(u) \geq d_0$ for $\|u\|_p = R$;*
- (ii) *there exists $\bar{u} \in W_0^{1,p}(\Omega)$ such that $\|\bar{u}\|_p > R$ and $J_\lambda(\bar{u}) < 0$.*

Proof. (i) By (8), the Hölder inequality, and the Sobolev embedding theorem, we have that

$$\begin{aligned}
 J_\lambda(u) &\geq \frac{1}{p} \|u\|_p^p - \frac{1}{p^*} \int_\Omega f(x) |u|^{p^*} dx \\
 &\quad - \frac{1}{r} \int_\Omega \lambda g(x) |u|^r dx \\
 &\geq \frac{1}{p} \|u\|_p^p - \frac{1}{p^*} S^{-p^*/p} \|u\|_p^{p^*} \\
 &\quad - \frac{1}{r} \lambda |g|_\infty |\Omega|^{(p^*-r)/p^*} S^{-r/p} \|u\|_p^r.
 \end{aligned}
 \tag{14}$$

Hence, there exist positive R and d_0 such that $J_\lambda(u) \geq d_0$ for $\|u\|_p = R$.

(ii) For any $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, from

$$\begin{aligned}
 J_\lambda(tu) &= \frac{t^p}{p} \|u\|_p^p + \frac{t^q}{q} \|u\|_q^q - \frac{t^{p^*}}{p^*} \int_\Omega f(x) |u|^{p^*} dx \\
 &\quad - \frac{t^r}{r} \int_\Omega \lambda g(x) |u|^r dx,
 \end{aligned}
 \tag{15}$$

we have $\lim_{t \rightarrow \infty} J_\lambda(tu) = -\infty$. For fixed some $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, there exist $\bar{t} > 0$ such that $\|\bar{t}u\|_p > R$ and $J_\lambda(\bar{t}u) < 0$. Let $\bar{u} = \bar{t}u$. \square

Define

$$\phi_\lambda(u) := \langle J'_\lambda(u), u \rangle. \tag{16}$$

Then for $u \in \mathcal{M}_\lambda$,

$$\begin{aligned}
 \langle \phi'_\lambda(u), u \rangle &= p \|u\|_p^p + q \|u\|_q^q \\
 &\quad - p^* \int_\Omega f(x) |u|^{p^*} dx - r \int_\Omega \lambda g(x) |u|^r dx \\
 &= (p^* - r) \int_\Omega \lambda g(x) |u|^r dx \\
 &\quad - (p^* - p) \|u\|_p^p - (p^* - q) \|u\|_q^q \\
 &= (p - r) \|u\|_p^p + (q - r) \|u\|_q^q \\
 &\quad + (r - p^*) \int_\Omega f(x) |u|^{p^*} dx < 0.
 \end{aligned}
 \tag{17}$$

Lemma 5. *Suppose that $1 < q < p < r < p^*$ and (H2) holds. If $u_0 \in \mathcal{M}_\lambda$ satisfies*

$$J_\lambda(u_0) = \min_{u \in \mathcal{M}_\lambda} J_\lambda(u) = \alpha_\lambda, \tag{18}$$

then u_0 is a solution of (E_λ) .

Proof. By (17), $\langle \phi'_\lambda(u), u \rangle < 0$ for $u \in \mathcal{M}_\lambda$. Since $J_\lambda(u_0) = \min_{u \in \mathcal{M}_\lambda} J_\lambda(u)$, by the Lagrange multiplier theorem, there is $\tau \in \mathbf{R}$ such that $J'_\lambda(u_0) = \tau \phi'_\lambda(u_0)$ in $W^1(\Omega)$. This implies that

$$0 = \langle J'_\lambda(u_0), u_0 \rangle = \tau \langle \phi'_\lambda(u_0), u_0 \rangle. \tag{19}$$

It then follows that $\tau = 0$ and $J'_\lambda(u_0) = 0$ in $W^1(\Omega)$. Thus, u_0 is a nontrivial solution of (E_λ) and $J_\lambda(u_0) = \alpha_\lambda$. \square

Lemma 6. *Suppose that $1 < q < p < r < p^*$ and (H2) holds. For each $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, there exists a unique positive number t_u such that $t_u u \in \mathcal{M}_\lambda$ and $J_\lambda(t_u u) = \sup_{t \geq 0} J_\lambda(tu)$ for any $\lambda > 0$.*

Proof. For fixed $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, consider

$$h(t) = J_\lambda(tu) = \frac{t^p}{p} \|u\|_p^p + \frac{t^q}{q} \|u\|_q^q \tag{20}$$

$$- \frac{t^{p^*}}{p^*} \int_\Omega f(x) |u|^{p^*} dx - \frac{t^r}{r} \int_\Omega \lambda g(x) |u|^r dx.$$

Since $h(0) = 0$, $\lim_{t \rightarrow \infty} h(t) = -\infty$, by Lemma 4(i), then it is easy to see that there exists a unique positive number t_u such that $\sup_{t \geq 0} h(t)$ is achieved at t_u . This means that $h'(t_u) = 0$; that is, $t_u u \in \mathcal{M}_\lambda$. \square

We will denote by $\tilde{\alpha}_\lambda$ the MP level:

$$\tilde{\alpha}_\lambda := \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \sup_{t \geq 0} J_\lambda(tu). \tag{21}$$

Then we have the following result.

Lemma 7. *Suppose that $1 < q < p < r < p^*$ and (H2) holds, then $\alpha_\lambda = \tilde{\alpha}_\lambda$ for any $\lambda > 0$.*

Proof. By Lemma 6, we have

$$\begin{aligned} \tilde{\alpha}_\lambda &= \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \sup_{t \geq 0} J_\lambda(tu) = \inf_{t, u \in \mathcal{M}_\lambda} J_\lambda(tu) \\ &\geq \inf_{u \in \mathcal{M}_\lambda} J_\lambda(u) = \alpha_\lambda. \end{aligned} \quad (22)$$

On the other hand, for $u \in \mathcal{M}_\lambda$, by Lemma 6, we have $t_u = 1$ and $J_\lambda(u) = \sup_{t \geq 0} J_\lambda(tu)$. Hence,

$$\begin{aligned} \alpha_\lambda &= \inf_{u \in \mathcal{M}_\lambda} J_\lambda(u) = \inf_{u \in \mathcal{M}_\lambda} \sup_{t \geq 0} J_\lambda(tu) \\ &\geq \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \sup_{t \geq 0} J_\lambda(tu) = \tilde{\alpha}_\lambda. \end{aligned} \quad (23)$$

Now the desired result follows from (22) and (23). \square

Remark 8. By Lemma 7 and the definition, it is apparent that $\alpha_{\lambda_1} \leq \alpha_{\lambda_2}$ if $\lambda_1 \geq \lambda_2$; that is, α_λ is nonincreasing in λ . Moreover, by Lemma 4(i), for any $\lambda_0 > 0$, there exists a $d = d(\lambda_0)$, related to the MP geometry, such that

$$0 < d \leq \alpha_\lambda \leq \alpha_0 \quad \forall \lambda \in [0, \lambda_0]. \quad (24)$$

Here α_0 is the MP level associated to the functional

$$J_0(u) = \frac{1}{p} \|u\|_p^p + \frac{1}{q} \|u\|_q^q - \frac{1}{p^*} \int_\Omega f(x) |u|^{p^*} dx. \quad (25)$$

3. (PS)_c-Condition in $W_0^{1,p}(\Omega)$ for J_λ

First, we define the Palais-Smale (denote by (PS)) sequence, (PS)-value, and (PS)-conditions in $W_0^{1,p}(\Omega)$ for J_λ .

Definition 9. (i) For $c \in \mathbf{R}$, a sequence $\{u_n\}$ is a (PS)_c-sequence in $W_0^{1,p}(\Omega)$ for J_λ if $J_\lambda(u_n) = c + o_n(1)$ and $J'_\lambda(u_n) = o_n(1)$ strongly in W' (Ω) as $n \rightarrow \infty$.

(ii) $c \in \mathbf{R}$ is a (PS)-value in $W_0^{1,p}(\Omega)$ for J_λ if there exists a (PS)_c-sequence in $W_0^{1,p}(\Omega)$ for J_λ .

(iii) J_λ satisfies the (PS)_c-condition in $W_0^{1,p}(\Omega)$ if every (PS)_c-sequence $\{u_n\}$ in $W_0^{1,p}(\Omega)$ for J_λ contains a convergent subsequence.

Applying Ekeland's variational principle and using the same argument as in Cao and Zhou [17] or Tarantello [18], we have the following lemma.

Lemma 10. *Suppose that $1 < q < p < r < p^*$ and (H2) holds. Then for any $\lambda > 0$, there exists a (PS) _{α_λ} -sequence $\{u_n\}$ in \mathcal{M}_λ for J_λ .*

To prove the existence of positive solutions, we claim that J_λ satisfies the (PS)_c-condition in $W_0^{1,p}(\Omega)$ for $c \in (0, (1/N)S^{N/p})$.

Lemma 11. *Suppose that $1 < q < p < r < p^*$, $|f|_\infty = 1$, and (H2) holds. Then for any $\lambda > 0$, J_λ satisfies the (PS)_c-condition in $W_0^{1,p}(\Omega)$ for all $c \in (0, (1/N)S^{N/p})$.*

Proof. Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a (PS)_c-sequence for J_λ which satisfies

$$J_\lambda(u_n) = c + o_n(1), \quad J'_\lambda(u_n) = o_n(1) \quad \text{in } W'(\Omega). \quad (26)$$

Then

$$\begin{aligned} c + s_n + \frac{t_n \|u_n\|}{p} &\geq J_\lambda(u_n) - \frac{1}{r} \langle J'_\lambda(u_n), u_n \rangle \\ &= \left(\frac{1}{p} - \frac{1}{r} \right) \|u_n\|_p^p + \left(\frac{1}{q} - \frac{1}{r} \right) \|u_n\|_q^q \\ &\quad + \left(\frac{1}{r} - \frac{1}{p^*} \right) \int_\Omega f(x) |u_n|^{p^*} dx \\ &\geq \frac{r-p}{rp} \|u_n\|_p^p, \end{aligned} \quad (27)$$

where $s_n = o_n(1)$, $t_n = o_n(1)$, as $n \rightarrow \infty$. It follows that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, there exist a subsequence still denoted by $\{u_n\}$ and $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega), \\ u_n &\rightarrow u \quad \text{strongly in } L^s(\Omega) \quad \forall 1 \leq s < p^*, \\ u_n &\rightarrow u \quad \text{a.e. in } \Omega. \end{aligned} \quad (28)$$

Furthermore, we have that $J'_\lambda(u) = 0$ in $W'(\Omega)$. By g being continuous on $\bar{\Omega}$, we get

$$\lambda \int_\Omega g(x) |u_n|^r dx = \lambda \int_\Omega g(x) |u|^r dx + o_n(1). \quad (29)$$

Let $v_n = u_n - u$. Then by f being positive continuous on $\bar{\Omega}$ and Brézis-Lieb lemma (see [19]), we obtain

$$\begin{aligned} \|v_n\|_p^p &= \|u_n\|_p^p - \|u\|_p^p + o_n(1), \\ \|v_n\|_q^q &= \|u_n\|_q^q - \|u\|_q^q + o_n(1), \\ \int_\Omega f(x) |v_n|^{p^*} dx &= \int_\Omega f(x) |u_n|^{p^*} dx - \int_\Omega f(x) |u|^{p^*} dx + o_n(1). \end{aligned} \quad (30)$$

From (26)–(30), we can deduce that

$$\begin{aligned} \frac{1}{p} \|v_n\|_p^p + \frac{1}{q} \|v_n\|_q^q - \frac{1}{p^*} \int_\Omega f(x) |v_n|^{p^*} dx \\ = c - J_\lambda(u) + o_n(1), \end{aligned} \quad (31)$$

$$\|v_n\|_p^p + \|v_n\|_q^q = \int_\Omega f(x) |v_n|^{p^*} dx + o_n(1). \quad (32)$$

Without loss of generality, we may assume that

$$\|v_n\|_p^p = a + o_n(1), \quad \|v_n\|_q^q = b + o_n(1). \quad (33)$$

So (32) and $|f|_\infty = 1$ imply that

$$\int_\Omega |v_n|^{p^*} dx \geq \int_\Omega f(x) |v_n|^{p^*} dx = a + b + o_n(1). \quad (34)$$

By the Sobolev inequality and (33) and (34), we have $\|v_n\|_p^p \geq S(\int_\Omega |v_n|^{p^*} dx)^{p/p^*}$ and

$$a \geq S(a+b)^{p/p^*} \geq Sa^{p/p^*}. \quad (35)$$

If $a > 0$, then (35) implies that $a \geq S^{N/p}$, combined with (31), (33)–(35) and Lemma 3, $1 < q < p < p^*$, as $n \rightarrow \infty$; we get

$$c = \frac{a}{p} + \frac{b}{q} - \frac{a+b}{p^*} + J_\lambda(u) \geq \frac{1}{N}a \geq \frac{1}{N}S^{N/p}, \quad (36)$$

which is a contradiction. So, we have $a = 0$; J_λ satisfies the $(PS)_c$ -condition in $W_0^{1,p}(\Omega)$ for all $c \in (0, (1/N)S^{N/p})$. \square

4. Existence of k Positive Solutions

In this section, we first give some preliminary notations and useful lemmas.

Choose $r_0 > 0$ small enough such that $\overline{B_{r_0}(a^i)} \subset \Omega$ and $\overline{B_{r_0}(a^i)} \cap \overline{B_{r_0}(a^j)} = \emptyset$ for $i \neq j$, $i, j = 1, 2, \dots, k$.

Define

$$Q_i(u) = \frac{\int_\Omega \phi_i(x) |u|^{p^*} dx}{\int_\Omega |u|^{p^*} dx}, \quad (37)$$

$$\phi_i(x) = \min \{1, |x - a^i|\}, \quad 1 \leq i \leq k.$$

Then we have the following separation result.

Lemma 12. *If $Q_i(u) \leq r_0/3$ and $Q_j(u) \leq r_0/3$ for $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, then $i = j$.*

Proof. For any $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ satisfying $Q_i(u) \leq r_0/3$ ($1 \leq i \leq k$), we get

$$\begin{aligned} \frac{r_0}{3} \int_\Omega |u|^{p^*} dx &\geq \int_\Omega \phi_i(x) |u|^{p^*} dx \\ &\geq \int_{\Omega \setminus B_{r_0}(a^i)} \phi_i(x) |u|^{p^*} dx \\ &\geq r_0 \int_{\Omega \setminus B_{r_0}(a^i)} |u|^{p^*} dx, \end{aligned} \quad (38)$$

which implies that

$$\int_\Omega |u|^{p^*} dx \geq 3 \int_{\Omega \setminus B_{r_0}(a^i)} |u|^{p^*} dx, \quad 1 \leq i \leq k. \quad (39)$$

Hence, from (39), we obtain

$$\begin{aligned} 2 \int_\Omega |u|^{p^*} dx &\geq 3 \left(\int_{\Omega \setminus B_{r_0}(a^i)} |u|^{p^*} dx + \int_{\Omega \setminus B_{r_0}(a^j)} |u|^{p^*} dx \right) \\ &\geq 3 \int_\Omega |u|^{p^*} dx \quad \text{if } i \neq j, \end{aligned} \quad (40)$$

which is a contradiction. \square

For $i = 1, 2, \dots, k$, we set

$$\begin{aligned} \mathcal{N}_\lambda^i &:= \left\{ u \in \mathcal{M}_\lambda : Q_i(u) < \frac{r_0}{3} \right\} \\ \partial \mathcal{N}_\lambda^i &:= \left\{ u \in \mathcal{M}_\lambda : Q_i(u) = \frac{r_0}{3} \right\}, \end{aligned} \quad (41)$$

and define

$$\alpha_\lambda^i := \inf_{\mathcal{N}_\lambda^i} J_\lambda(u), \quad \tilde{\alpha}_\lambda^i := \inf_{\partial \mathcal{N}_\lambda^i} J_\lambda(u). \quad (42)$$

Now let us assume that (H1)–(H3) hold. From conditions (H2) and (H3), we can choose a $\rho \in (0, r_0/2)$ small enough and there exist some positive constants γ_1, γ_2 such that for $1 \leq i \leq k$, we have

$$\overline{\bigcup_{1 \leq i \leq k} B_{2\rho}(a^i)} \subset \Omega,$$

$$|f(x) - f(a^i)| \leq \gamma_1 |x - a^i|^\beta \quad \forall x \in \overline{B_{2\rho}(a^i)}, \quad (43)$$

$$g(x) \geq \gamma_2 \quad \forall x \in \overline{\bigcup_{1 \leq i \leq k} B_{2\rho}(a^i)},$$

for some $\beta > N/(p-1)$. For $i \in \{1, 2, \dots, k\}$ and $\varepsilon > 0$, we define

$$u_\varepsilon^{a^i}(x) = \frac{\eta_i(x)}{\left(\varepsilon + |x - a^i|^{p/(p-1)}\right)^{(N-p)/p}}, \quad (44)$$

$$v_\varepsilon^{a^i}(x) = \varepsilon^{(N-p)/p^2} u_\varepsilon^{a^i}(x),$$

where $\eta_i \in C_0^\infty(B_{2\rho}(a^i))$ is a function such that $0 \leq \eta_i(x) \leq 1$ and $\eta_i(x) \equiv 1$ on $B_\rho(a^i)$. Then we obtain the following estimates (see [20]):

$$\begin{aligned} &\int_\Omega \left| u_\varepsilon^{a^i} \right|^t dx \\ &= \begin{cases} K_1 \varepsilon^{(N(p-1)-t(N-p))/p} + O(1), & t > \frac{N(p-1)}{p}, \\ K_1 |\ln \varepsilon| + O(1), & t = \frac{N(p-1)}{p}, \\ O(1), & t < \frac{N(p-1)}{p}, \end{cases} \end{aligned} \quad (45)$$

$$\begin{aligned} &\int_\Omega \left| \nabla u_\varepsilon^{a^i} \right|^t dx \\ &= \begin{cases} K_2 \varepsilon^{(t+N(p-1)-tN)/p} + O(1), & t > \frac{N(p-1)}{p}, \\ K_2 |\ln \varepsilon| + O(1), & t = \frac{N(p-1)}{p}, \\ O(1), & t < \frac{N(p-1)}{p}. \end{cases} \end{aligned} \quad (46)$$

From (43)–(46) [13, Lemma 4.2] and conditions (H2)–(H3), we can deduce the following estimates:

$$\int_{\Omega} |\nabla v_{\varepsilon}^{a^i}|^p dx = K_2 + O(\varepsilon^{(N-p)/p}), \tag{47}$$

$$\int_{\Omega} |\nabla v_{\varepsilon}^{a^i}|^q dx = K_2 + O(\varepsilon^{q(N-p)/p^2}),$$

$$\int_{\Omega} f(x) |v_{\varepsilon}^{a^i}|^{p^*} dx = K_3^{p^*/p} + O(\varepsilon^{N/p}), \tag{48}$$

$$\int_{\Omega} |v_{\varepsilon}^{a^i}|^r dx = K_1 \varepsilon^{((p-1)/p)(N-r((N-p)/p))} + O(\varepsilon^{r(N-p)/p^2}), \tag{49}$$

where $K_1, K_2,$ and K_3 are positive constants independent of ε , and $S = K_2/K_3$ is the best Sobolev constant given in (8).

Next, we will investigate the effect of the coefficient $f(x)$ to find some Palais-Smale sequences which are used to prove Theorem 2.

Lemma 13. *If (H1)–(H3) hold, then for any $i \in \{1, 2, \dots, k\}$ and any $\lambda > 0$, there exists a $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ one has*

$$\sup_{t \geq 0} J_{\lambda}(tv_{\varepsilon}^{a^i}) < \frac{1}{N} S^{N/p} \quad \text{uniformly in } i. \tag{50}$$

In particular, $0 < \alpha_{\lambda} \leq \alpha_{\lambda}^i < (1/N)S^{N/p}$ for all $\lambda > 0$.

Proof. By Lemma 6, there exists a $t_{\varepsilon}^i > 0$ such that $t_{\varepsilon}^i v_{\varepsilon}^{a^i} \in \mathcal{M}_{\lambda}$. Furthermore,

$$\begin{aligned} Q_i(t_{\varepsilon}^i v_{\varepsilon}^{a^i}) &= \frac{\int_{\Omega} \phi_i(x) |v_{\varepsilon}^{a^i}|^{p^*} dx}{\int_{\Omega} |v_{\varepsilon}^{a^i}|^{p^*} dx} \\ &= \frac{\int_{\Omega_{\varepsilon}} \phi_i(a^i + \varepsilon y) |\eta_i(a^i + \varepsilon y) V(y)|^{p^*} dy}{\int_{\Omega_{\varepsilon}} |\eta_i(a^i + \varepsilon y) V(y)|^{p^*} dy} \tag{51} \\ &\rightarrow \phi_i(a^i) = 0 \quad \text{as } \varepsilon \rightarrow 0, \end{aligned}$$

where $\Omega_{\varepsilon} = \{x : \varepsilon x + a^i \in \Omega\}$ and $V(y) = 1/(1 + |y|^{p/(p-1)})^{(N-p)/p}$. Hence, there exists an $\varepsilon_1 > 0$ small enough such that for any $\varepsilon \in (0, \varepsilon_1)$, we have

$$Q_i(t_{\varepsilon}^i v_{\varepsilon}^{a^i}) < \frac{r_0}{3}, \tag{52}$$

which implies $t_{\varepsilon}^i v_{\varepsilon}^{a^i} \in \mathcal{N}_{\lambda}^i$ for any $\varepsilon \in (0, \varepsilon_1)$, and then

$$0 < \alpha_{\lambda} \leq \alpha_{\lambda}^i \leq J_{\lambda}(t_{\varepsilon}^i v_{\varepsilon}^{a^i}) \leq \sup_{t \geq 0} J_{\lambda}(tv_{\varepsilon}^{a^i}) = \sup_{t \geq 0} J_{\lambda}(tv_{\varepsilon}^{a^i}). \tag{53}$$

Set

$$\begin{aligned} h(t) &= J_{\lambda}(tv_{\varepsilon}^{a^i}) = \frac{t^p}{p} \|v_{\varepsilon}^{a^i}\|_p^p dx + \frac{t^q}{q} \|v_{\varepsilon}^{a^i}\|_q^q \\ &\quad - \frac{t^{p^*}}{p^*} \int_{\Omega} f(x) |v_{\varepsilon}^{a^i}|^{p^*} dx - \frac{t^r}{r} \int_{\Omega} \lambda g(x) |v_{\varepsilon}^{a^i}|^r dx. \end{aligned} \tag{54}$$

Since $h(0) = 0, \lim_{t \rightarrow +\infty} h(t) = -\infty$, then there exists a t_{ε} such that $\sup_{t \geq 0} J_{\lambda}(tv_{\varepsilon}^{a^i}) = J_{\lambda}(t_{\varepsilon} v_{\varepsilon}^{a^i})$ hold, and then t_{ε} satisfies

$$\begin{aligned} 0 &= h'(t_{\varepsilon}) = t_{\varepsilon}^{p-1} \|v_{\varepsilon}^{a^i}\|_p^p + t_{\varepsilon}^{q-1} \|v_{\varepsilon}^{a^i}\|_q^q \\ &\quad - t_{\varepsilon}^{p^*-1} \int_{\Omega} f(x) |v_{\varepsilon}^{a^i}|^{p^*} dx - t_{\varepsilon}^{r-1} \int_{\Omega} \lambda g(x) |v_{\varepsilon}^{a^i}|^r dx; \end{aligned} \tag{55}$$

then we have

$$\|v_{\varepsilon}^{a^i}\|_p^p + t_{\varepsilon}^{q-p} \|v_{\varepsilon}^{a^i}\|_q^q > t_{\varepsilon}^{p^*-p} \int_{\Omega} f(x) |v_{\varepsilon}^{a^i}|^{p^*} dx. \tag{56}$$

From (47) and (48), fixing any $\varepsilon_2 > 0$ small enough, there exists $T_1 > 0$ such that

$$t_{\varepsilon} \leq T_1 \quad \text{for any } \varepsilon \in (0, \varepsilon_2). \tag{57}$$

Also, from (55), we obtain

$$\|v_{\varepsilon}^{a^i}\|_p^p < t_{\varepsilon}^{p^*-p} \int_{\Omega} f(x) |v_{\varepsilon}^{a^i}|^{p^*} dx + t_{\varepsilon}^{r-p} |g|_{\infty} \int_{\Omega} \lambda |v_{\varepsilon}^{a^i}|^r dx. \tag{58}$$

From (47)–(49) and (58), there exist $\varepsilon_3 > 0$ and $T_2 > 0$ such that

$$t_{\varepsilon} \geq T_2 \quad \text{for any } \varepsilon \in (0, \varepsilon_3). \tag{59}$$

Let $\varepsilon_4 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} > 0$; then

$$0 < T_2 \leq t_{\varepsilon} \leq T_1 \quad \forall \varepsilon \in (0, \varepsilon_4), \tag{60}$$

where T_1 and T_2 are independent of ε . From [13, Lemma 4.2] and conditions (H2)–(H3), we also have

$$\begin{aligned} \sup_{t \geq 0} \left(\frac{t^p}{p} \|v_{\varepsilon}^{a^i}\|_p^p - \frac{t^{p^*}}{p^*} \int_{\Omega} f(x) |v_{\varepsilon}^{a^i}|^{p^*} dx \right) \\ = \frac{1}{N} S^{N/p} + O(\varepsilon^{(N-p)/p}). \end{aligned} \tag{61}$$

By (43), (46)–(49), (60), and (61), for $\varepsilon \in (0, \varepsilon_4)$, we obtain

$$\begin{aligned}
 h(t_\varepsilon) &\leq \sup_{t \geq 0} \left(\frac{t^p}{p} \|v_\varepsilon^i\|_p^p - \frac{t^{p^*}}{p^*} \int_\Omega f(x) |v_\varepsilon^i|^{p^*} dx \right) \\
 &\quad + \frac{t^q}{q} \|v_\varepsilon^i\|_q^q - \frac{t^r}{r} \int_\Omega \lambda g(x) |v_\varepsilon^i|^r dx \\
 &\leq \frac{1}{N} S^{N/p} + O(\varepsilon^{(N-p)/p}) + \frac{T_1^q}{q} \|v_\varepsilon^i\|_q^q \\
 &\quad - \frac{T_2^r}{r} \lambda \gamma_2 \int_\Omega |v_\varepsilon^i|^r dx \\
 &\leq \frac{1}{N} S^{N/p} + C_1 \varepsilon^{(N-p)/p} \\
 &\quad + C_2 \varepsilon^{q(N-p)/p^2} - C_3 \varepsilon^{(p(p-1))(N-r((N-p)/p))},
 \end{aligned} \tag{62}$$

where C_1, C_2, C_3 are positive constants independent of ε . Since $1 < q < N(p-1)/(N-1) < p \leq \max\{p, p^* - q/(p-1)\} < r < p^*$, we obtain that

$$\frac{N-p}{p} > \frac{q(N-p)}{p^2} > \frac{p}{p-1} \left(N - r \frac{N-p}{p} \right); \tag{63}$$

then there exists an $\varepsilon_0 \in (0, \varepsilon_3)$ such that $h(t_\varepsilon) = \sup_{t \geq 0} J_\lambda(t v_\varepsilon^i) < (1/N) S^{N/p}$ uniformly in i for all $\varepsilon \in (0, \varepsilon_0)$. Moreover, from (53), we have $0 < \alpha_\lambda \leq \alpha_\lambda^i < (1/N) S^{N/p}$ for all $1 \leq i \leq k$ and $\lambda > 0$. This completes the proof. \square

Proof of Theorem 1. From Lemmas 5, 10, 11, and 13, we get for all $\lambda > 0$ that there exists a u_0 such that $J'_\lambda(u_0) = 0$ and $J_\lambda(u_0) = \alpha_\lambda$. Set $u_+ = \max\{u, 0\}$. Replace the terms $\int_\Omega f(x)|u|^{p^*} dx$ and $\int_\Omega g(x)|u|^r dx$ of the functional J_λ by $\int_\Omega f(x)u_+^{p^*} dx$ and $\int_\Omega g(x)u_+^r dx$, respectively. It then follows that u_0 is a nonnegative solution of (E_λ) . Applying the maximum principle, (E_λ) admits at least one positive solution u_0 in $W_0^{1,p}(\Omega)$. \square

By studying the argument as in [21, Theorem III 3.1] and [22], we obtain the following lemma.

Lemma 14. *Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a nonnegative function sequence with $|u_n|_{p^*} = 1$ and $\|u_n\|_p^p \rightarrow S$. Then there exists a sequence $(y_n, \sigma_n) \in \Omega \times \mathbf{R}^+$ such that*

$$v_n(x) := \sigma_n^{(N-p)/p} u_n(\sigma_n x + y_n) \tag{64}$$

contains a convergent subsequence denoted again by $\{v_n\}$ such that

$$v_n \rightarrow v \text{ in } D^{1,p}(\mathbf{R}^N), \tag{65}$$

where $v(x) > 0$ in \mathbf{R}^N . Moreover, we have $\sigma_n \rightarrow 0$, $(1/\sigma_n) \text{dist}(y_n, \partial\Omega) \rightarrow \infty$, and $y_n \rightarrow y \in \bar{\Omega}$ as $n \rightarrow \infty$.

Lemma 15. *Suppose that (H2) and (H3) hold. Then for any $i \in \{1, 2, \dots, k\}$, there exists $\tilde{\lambda}_i > 0$ such that*

$$\tilde{\alpha}_\lambda^i > \frac{1}{N} S^{N/p} \quad \forall \lambda \in (0, \tilde{\lambda}_i). \tag{66}$$

Proof. Fix $i \in \{1, 2, \dots, k\}$. Assume the contrary; that is, there then exists a sequence $\{\lambda_n\}$ with $\lambda_n \rightarrow 0^+$ as $n \rightarrow \infty$ such that $\tilde{\alpha}_{\lambda_n}^i \rightarrow c \leq (1/N) S^{N/p}$. Consequently, there exists a sequence $\{u_n\} \subset \partial\mathcal{N}_{\lambda_n}^i$ such that, as $n \rightarrow \infty$,

$$\begin{aligned}
 \|u_n\|_p^p + \|u_n\|_q^q &= \int_\Omega f(x) |u_n|^{p^*} dx \\
 &\quad + \lambda_n \int_\Omega g(x) |u_n|^r dx + o_n(1),
 \end{aligned} \tag{67}$$

and by Remark 8, we have that there exists a $d > 0$ such that

$$0 < d \leq \liminf_{n \rightarrow \infty} \alpha_{\lambda_n} \leq \lim_{n \rightarrow \infty} J_{\lambda_n}(u_n) = c \leq \frac{1}{N} S^{N/p}, \tag{68}$$

where d is independent of λ_n for all n . It then follows easily that $\{u_n\}$ is uniformly bounded in $W_0^{1,p}(\Omega)$, and since $g(x)$ is continuous on $\bar{\Omega}$, we obtain

$$\lambda_n \int_\Omega g(x) |u_n|^r dx = o_n(1) \quad \text{as } n \rightarrow \infty. \tag{69}$$

From (67)–(69), we may assume that there exist $a \geq 0$ and $b \geq 0$ such that

$$\|u_n\|_p^p = a + o_n(1), \quad \|u_n\|_q^q = b + o_n(1). \tag{70}$$

So (70) and $|f|_\infty = 1$ imply that

$$|u_n|_{p^*}^{p^*} \geq \int_\Omega f(x) |u_n|^{p^*} dx = a + b + o_n(1). \tag{71}$$

By (70), (71), and the Sobolev inequality, we have

$$\begin{aligned}
 a &= \lim_{n \rightarrow \infty} \|u_n\|_p^p \geq S \lim_{n \rightarrow \infty} |u_n|_{p^*}^p \\
 &\geq S \lim_{n \rightarrow \infty} \left(\int_\Omega f(x) |u_n|^{p^*} dx \right)^{p/p^*} \\
 &\geq S(a+b)^{p/p^*} \geq Sa^{p/p^*},
 \end{aligned} \tag{72}$$

which implies $a = 0$ or $a \geq S^{N/p}$. If $a = 0$, then by (72) we have that $b = 0$. From $a = b = 0$, we can deduce that $c = 0$ which is a contradiction. Hence,

$$a \geq S^{N/p}. \tag{73}$$

On the other hand, by $J_{\lambda_n}(u_n) = c + o_n(1)$, $c \leq (1/N) S^{N/p}$, and (69)–(71), we get

$$\frac{a}{p} + \frac{b}{q} - \frac{a+b}{p^*} = \lim_{n \rightarrow \infty} J_{\lambda_n}(u_n) = c \leq \frac{1}{N} S^{N/p}. \tag{74}$$

This implies that

$$\frac{a}{N} \leq \left(\frac{a}{p} - \frac{a}{p^*} \right) + \left(\frac{b}{q} - \frac{b}{p^*} \right) = c \leq \frac{1}{N} S^{N/p}. \tag{75}$$

Hence, together with (73), we get $a = S^{N/p}$ and $b = 0$, and then, from (71) and (72), we also have

$$\lim_{n \rightarrow \infty} |u_n|_{p^*}^{p^*} = \lim_{n \rightarrow \infty} \int_\Omega f(x) |u_n|^{p^*} dx = a = S^{N/p}. \tag{76}$$

Set $w_n = u_n/|u_n|_{p^*}$; then we have

$$|w_n|_{p^*} = 1, \quad \lim_{n \rightarrow \infty} \|w_n\|_p^p = \lim_{n \rightarrow \infty} \frac{\|u_n\|_p^p}{|u_n|_{p^*}^p} = S. \quad (77)$$

Using Lemma 14, there exists a sequence $(y_n, \sigma_n) \in \Omega \times \mathbf{R}^+$ such that the sequence

$$v_n(x) := \sigma_n^{(N-p)/p} w_n(\sigma_n x + y_n) \quad (78)$$

converges strongly to $v \in D^{1,p}(\mathbf{R}^N)$, $\sigma_n \rightarrow 0$, $y_n \rightarrow y \in \bar{\Omega}$, and $(1/\sigma_n) \text{dist}(y_n, \partial\Omega) \rightarrow \infty$ as $n \rightarrow \infty$.

Let $\Omega_n = \{x : \sigma_n x + y_n \in \Omega\}$. Since $\sigma_n \rightarrow 0$, $y_n \rightarrow y \in \bar{\Omega}$, and $(1/\sigma_n) \text{dist}(y_n, \partial\Omega) \rightarrow \infty$ as $n \rightarrow \infty$, then $\Omega_n \rightarrow \mathbf{R}^N$ as $n \rightarrow \infty$. Observe that $Q_i(w_n) = Q_i(u_n) = r_0/3$. By the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \frac{r_0}{3} &= \lim_{n \rightarrow \infty} Q_i(w_n) = \lim_{n \rightarrow \infty} \frac{\int_{\Omega} \phi_i(x) |w_n|^p dx}{\int_{\Omega} |w_n|_{p^*}^p dx} \\ &= \lim_{n \rightarrow \infty} \frac{\int_{\Omega} \phi_i(x) |v_n((x - y_n)/\sigma_n)|^p dx}{\int_{\Omega} |v_n((x - y_n)/\sigma_n)|_{p^*}^p dx} \\ &= \lim_{n \rightarrow \infty} \frac{\int_{\Omega_n} \phi_i(\sigma_n x + y_n) |v_n(x)|^p dx}{\int_{\Omega_n} |v_n(x)|_{p^*}^p dx} = \phi_i(y), \end{aligned} \quad (79)$$

which implies that $y \neq a^i$ by the definition of $\phi_i(x)$. On the other hand, by the Lebesgue dominated convergence theorem again and (76), we get

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \int_{\Omega} f(x) |w_n(x)|^p dx \\ &= \lim_{n \rightarrow \infty} \int_{\Omega_n} f(\sigma_n x + y_n) |v_n(x)|^p dx = f(y), \end{aligned} \quad (80)$$

which is impossible, because $f(x)$ is not a constant function by condition (H3). \square

According to Lemma 13, we have

$$0 < \alpha_\lambda \leq \alpha_\lambda^i < \frac{1}{N} S^{N/p} \quad \forall \lambda > 0. \quad (81)$$

According to Lemma 15, for each $i \in \{1, 2, \dots, k\}$, there exists $\tilde{\lambda}_i > 0$ such that

$$\tilde{\alpha}_\lambda^i > \frac{1}{N} S^{N/p} \quad \forall \lambda \in (0, \tilde{\lambda}_i). \quad (82)$$

Let $\lambda_0 = \min_{1 \leq i \leq k} \tilde{\lambda}_i > 0$. Then for each $i \in \{1, 2, \dots, k\}$, by (81) and (82), we obtain that

$$\alpha_\lambda^i < \tilde{\alpha}_\lambda^i \quad \forall \lambda \in (0, \lambda_0). \quad (83)$$

Hence

$$\alpha_\lambda^i = \inf_{u \in \mathcal{N}_\lambda^i \cup \partial \mathcal{N}_\lambda^i} J_\lambda(u) \quad \forall \lambda \in (0, \lambda_0). \quad (84)$$

Applying Ekeland's variational principle and using the standard computation, we have the following lemma.

Lemma 16. *If $\lambda \in (0, \lambda_0)$, then for each $i \in \{1, 2, \dots, k\}$, there exists a $(PS)_{\alpha_\lambda^i}$ -sequence $\{u_n^i\} \subset \mathcal{N}_\lambda^i$ in $W_0^{1,p}(\Omega)$ for J_λ .*

Proof. See Cao and Zhou [17] or Tarantello [18]. \square

Proof of Theorem 2. By Lemma 16, for all $\lambda \in (0, \lambda_0)$, there exists a $(PS)_{\alpha_\lambda^i}$ -sequence $\{u_n^i\} \subset \mathcal{N}_\lambda^i$ in $W_0^{1,p}(\Omega)$ for J_λ where $1 \leq i \leq k$. From (81), we have

$$\alpha_\lambda^i \in \left(0, \frac{1}{N} S^{N/p}\right). \quad (85)$$

Note that J_λ satisfies the $(PS)_c$ -condition for $c \in (0, (1/N)S^{N/p})$. Hence, we obtain that J_λ at least k critical points in \mathcal{M}_λ for all $\lambda \in (0, \lambda_0)$. Set $u_+ = \max\{u, 0\}$. Replace the terms $\int_{\Omega} f(x)|u|^p dx$ and $\int_{\Omega} g(x)|u|^r dx$ of the functional J_λ by $\int_{\Omega} f(x)u_+^p dx$ and $\int_{\Omega} g(x)u_+^r dx$, respectively. It then follows that (E_λ) has k nonnegative solutions. Applying the maximum principle, (E_λ) admits at least k positive solutions. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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