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Research Article

On Growth of Meromorphic Solutions of Complex Functional Difference Equations

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The main purpose of this paper is to investigate the growth order of the meromorphic solutions of complex functional difference equation of the form $(\sum_{\lambda\in I}\alpha_\lambda(z)(\prod_{\nu=1}^{n}f(z+c_\nu)^{l_{\lambda,\nu}}))/(\sum_{\mu\in J}\beta_\mu(z)(\prod_{\nu=1}^{n}f(z+c_\nu)^{m_{\mu,\nu}})) = Q(z,f(p(z)))$, where $I=\{\lambda=(l_{\lambda,1},l_{\lambda,2},\dots,l_{\lambda,n})\mid l_{\lambda,\nu}\in\mathbb{N}\bigcup\{0\},\ \nu=1,2,\dots,n\}$ and $J=\{\mu=(m_{\mu,1},m_{\mu,2},\dots,m_{\mu,n})\mid m_{\mu,\nu}\in\mathbb{N}\bigcup\{0\},\ \nu=1,2,\dots,n\}$ are two finite index sets, c_ν $(\nu=1,2,\dots,n)$ are distinct complex numbers, $\alpha_\lambda(z)$ $(\lambda\in I)$ and $\beta_\mu(z)$ $(\mu\in J)$ are small functions relative to $f(z)$, and $Q(z,u)$ is a rational function in u with coefficients which are small functions of $f(z)$, $p(z)=p_kz^k+p_{k-1}z^{k-1}+\cdots+p_0\in\mathbb{C}[z]$ of degree $k\ge1$. We also give some examples to show that our results are sharp.

1. Introduction and Main Results

Let $f(z)$ be a function meromorphic in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations and results in Nevanlinna's value distribution theory of meromorphic functions such as the characteristic function $T(r,f)$, proximity function $m(r,f)$, counting function $N(r,f)$, and the first and second main theorems (see, e.g., [1–4]). We also use $\overline{N}(r,f)$ to denote the counting function of the poles of $f(z)$ whose every pole is counted only once. The notations $\rho(f)$ and $\mu(f)$ denote the order and the lower order of $f(z)$, respectively. $S(r,f)$ denotes any quantity that satisfies the condition: $S(r,f)=o(T(r,f))$ as $r\to\infty$ possibly outside an exceptional set of r of finite linear measure. A meromorphic function $a(z)$ is called a small function of $f(z)$ or a small function relative to $f(z)$ if and only if $T(r,a(z))=S(r,f)$.

Recently, some papers (see, e.g., [5–7]) focusing on complex difference and functional difference equations emerged. In 2005, Laine et al. [5] firstly considered the growth of meromorphic solutions of the complex functional difference equations by utilizing Nevanlinna theory. They obtained the following result.

Theorem A. *Suppose that f is a transcendental meromorphic solution of the equation*

$$\sum_{\{J\}}\alpha_J(z)\left(\prod_{j\in J}f\left(z+c_j\right)\right)=f\left(p\left(z\right)\right), \tag{1}$$

where $\{J\}$ is a collection of all subsets of $\{1,2,\dots,n\}$, c_j's are distinct complex constants, and $p(z)$ is a polynomial of degree $k\ge2$. Moreover, we assume that the coefficients $\alpha_J(z)$ are small functions relative to f and that $n\ge k$. Then

$$T\left(r,f\right)=O\left(\left(\log r\right)^{\alpha+\varepsilon}\right), \tag{2}$$

where $\alpha=\log n/\log k$.

In 2007, Rieppo [6] gave an estimation of growth of meromorphic solutions of complex functional equations as follows.

Theorem B. *Suppose that f is a transcendental meromorphic function. Let $Q(z,f)$, $R(z,f)$ be rational functions in f with small meromorphic coefficients relative to f such that $0<q:=\deg_fQ\le d:=\deg_fR$ and $p(z)=p_kz^k+p_{k-1}z^{k-1}+$*

$\cdots + p_0 \in \mathbb{C}[z]$ *of degree* $k > 1$. *If* f *is a solution of the functional equation*

$$R(z, f(z)) = Q(z, f(p(z))), \tag{3}$$

then $qk \le d$, *and for any* $\varepsilon, 0 < \varepsilon < 1$, *there exist positive real constants* K_1 *and* K_2 *such that*

$$K_1(\log r)^{\alpha-\varepsilon} \le T(r, f) \le K_2(\log r)^{\alpha+\varepsilon}, \quad \alpha = \frac{\log d - \log q}{\log k}, \tag{4}$$

when r *is large enough.*

Rieppo [6] also considered the growth order of meromorphic solutions of functional equation (3) when $k = 1$ and got the following.

Theorem C. *Suppose that* f *is a transcendental meromorphic solution of* (3), *where* $p(z) = az+b$, $a, b \in \mathbb{C}$, $a \ne 0$ *and* $|a| \ne 1$. *Then*

$$\mu(f) = \rho(f) = \frac{\log d - \log q}{\log |a|}. \tag{5}$$

Two years later, Zheng et al. [7] extended Theorem A to more general type and obtained a similar result of Theorem C. In fact, they got the following two results.

Theorem D. *Suppose that* f *is a transcendental meromorphic solution of the equation*

$$\sum_{\{J\}} \alpha_J(z) \left(\prod_{j\in J} f(z + c_j) \right) = Q(z, f(p(z))), \tag{6}$$

where $\{J\}$ *is a collection of all nonempty subsets of* $\{1, 2, \ldots, n\}$, c_j $(j = 1, \ldots, n)$ *are distinct complex constants,* $p(z) = p_k z^k + p_{k-1} z^{k-1} + \cdots + p_0 \in \mathbb{C}[z]$ *of degree* $k > 1$, *and* $Q(z, u)$ *is a rational function in* u *of* $\deg_u Q = q(> 0)$. *Also suppose that all the coefficients of* (6) *are small functions relative to* f. *Then* $qk \le n$, *and*

$$T(r, f) = O\left((\log r)^{\alpha+\varepsilon}\right), \tag{7}$$

where $\alpha = (\log n - \log q)/\log k$.

Theorem E. *Suppose that* f *is a transcendental meromorphic solution of* (6), *where* $\{J\}$ *is a collection of all nonempty subsets of* $\{1, 2, \ldots, n\}$, c_j $(j = 1, \ldots, n)$ *are distinct complex constants,* $p(z) = az + b$, $a, b \in \mathbb{C}$, *and* $Q(z, u)$ *is a rational function in* u *of* $\deg_u Q = q(> 0)$. *Also suppose that all the coefficients of* (6) *are small functions relative to* f.

(i) *If* $0 < |a| < 1$, *then we have*

$$\mu(f) \ge \frac{\log q - \log n}{-\log |a|}. \tag{8}$$

(ii) *If* $|a| > 1$, *then we have* $q \le n$ *and*

$$\rho(f) \le \frac{\log n - \log q}{\log |a|}. \tag{9}$$

(iii) *If* $|a| = 1, q > n$, *then we have* $\rho(f) = \mu(f) = \infty$.

In this paper, we will consider a more general class of complex functional difference equations. We prove the following results, which generalize the above related results.

Theorem 1. *Suppose that* $f(z)$ *is a transcendental meromorphic solution of the functional difference equation*

$$\frac{\sum_{\lambda\in I} \alpha_\lambda(z) \left(\prod_{\nu=1}^n f(z + c_\nu)^{l_{\lambda,\nu}}\right)}{\sum_{\mu\in J} \beta_\mu(z) \left(\prod_{\nu=1}^n f(z + c_\nu)^{m_{\mu,\nu}}\right)} = Q(z, f(p(z))), \tag{10}$$

where c_ν $(\nu = 1, \ldots, n)$ *are distinct complex constants,* $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \ldots, l_{\lambda,n}) \mid l_{\lambda,\nu} \in \mathbb{N}\bigcup\{0\}, \nu = 1, 2, \ldots, n\}$ *and* $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \ldots, m_{\mu,n}) \mid m_{\mu,\nu} \in \mathbb{N}\bigcup\{0\}, \nu = 1, 2, \ldots, n\}$ *are two finite index sets,* $p(z) = p_k z^k + p_{k-1} z^{k-1} + \cdots + p_0 \in \mathbb{C}[z]$ *of degree* $k > 1$, *and* $Q(z, u)$ *is a rational function in* u *of* $\deg_u Q = q(> 0)$. *Also suppose that all the coefficients of* (10) *are small functions relative to* f. *Denoting*

$$\sigma_\nu = \max_{\lambda,\mu} \{l_{\lambda,\nu}, m_{\mu,\nu}\} \quad (\nu = 1, 2, \ldots, n), \; \sigma = \sum_{\nu=1}^{n} \sigma_\nu. \tag{11}$$

Then $qk \le \sigma$, *and*

$$T(r, f) = O\left((\log r)^{\alpha+\varepsilon}\right), \tag{12}$$

where $\alpha = (\log \sigma - \log q)/\log k$.

Theorem 2. *Suppose that* f *is a transcendental meromorphic solution of the equation*

$$\frac{\sum_{\lambda\in I} \alpha_\lambda(z) \left(\prod_{\nu=1}^n f(z + c_\nu)^{l_{\lambda,\nu}}\right)}{\sum_{\mu\in J} \beta_\mu(z) \left(\prod_{\nu=1}^n f(z + c_\nu)^{m_{\mu,\nu}}\right)} = Q(z, f(az + b)), \tag{13}$$

where c_ν $(\nu = 1, \ldots, n)$ *are distinct complex constants,* $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \ldots, l_{\lambda,n}) \mid l_{\lambda,\nu} \in \mathbb{N}\bigcup\{0\}, \nu = 1, 2, \ldots, n\}$ *and* $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \ldots, m_{\mu,n}) \mid m_{\mu,\nu} \in \mathbb{N}\bigcup\{0\}, \nu = 1, 2, \ldots, n\}$ *are two finite index sets,* $a, b \in \mathbb{C}$, *and* $Q(z, u)$ *is a rational function in* u *of* $\deg_u Q = q(> 0)$. *Also suppose that all the coefficients of* (10) *are small functions relative to* f. *Denoting*

$$\sigma_\nu = \max_{\lambda,\mu} \{l_{\lambda,\nu}, m_{\mu,\nu}\} \quad (\nu = 1, 2, \ldots, n), \; \sigma = \sum_{\nu=1}^{n} \sigma_\nu. \tag{14}$$

(i) *If* $0 < |a| < 1$, *then we have*

$$\mu(f) \ge \frac{\log q - \log \sigma}{-\log |a|}. \tag{15}$$

(ii) *If* $|a| > 1$, *then we have* $q \le \sigma$ *and*

$$\rho(f) \le \frac{\log \sigma - \log q}{\log |a|}. \tag{16}$$

(iii) *If* $|a| = 1$ *and* $q > \sigma$, *then we have* $\mu(f) = \rho(f) = \infty$.

Next we will give some examples to show that our results are best in some extent.

Example 3. Let $c_1 = \arctan 2$, $c_2 = -\pi/4$. Then it is easy to check that $f(z) = \tan z$ solves the following equation:

$$\begin{aligned}&\frac{f(z+c_1)^2 f(z+c_2)}{f(z+c_1)+f(z+c_2)^2}\\&=\left(-4f\left(\frac{z}{2}\right)^8+8f\left(\frac{z}{2}\right)^7+28f\left(\frac{z}{2}\right)^6-56f\left(\frac{z}{2}\right)^5\right.\\&\quad\left.-32f\left(\frac{z}{2}\right)^4+56f\left(\frac{z}{2}\right)^3+28f\left(\frac{z}{2}\right)^2-8f\left(\frac{z}{2}\right)-4\right)\\&\quad\times\left(3f\left(\frac{z}{2}\right)^8+10f\left(\frac{z}{2}\right)^7+16f\left(\frac{z}{2}\right)^6+122f\left(\frac{z}{2}\right)^5\right.\\&\quad\left.-6f\left(\frac{z}{2}\right)^4-122f\left(\frac{z}{2}\right)^3+16f\left(\frac{z}{2}\right)^2-10f\left(\frac{z}{2}\right)+3\right)^{-1}.\end{aligned}\tag{17}$$

Obviously, we have

$$\mu(f)=\rho(f)=1=\frac{\log q-\log\sigma}{-\log|a|},\tag{18}$$

where $q = 8$, $\sigma = 4$ and $a = 1/2$.

Example 3 shows that the estimate in Theorem 2(i) is sharp.

Example 4. It is easy to check that $f(z) = \tan z$ satisfies the equation

$$\begin{aligned}&\frac{f(z+(\pi/3))^2 f(z+(\pi/6))-f(z+(\pi/6))}{f(z+(\pi/3))f(z+(\pi/6))^2-f(z+(\pi/3))}\\&=\frac{\sqrt{3}f(2z)^2+4f(2z)+\sqrt{3}}{-\sqrt{3}f(2z)^2+4f(2z)-\sqrt{3}}.\end{aligned}\tag{19}$$

Clearly, we have

$$\mu(f)=\rho(f)=1=\frac{\log\sigma-\log q}{\log|a|},\tag{20}$$

where $\sigma = 4$, $q = 2$ and $a = 2$.

Example 4 shows that the estimate in Theorem 2(ii) is sharp.

Example 5. $f(z) = \tan z$ satisfies the equation of the form

$$\begin{aligned}&\frac{f(z+(\pi/4))^2}{f(z+(\pi/4))+f(z-(\pi/4))^2}\\&=\frac{-\left(f(z/2)^2-2f(z/2)-1\right)^3}{8f(z/2)\left(f(z/2)^2-1\right)\left(f(z/2)^2+2f(z/2)-1\right)},\end{aligned}\tag{21}$$

where $\sigma = 4$, $q = 6$, and $a = 1/2$. $\rho(f) = \mu(f) = 1 > \log(3/2)/\log 2 = (\log q - \log\sigma)/-\log|a|$.

Example 5 shows that the strict inequality in Theorem 2 may occur. Therefore, we do not have the same estimation as in Theorem C for the growth order of meromorphic solutions of (13).

The following Example shows that the restriction $q > \sigma$ in case (iii) in Theorem 2 is necessary.

Example 6. Meromorphic function $f(z) = \tan z$ solves the following equation:

$$\frac{f(z+(\pi/4))^2}{f(z+(\pi/4))+f(z-(\pi/4))^2}=\frac{(f(z)+1)^3}{4f(z)(1-f(z))},\tag{22}$$

where $a = 1$ and $4 = \sigma > q = 3$, but $\rho(f) = \mu(f) = 1$.

Next, we give an example to show that case (iii) in Theorem 2 may hold.

Example 7. Function $f(z) = ze^{e^z}$ satisfies the following equation:

$$\begin{aligned}&\frac{(z+\log 6)(z+\log 2)^5\left[f(z+\log 4)^4+f(z+\log 4)\right]}{(z+\log 4)f(z+\log 6)}\\&=\frac{(z+\log 4)^3 f(z+\log 2)^6+(z+\log 2)^6}{f(z+\log 2)},\end{aligned}\tag{23}$$

where $a = 1$ and $q = 6 > 5 = \sigma$. Obviously, $\rho(f) = \mu(f) = \infty$.

2. Main Lemmas

In order to prove our results, we need the following lemmas.

Lemma 1 (see [4, 8]). *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z,f)=\frac{P(z,f)}{Q(z,f)}=\frac{\sum_{i=0}^{p}a_i(z)f^i}{\sum_{j=0}^{q}b_j(z)f^j},\tag{24}$$

such that the meromorphic coefficients $a_i(z)$, $b_j(z)$ satisfy

$$\begin{aligned}T(r,a_i)&=S(r,f),\quad i=0,1,\ldots,p,\\T(r,b_j)&=S(r,f),\quad j=0,1,\ldots,q;\end{aligned}\tag{25}$$

then one has

$$T(r,R(z,f))=\max\{p,q\}\cdot T(r,f)+S(r,f).\tag{26}$$

From the proof of Theorem 1 in [9], we have the following estimate for the Nevanlinna characteristic.

Lemma 2. *Let $f_1, f_2, \ldots, f_n$ be distinct meromorphic functions and*

$$F(z)=\frac{P(z)}{Q(z)}=\frac{\sum_{\lambda\in I}\alpha_\lambda(z)f_1^{l_{\lambda,1}}f_2^{l_{\lambda,2}}\ldots f_n^{l_{\lambda,n}}}{\sum_{\mu\in J}\beta_\mu(z)f_1^{m_{\mu,1}}f_2^{m_{\mu,2}}\ldots f_n^{m_{\mu,n}}}.\tag{27}$$

Then

$$T(r, F(z)) \leq \sum_{\nu=1}^{n} \sigma_\nu T(r, f_\nu) + S(r, f), \quad (28)$$

where $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \ldots, l_{\lambda,n}) \mid l_{\lambda,\nu} \in \mathbb{N}\bigcup\{0\}, \nu = 1, 2, \ldots, n\}$ *and* $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \ldots, m_{\mu,n}) \mid m_{\mu,\nu} \in \mathbb{N}\bigcup\{0\}, \nu = 1, 2, \ldots, n\}$ *are two finite index sets,* $\sigma_\nu = \max_{\lambda,\mu}\{l_{\lambda,\nu}, m_{\mu,\nu}\}$ $(\nu = 1, 2, \ldots, n)$. $\alpha_\lambda(z) = o(T(r, f_\nu)(\lambda \in I))$ *and* $\beta_\mu(z) = o(T(r, f_\nu)(\mu \in J))$ *hold for all* $\nu \in \{1, 2, \ldots, n\}$ *and satisfy* $T(r, \alpha_\lambda) = S(r, f)$ $(\lambda \in I)$ *and* $T(r, \beta_\mu) = S(r, f)$ $(\mu \in J)$.

Lemma 3 (see [7]). *Let c be a complex constant. Given* $\varepsilon > 0$ *and a meromorphic function* f*, one has*

$$T(r, f(z \pm c)) \leq (1 + \varepsilon) T(r + |c|, f), \quad (29)$$

for all $r > r_0$*, where* r_0 *is some positive constant.*

Lemma 4 (see [4]). *Let* $g : (0, +\infty) \to \mathbb{R}$, $h : (0, +\infty) \to \mathbb{R}$ *be monotone increasing functions such that* $g(r) \leq h(r)$ *outside of an exceptional set E of finite linear measure. Then, for any* $\alpha > 1$*, there exists* $r_0 > 0$ *such that* $g(r) \leq h(\alpha r)$ *for all* $r > r_0$.

Lemma 5 (see [10]). *Let* f *be a transcendental meromorphic function, and* $p(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0$, $a_k \neq 0$, *be a nonconstant polynomial of degree k. Given* $0 < \delta < |a_k|$*, denote* $\lambda = |a_k| + \delta$ *and* $\mu = |a_k| - \delta$*. Then given* $\varepsilon > 0$ *and* $a \in \mathbb{C}\bigcup\{\infty\}$*, one has*

$$\begin{gathered} kn(\mu r^k, a, f) \leq n(r, a, f(p(z))) \leq kn(\lambda r^k, a, f) \\ N(\mu r^k, a, f) + O(\log r) \leq N(r, a, f(p(z))) \\ \leq N(\lambda r^k, a, f) + O(\log r) \\ (1 - \varepsilon) T(\mu r^k, f) \leq T(r, f(p(z))) \leq (1 + \varepsilon) T(\lambda r^k, f), \end{gathered} \quad (30)$$

for all r large enough.

Lemma 6 (see [11]). *Let* $\phi : [r_0, +\infty) \to (0, +\infty)$ *be positive and bounded in every finite interval, and suppose that* $\phi(\mu r^m) \leq A\phi(r) + B$ *holds for all r large enough, where* $\mu > 0$, $m > 1$, $A > 1$ *and B are real constants. Then*

$$\phi(r) = O((\log r)^\alpha), \quad (31)$$

where $\alpha = \log A / \log m$.

Lemma 7 (see [6]). *Let* $\phi : (r_0, \infty) \to (1, \infty)$*, where* $r_0 \geq 1$*, be a monotone increasing function. If for some real constant* $\alpha > 1$*, there exists a real number* $K > 1$ *such that* $\phi(\alpha r) \geq K\phi(r)$*, then*

$$\varliminf_{r \to \infty} \frac{\log \phi(r)}{\log r} \geq \frac{\log K}{\log \alpha}. \quad (32)$$

Lemma 8 (see [12]). *Let* $\phi : (1, \infty) \to (0, \infty)$ *be a monotone increasing function and let* f *be a nonconstant meromorphic function. If, for some real constant* $\alpha \in (0, 1)$*, there exist real constants* $K_1 > 0$ *and* $K_2 \geq 1$ *such that*

$$T(r, f) \leq K_1 \phi(\alpha r) + K_2 T(\alpha r, f) + S(\alpha r, f), \quad (33)$$

then

$$\rho(f) \leq \frac{\log K_2}{-\log \alpha} + \varlimsup_{r \to \infty} \frac{\log \phi(r)}{\log r}. \quad (34)$$

3. Proof of Theorems

Proof of Theorem 1. We assume $f(z)$ is a transcendental meromorphic solution of (10). Denoting $C = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$. According to Lemmas 1, 2, and 3 and the last assertion of Lemma 5, we get that for any $\varepsilon_1 > 0$,

$$\begin{aligned} & q(1 - \varepsilon_1) T(\mu r^k, f) + S(r, f) \\ & \leq qT(r, f(p(z))) + S(r, f) \\ & = T(r, Q(z, f(p(z)))) \\ & = T\left(r, \frac{\sum_{\lambda \in I} \alpha_\lambda(z) \left(\prod_{\nu=1}^{n} f(z + c_\nu)^{l_{\lambda,\nu}}\right)}{\sum_{\mu \in J} \beta_\mu(z) \left(\prod_{\nu=1}^{n} f(z + c_\nu)^{m_{\mu,\nu}}\right)}\right) \\ & \leq \sum_{\nu=1}^{n} \sigma_\nu T(r, f(z + c_\nu)) + S(r, f) \\ & \leq \sum_{\nu=1}^{n} \sigma_\nu (1 + \varepsilon_1) T(r + C, f(z)) + S(r, f) \\ & = \left(\sum_{\nu=1}^{n} \sigma_\nu\right)(1 + \varepsilon_1) T(r + C, f(z)) + S(r, f) \\ & = \sigma(1 + \varepsilon_1) T(r + C, f(z)) + S(r, f), \end{aligned} \quad (35)$$

where r is large enough and $\mu = |p_k| - \delta$ for some $0 < \delta < |p_k|$. Since $T(r + C, f) \leq T(\beta r, f)$ holds for r large enough for $\beta > 1$, we may assume r to be large enough to satisfy

$$q(1 - \varepsilon_1) T(\mu r^k, f) \leq \sigma(1 + \varepsilon_1) T(\beta r, f) \quad (36)$$

outside a possible exceptional set of finite linear measure. By Lemma 4, we know that whenever $\gamma > 1$,

$$q(1 - \varepsilon_1) T(\mu r^k, f) \leq \sigma(1 + \varepsilon_1) T(\gamma\beta r, f) \quad (37)$$

holds for all r large enough. Denote $t = \gamma\beta r$; thus the inequality (37) may be written in the form

$$T\left(\frac{\mu}{(\gamma\beta)^k} t^k, f\right) \leq \frac{\sigma(1 + \varepsilon_1)}{q(1 - \varepsilon_1)} T(t, f). \quad (38)$$

By Lemma 6, we have

$$T(r, f) = O((\log r)^{\alpha_1}), \quad (39)$$

where

$$\alpha_1 = \frac{\log(\sigma(1+\varepsilon_1)/q(1-\varepsilon_1))}{\log k} = \frac{\log\sigma - \log q}{\log k} + \frac{\log((1+\varepsilon_1)/(1-\varepsilon_1))}{\log k}. \quad (40)$$

Denoting now $\alpha = (\log\sigma - \log q)/\log k$ and $\varepsilon = \log((1+\varepsilon_1)/(1-\varepsilon_1))/\log k$; thus we obtain the required form.

Finally, we show that $qk \le \sigma$. If $qk > \sigma$, then we have $\alpha < 1$. For sufficiently small $\varepsilon > 0$, we have $\alpha + \varepsilon < 1$, which contradicts with the transcendency of f. Thus Theorem 1 is proved. □

Proof of Theorem 2. Suppose $f(z)$ is a transcendental meromorphic solution of (13). Denoting $C = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$.

(i) $0 < |a| < 1$. We may assume that $q > \sigma$, since the case $q \le \sigma$ is trivial by the fact that $\mu(f) \ge 0$. By Lemmas 1–3, we have for any $\varepsilon > 0$ and $\beta > 1$,

$$\begin{aligned} qT(r, f(p(z))) + S(r, f) &= T(r, Q(z, f(p(z)))) \\ &= T\left(r, \frac{\sum_{\lambda\in I}\alpha_\lambda(z)\left(\prod_{\nu=1}^{n} f(z+c_\nu)\right)^{l_{\lambda,\nu}}}{\sum_{\mu\in J}\beta_\mu(z)\left(\prod_{\nu=1}^{n} f(z+c_\nu)\right)^{m_{\mu,\nu}}}\right) \\ &\le \sum_{\nu=1}^{n}\sigma_\nu T(r, f(z+c_\nu)) + S(r, f) \\ &\le \sum_{\nu=1}^{n}\sigma_\nu(1+\varepsilon)T(r+C, f(z)) + S(r, f) \\ &= \left(\sum_{\nu=1}^{n}\sigma_\nu\right)(1+\varepsilon)T(r+C, f(z)) + S(r, f) \\ &= \sigma(1+\varepsilon)T(r+C, f(z)) + S(r, f) \\ &\le \sigma(1+\varepsilon)T(\beta r, f) + S(r, f), \end{aligned} \quad (41)$$

where r is large enough.

By the last assertion of Lemma 5 and (41), we obtain that, for $\mu = |a| - \delta$ $(0 < \delta < |a|, 0 < \mu < 1)$, the following inequality

$$q(1-\varepsilon)T(\mu r, f) \le \sigma(1+\varepsilon)T(\beta r, f) \quad (42)$$

holds, where r is large enough outside of a possible set of finite linear measure. By Lemma 4, we get that for any $\gamma > 1$ and sufficiently large r,

$$q(1-\varepsilon)T(\mu r, f) \le \sigma(1+\varepsilon)T(\gamma\beta r, f). \quad (43)$$

Therefore,

$$\frac{q(1-\varepsilon)}{\sigma(1+\varepsilon)}T(r, f) \le T\left(\frac{\gamma\beta}{\mu}r, f\right). \quad (44)$$

Since $\beta > 1, \gamma > 1, 0 < \mu < 1$ and $q > \sigma$, we have $\beta\gamma/\mu > 1$ and $q(1-\varepsilon)/\sigma(1+\varepsilon) > 1$ when ε is small enough. Using Lemma 7, we see that

$$\mu(f) \ge \frac{\log q(1-\varepsilon) - \log\sigma(1+\varepsilon)}{\log\gamma\beta - \log\mu}. \quad (45)$$

Letting $\varepsilon \to 0, \delta \to 0, \beta \to 1$ and $\gamma \to 1$, we have

$$\mu(f) \ge \frac{\log q - \log\sigma}{-\log|a|}. \quad (46)$$

(ii) $|a| > 1$. By the similar reasoning as is (i), we easily obtain that

$$\begin{aligned} q(1-\varepsilon)T(\mu r, f) &\le qT(r, f(p(z))) \\ &\le \sigma(1+\varepsilon)T(r+C, f(z)) + S(r, f) \end{aligned} \quad (47)$$

for all r large enough. We may select sufficiently small numbers $\delta > 0$ and $\varepsilon > 0$, such that $\mu = |a| - \delta > 1$ and $(1/\mu) + \varepsilon < 1$. Thus we have

$$T(\mu r, f) \le \frac{\sigma(1+\varepsilon)}{q(1-\varepsilon)}T(r+C, f(z)) + S(r, f); \quad (48)$$

namely,

$$T(\mu r, f) \le \frac{\sigma(1+\varepsilon)}{q(1-\varepsilon)}T(r+C, f(z)), \quad (49)$$

where r is large enough possibly outside of a set of finite linear measure. By Lemma 4, we have for any $1 < \gamma < \mu$,

$$T(\mu r, f) \le \frac{\sigma(1+\varepsilon)}{q(1-\varepsilon)}T(\gamma r, f(z)); \quad (50)$$

that is,

$$T(r, f) \le \frac{\sigma(1+\varepsilon)}{q(1-\varepsilon)}T\left(\frac{\gamma}{\mu}r, f(z)\right) \quad (51)$$

holds for all sufficiently large r. By Lemma 8, we obtain

$$\rho(f) \le \frac{\log\sigma - \log q + \log(1+\varepsilon) - \log(1-\varepsilon)}{-\log(\gamma/\mu)}. \quad (52)$$

Letting $\varepsilon \to 0, \delta \to 0$ and $\gamma \to 1$, we have

$$\rho(f) \le \frac{\log\sigma - \log q}{\log|a|}. \quad (53)$$

(iii) $|a| = 1$ and $q > \sigma$. The proof of this case is completely similar as in the case in (i). In fact, we set $\mu = |a| - \delta = 1 - \delta$ $(0 < \delta < 1, 0 < \mu < 1)$. Similarly, we can get

$$\mu(f) \ge \frac{\log q - \log\sigma}{-\log|a|}. \quad (54)$$

Since $|a| = 1$, we have $\mu(f) = \rho(f) = \infty$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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