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Research Article

Poincaré Inequalities for Composition Operators with L^{φ}-Norm

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We establish the Poincaré-type inequalities for the composition of the homotopy operator, exterior derivative operator, and the projection operator with L^{φ}-norm applied to the nonhomogeneous A-harmonic equation in $L^{\varphi}(\Omega)$-averaging domains.

1. Introduction

The purpose of the paper is to develop the Poincaré-type inequalities for the composition of the homotopy operator T, exterior derivative operator d, and the projection operator H with L^{φ}-norm. These operators play critical roles in investigating the properties of the solutions to PDEs and in controlling oscillatory behavior of the solutions in domains [1–6]. We first establish the local Poincaré inequalities for the composition $T \circ d \circ H$ in $L^{\varphi}(\Omega)$-averaging domains. Then, we prove the global Poincaré inequalities for the composition of $T \circ d \circ H$ in $L^{\varphi}(\Omega)$-averaging domains.

In this paper, we assume Ω is a bounded and convex domain in $\mathbb{R}^n, n \geq 2$ and $B = B(x_0, r)$ is a ball that is centred at x_0 with r as its radius. For any $\sigma > 0$, we use σB to denote the ball with centred at x_0 with radius σr. We do not distinguish the balls from the cubes in this paper. We use $|E|$ to denote the Lebesgue measure of a set $E \subset \mathbb{R}^n$. We call ω a weight if $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\omega > 0$ a.e. For a function u, we denote the average of u over B by $u_B = (1/|B|) \int_B u\, dx$. Differential forms are extensions of functions in $\mathbb{R}^n$. For example, the function $u(x_1, x_2, \ldots, x_n)$ is called a 0-form. Moreover, if $u(x_1, x_2, \ldots, x_n)$ is differentiable, it is called a differential 0-form. The 1-form $u(x)$ in $\mathbb{R}^n$ can be written as $u(x) = \sum_{i=1}^{n} u_i(x_1, x_2, \ldots, x_n) dx_i$. If the coefficient functions $u_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, are differentiable, $u(x)$ is called a differential 1-form. Similarly, a differential k-form $u(x)$ is generated by $\{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}\}$, $k = 1, 2, \ldots, n$, that is, $u(x) = \sum_I u_I(x) dx_I = \sum u_{i_1 i_2 \cdots i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, where $I = (i_1, i_2, \ldots, i_k)$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l-forms in $\mathbb{R}^n$, $D'(\Omega, \wedge^l)$ be the space of all differential l-forms on Ω and $L^p(\Omega, \wedge^l)$ be the l-forms $u(x) = \sum_I u_I(x) dx_I$ on Ω satisfying $\int_\Omega |u_I|^p < \infty$ for all ordered l-tuples I, $l = 1, 2, \ldots, n$. We denote the exterior derivative by $d : D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ for $l = 0, 1, \ldots, n-1$, and define the Hodge star operator $* : \wedge^k \to \wedge^{n-k}$ as follows: if $u = u_{i_1 i_2 \cdots i_k}(x_1, x_2, \ldots, x_n) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} = u_I dx_I$, $i_1 < i_2 < \cdots < i_k$, is a differential k-form, then $*u = *(u_{i_1 i_2 \cdots i_k} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = (-1)^{\sum(I)} u_I dx_J$, where $I = (i_1, i_2, \ldots, i_k)$, $J = \{1, 2, \ldots, n\} - I$, and $\sum(I) = (k(k+1)/2) + \sum_{i=1}^{k} i_j$. The Hodge codifferential operator $d* : D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$ is given by $d* = (-1)^{nl+1} * d*$ on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n-1$.

We use M to denote a bounded and convex domain on $\mathbb{R}^n$. Let $\wedge^l M$ be the lth exterior power of the cotangent bundle, let $C^\infty(\wedge^l M)$ be the space of smooth l-forms on M, and $\mathscr{W}(\wedge^l M) = \{u \in L^1_{\text{loc}}(\wedge^l M) : u \text{ has generalized gradient}\}$. The harmonic l-fields are defined by $\mathscr{H}(\wedge^l M) = \{u \in \mathscr{W}(\wedge^l M) : du = d^* u = 0, u \in L^p \text{ for some } 1 < p < \infty\}$. The orthogonal complement of $\mathscr{H}$ in L^1 is defined by $\mathscr{H}^\perp = \{u \in L^1 : \langle u, h \rangle = 0 \text{ for all } h \in \mathscr{H}\}$. Then, the Green's operator G is defined as $G : C^\infty(\wedge^l M) \to \mathscr{H}^\perp \cap C^\infty(\wedge^l M)$ by assigning $G(u)$ as the unique element of $\mathscr{H}^\perp \cap C^\infty(\wedge^l M)$ satisfying Poisson's equation $\Delta G(u) = u - H(u)$, where H is the harmonic projection operator that maps $C^\infty(\wedge^l M)$ onto $\mathscr{H}$ so that $H(u)$ is the harmonic part of u. See [7, 8] for more properties of these operators. The differential forms can be used to describe various systems of PDEs and to express different geometric structures on manifolds. See [9, 10].

The operator K_y with the case $y = 0$ was first introduced by Cartan in [11]. Then, it was extended to the following version in [12]. To each $y \in \Omega$ there corresponds a linear operator $K_y : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ defined by $(K_y u)(x; \xi_1, \ldots, \xi_{l-1}) = \int_0^1 t^{l-1} u(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1}) dt$ and the decomposition $u = d(K_y u) + K_y(du)$. A homotopy operator $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ is defined by averaging K_y over all points $y \in \Omega$: $Tu = \int_\Omega \phi(y) K_y(du)$, where $\phi \in C_0^\infty(\Omega)$ is normalized so that $\int \phi(y)dy = 1$.

We are particularly interested in a class of differential forms satisfying the well-known nonhomogeneous A-harmonic equation

$$d^* A(x, du) = B(x, du), \tag{1}$$

where $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ and $B : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions

$$\left|A(x, \xi)\right| \le a|\xi|^{p-1}, \qquad A(x, \xi) \cdot \xi \ge |\xi|^p, \qquad \left|B(x, \xi)\right| \le b|\xi|^{p-1} \tag{2}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a > 0$ and $b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1). A solution to (1) is an element of the Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega, \wedge^{l-1})$ such that $\int_\Omega A(x, du) \cdot d\varphi + B(x, du) \cdot \varphi = 0$ for all $\varphi \in W^{1,p}_{\mathrm{loc}}(\Omega, \wedge^{l-1})$ with compact support. If u is a function (0-form) in $\mathbb{R}^n$, (1) reduces to

$$\operatorname{div} A(x, \nabla u) = B(x, \nabla u). \tag{3}$$

If the operator $B = 0$, (1) becomes

$$d^* A(x, du) = 0, \tag{4}$$

which is called the homogeneous A-harmonic equation. Let $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ be defined by $A(x, \xi) = \xi|\xi|^{p-2}$ with $p > 1$. Then, A satisfies the required conditions and $d^* A(x, du) = 0$ becomes the p-harmonic equation $d^*(du|du|^{p-2}) = 0$ for differential forms. Some results have been obtained in recent years about different versions of the A-harmonic equation; see [1, 2, 8, 9, 13–15].

2. Main Results and Proofs

Definition 1. Let φ be a continuously increasing convex function on $[0, \infty)$ with $\varphi(0) = 0$, and let Ω be a domain with $\mu(\Omega) < \infty$. If u is a measurable function in Ω, then we define the Orlicz norm of u by

$$\|u\|_{\varphi,\Omega} = \inf\left\{k > 0 : \int_\Omega \varphi\left(\frac{|u(x)|}{k}\right) dx \le 1\right\}. \tag{5}$$

A continuously increasing function $\varphi : [0, \infty)$ with $\varphi(0) = 0$ is called an Orlicz function, and a convex Orlicz function φ is often called a Young function.

From Definition 1, it is easy to see that for any domain $\Omega \subset \mathbb{R}^n$

$$\int_\Omega \varphi\left(\frac{|u(x)|}{\|u\|_{\varphi,\Omega}}\right) dx \le 1 \tag{6}$$

if $\|u\|_{\varphi,\Omega}$ is finite.

Definition 2. Let φ be an increasing convex function on $[0, \infty)$ with $\varphi(0) = 0$. We call a proper subdomain $\Omega \subset \mathbb{R}^n$ an Orlicz space $L^\varphi(\Omega)$, if $\mu(\Omega) < \infty$ and there exists a constant C such that

$$\left\|u - u_{B_0}\right\|_{\varphi,\Omega} \le C \sup_{B \subset \Omega} \left\|u - u_B\right\|_{\varphi,B} \tag{7}$$

for some ball $B_0 \subset \Omega$ and all integrable functions u in Ω, where the supremum is over all balls B with $B \subset \Omega$.

Definition 3 (see [15]). We say that a Young function φ lies in the class $G(p, q, C)$, $1 \le p < q < \infty$, $C \ge 1$, if (i) $1/C \le \varphi(t^{1/p})/g(t) \le C$ and (ii) $1/C \le \varphi(t^{1/p})/h(t) \le C$ for all $t > 0$, where g is a convex increasing function and h is a concave increasing function on $[0, \infty)$.

From [15], we know that the class $G(p, q, C)$ contains some very interesting functions, such as $\varphi(t) = t^p$ and $\varphi(t) = t^p \log_+^\alpha(t)$, $p \ge 1$, $\alpha \in \mathbb{R}$, and each of φ, g and h is doubling in the sense that its values at t and $2t$ are uniformly comparable for all $t > 0$, and the consequent fact that

$$C_1 t^q \le h^{-1}(\varphi(t)) \le C_2 t^q, \qquad C_1 t^p \le g^{-1}(\varphi(t)) \le C_2 t^p, \tag{8}$$

where C_1 and C_2 are constants. We will need the following reverse Hölder inequality.

Lemma 4 (see [4]). *Let u be a solution of the nonhomogeneous A-harmonic equation* (1) *in a bounded and convex domain Ω and $0 < s, t < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \le C|B|^{(t-s)/st} \|u\|_{t,\sigma B} \tag{9}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$.

Lemma 5 (see [1]). *Let u be a solution of the nonhomogeneous A-harmonic equation* (1) *in a bounded and convex domain Ω. Let H be the projection operator, and let $T : C^\infty(\Omega, \Lambda^l) \to C^\infty(\Omega, \Lambda^{l-1})$ be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\left\|T(d(H(u))) - (T(d(H(u))))_B\right\|_{s,B} \le C|B| \operatorname{diam}(B) \|du\|_{s,B} \tag{10}$$

for all balls B with $B \subset \Omega$.

Lemma 6 (see [1]). *Let u be a solution of the nonhomogeneous A-harmonic equation* (1) *in a bounded and convex domain Ω. Let H be the projection operator, and let $T : C^\infty(\Omega, \Lambda^l) \to$*

$C^\infty(\Omega, \Lambda^{l-1})$ be the homotopy operator. Then, there exists a constant C, independent of u, such that

$$\|T(d(H(u))) - (T(d(H(u))))_B\|_{s,B} \le C|B|\operatorname{diam}(B)\|u\|_{s,\sigma B} \tag{11}$$

for all balls B with $\sigma B \subset \Omega$, where $\sigma > 1$ is a constant.

Theorem 7. *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, and let Ω be a bounded convex domain. Assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$ and u is a solution of the nonhomogeneous A-harmonic (1) in Ω, $\varphi(|du|) \in L^1_{\mathrm{loc}}(\Omega)$. Let H be the projection operator, and let $T : C^\infty(\Omega, \Lambda^l) \to C^\infty(\Omega, \Lambda^{l-1})$ be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\|T(d(H(u))) - (T(d(H(u))))_B\|_{\varphi,B} \le C|B|\operatorname{diam}(B)\|du\|_{\varphi,\sigma B} \tag{12}$$

for some $\sigma > 1$ and all balls B with $\sigma B \subset \Omega$.

Proof. For any constant $k > 0$, from Lemmas 4 and 5, (i) in Definition 3, using the fact that φ is an increasing function, Jensen's inequality, and noticing that φ and g are doubling, we have

$$\begin{aligned}
&\varphi\left(\frac{1}{k}\left(\int_B |T(d(H(u))) - (T(d(H(u))))_B|^q dx\right)^{1/q}\right) \\
&\quad\le \varphi\left(\frac{1}{k}C_1|B|^{(p-q)/pq}\left(\int_{\sigma B}|T(d(H(u))) - (T(d(H(u))))_B|^p dx\right)^{1/p}\right) \\
&\quad\le \varphi\left(\frac{1}{k}C_2|B|^{1+(p-q)/pq}\operatorname{diam}(B)\left(\int_{\sigma B}|du|^p dx\right)^{1/p}\right) \\
&\quad\le \varphi\left(\left(\frac{1}{k^p}C_2^p|B|^{p+(p-q)/q}(\operatorname{diam}(B))^p\int_{\sigma B}|du|^p dx\right)^{1/p}\right) \\
&\quad\le C_3 g\left(\frac{1}{k^p}C_2^p|B|^{p+(p-q)/q}(\operatorname{diam}(B))^p\int_{\sigma B}|du|^p dx\right) \\
&\quad= C_3 g\left(\int_{\sigma B}\frac{1}{k^p}C_2^p|B|^{p+(p-q)/q}(\operatorname{diam}(B))^p|du|^p dx\right) \\
&\quad\le C_3\int_{\sigma B} g\left(\frac{1}{k^p}C_2^p|B|^{p+(p-q)/q}(\operatorname{diam}(B))^p|du|^p\right)dx.
\end{aligned} \tag{13}$$

Since $p \ge 1$, then, $1 + (p-q)/pq > 0$. Hence, we have $|B|^{1+(p-q)/pq} \le |\Omega|^{1+(p-q)/pq} \le C_4$. From (i) in Definition 3, we find that $g(t) \le C_5\varphi(t^{1/p})$. Thus,

$$\begin{aligned}
&\int_{\sigma B} g\left(\frac{1}{k^p}C_2^p|B|^{p+(p-q)/q}(\operatorname{diam}(B))^p|du|^p\right)dx \\
&\quad\le C_5\int_{\sigma B}\varphi\left(\frac{1}{k}C_2|B|^{1+(p-q)/pq}\operatorname{diam}(B)|du|\right)dx \\
&\quad\le C_5\int_{\sigma B}\varphi\left(\frac{1}{k}C_2|B|\operatorname{diam}(B)|du|\right)dx.
\end{aligned} \tag{14}$$

Combining (13) and (14) yields

$$\begin{aligned}
&\varphi\left(\frac{1}{k}\left(\int_B |T(d(H(u))) - (T(d(H(u))))_B|^q dx\right)^{1/q}\right) \\
&\quad\le C_6\int_{\sigma B}\varphi\left(\frac{1}{k}C_2|B|\operatorname{diam}(B)|du|\right)dx.
\end{aligned} \tag{15}$$

Using Jensen's inequality for h^{-1}, (8), and noticing that φ and h are doubling, we obtain

$$\begin{aligned}
&\int_B \varphi\left(\frac{|T(d(H(u))) - (T(d(H(u))))_B|}{k}\right)dx \\
&\quad= h\left(h^{-1}\left(\int_B \varphi\left(\left(|T(d(H(u))) - (T(d(H(u))))_B|\right)\times(k)^{-1}\right)dx\right)\right) \\
&\quad\le h\left(\int_B h^{-1}\left(\varphi\left(\left(|T(d(H(u))) - (T(d(H(u))))_B|\right)\times(k)^{-1}\right)\right)dx\right) \\
&\quad\le h\left(C_7\int_B\left(\left(|T(d(H(u))) - (T(d(H(u))))_B|\right)\times(k)^{-1}\right)^q dx\right) \\
&\quad\le C_8\varphi\left(\left(C_7\int_B\left(\left(|T(d(H(u))) - (T(d(H(u))))_B|\right)\times(k)^{-1}\right)^q dx\right)^{1/q}\right) \\
&\quad\le C_8\varphi\left(\frac{1}{k}\left(C_7\int_B\left(|T(d(H(u))) - (T(d(H(u))))_B|\right)^q dx\right)^{1/q}\right) \\
&\quad\le C_9\varphi\left(\frac{1}{k}\left(\int_B\left(|T(d(H(u))) - (T(d(H(u))))_B|\right)^q dx\right)^{1/q}\right).
\end{aligned} \tag{16}$$

Substituting (15) into (16) and noticing that φ is doubling, we have

$$\begin{aligned}\int_B \varphi\left(\frac{|T(d(H(u)))-(T(d(H(u))))_B|}{k}\right)dx &\le C_{10}\int_{\sigma B}\varphi\left(\frac{1}{k}C_2|B|\operatorname{diam}(B)|du|\right)dx \\ &\le C_{11}\int_{\sigma B}\varphi\left(\frac{1}{k}|B|\operatorname{diam}(B)|du|\right)dx.\end{aligned} \tag{17}$$

From Definition 2 and (17), we have the following version of Poincaré inequality with the Orlicz norm:

$$\begin{aligned}&\|T(d(H(u)))-(T(d(H(u))))_B\|_{\varphi,B} \\ &\quad\le C|B|\operatorname{diam}(B)\|du\|_{\varphi,\sigma B}.\end{aligned} \tag{18}$$

We have completed the proof of Theorem 7. □

Theorem 8. *Let φ be a Young function in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, and let Ω be a bounded convex domain. Assume that $\varphi(|u|) \in L^1_{\mathrm{loc}}(\Omega)$ and u is a solution of the non-homogeneous A-harmonic (1) in Ω, $\varphi(|du|) \in L^1_{\mathrm{loc}}(\Omega)$. Let H be the projection operator, and let $T : C^\infty(\Omega,\Lambda^l) \to C^\infty(\Omega,\Lambda^{l-1})$ be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\begin{aligned}&\|T(d(H(u)))-(T(d(H(u))))_B\|_{\varphi,B} \\ &\quad\le C|B|\operatorname{diam}(B)\|du\|_{\varphi,B}\end{aligned} \tag{19}$$

for some $\sigma > 1$ and all balls B with $\sigma B \subset \Omega$.

Proof. For any constant $k > 0$, from Lemma 5, (i) in Definition 3, using the fact that φ is an increasing function, Jensen's inequality, and noticing that φ and g are doubling, we have

$$\begin{aligned}&\varphi\left(\frac{1}{k}\left(\int_B|T(d(H(u)))-(T(d(H(u))))_B|^p dx\right)^{1/p}\right) \\ &\quad\le \varphi\left(\frac{1}{k}C_1|B|\operatorname{diam}(B)\left(\int_B|du|^p dx\right)^{1/p}\right) \\ &\quad\le \varphi\left(\left(\frac{1}{k^p}C_1^p|B|^p(\operatorname{diam}(B))^p\int_B|du|^p dx\right)^{1/p}\right) \\ &\quad\le C_2 g\left(\frac{1}{k^p}C_1^p|B|^p(\operatorname{diam}(B))^p\int_B|du|^p dx\right) \\ &\quad= C_2 g\left(\int_B\frac{1}{k^p}C_1^p|B|^p(\operatorname{diam}(B))^p|du|^p dx\right) \\ &\quad\le C_2\int_B g\left(\frac{1}{k^p}C_1^p|B|^p(\operatorname{diam}(B))^p|du|^p\right)dx.\end{aligned} \tag{20}$$

Since $p \ge 1$, then $|B| \le |\Omega| \le C_3$. From (i) in Definition 3, we find that $g(t) \le C_4\varphi(t^{1/p})$. Thus,

$$\begin{aligned}&\int_B g\left(\frac{1}{k^p}C_1^p|B|^p(\operatorname{diam}(B))^p|du|^p\right)dx \\ &\quad\le C_4\int_B\varphi\left(\frac{1}{k}C_1|B|\operatorname{diam}(B)|du|\right)dx.\end{aligned} \tag{21}$$

Combining (20) and (21) yields

$$\begin{aligned}&\varphi\left(\frac{1}{k}\left(\int_B|T(d(H(u)))-(T(d(H(u))))_B|^p dx\right)^{1/p}\right) \\ &\quad\le C_5\int_B\varphi\left(\frac{1}{k}C_1|B|\operatorname{diam}(B)|du|\right)dx.\end{aligned} \tag{22}$$

Using Jensen's inequality for g^{-1}, (8), and noticing that φ and h are doubling, we obtain

$$\begin{aligned}&\int_B\varphi\left(\frac{|T(d(H(u)))-(T(d(H(u))))_B|}{k}\right)dx \\ &\quad= g\left(g^{-1}\left(\int_B\varphi\left((|T(d(H(u))) -(T(d(H(u))))_B|)\times(k)^{-1}\right)dx\right)\right) \\ &\quad\le g\left(\int_B g^{-1}\left(\varphi\left((|T(d(H(u))) -(T(d(H(u))))_B|)\times(k)^{-1}\right)\right)dx\right) \\ &\quad\le g\left(C_6\int_B\left((|T(d(H(u))) -(T(d(H(u))))_B|)\times(k)^{-1}\right)^p dx\right) \\ &\quad\le C_7\varphi\left(\left(C_6\int_B\left((|T(d(H(u))) -(T(d(H(u))))_B|)\times(k)^{-1}\right)^p dx\right)^{1/p}\right) \\ &\quad\le C_7\varphi\left(\frac{1}{k}\left(C_6\int_B(|T(d(H(u))) -(T(d(H(u))))_B|)^p dx\right)^{1/p}\right) \\ &\quad\le C_8\varphi\left(\frac{1}{k}\left(\int_B(|T(d(H(u))) -(T(d(H(u))))_B|)^p dx\right)^{1/p}\right).\end{aligned} \tag{23}$$

Substituting (22) into (23) and noticing that φ is doubling, we have

$$\int_B \varphi\left(\frac{\left|T(d(H(u))) - (T(d(H(u))))_B\right|}{k}\right) dx \leq C_9 \int_B \varphi\left(\frac{1}{k} C_1 |B| \operatorname{diam}(B) |du|\right) dx \leq C_{10} \int_B \varphi\left(\frac{1}{k} |B| \operatorname{diam}(B) |du|\right) dx. \tag{24}$$

From Definition 2 and (24), we have the following version of Poincaré inequality with the Orlicz norm:

$$\left\|T(d(H(u))) - (T(d(H(u))))_B\right\|_{\varphi,B} \leq C |B| \operatorname{diam}(B) \|du\|_{\varphi,B}. \tag{25}$$

We have completed the proof of Theorem 8. □

Using a similar method to the proof of Theorem 8, we can establish the following version of Poincaré inequality with the Orlicz norm.

Theorem 9. *Let φ be a Young function in the class $G(p,q,C)$, $1 \leq p < q < \infty, C \geq 1$, and let Ω be a bounded convex domain. Assume that $\varphi(|u|) \in L^1_{loc}(\Omega)$ and u is a solution of the non-homogeneous A-harmonic* (1) *in Ω, $\varphi(|du|) \in L^1_{loc}(\Omega)$. Let H be the projection operator, and let $T : C^\infty(\Omega, \Lambda^l) \to C^\infty(\Omega, \Lambda^{l-1})$ be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\left\|T(d(H(u))) - (T(d(H(u))))_B\right\|_{\varphi,B} \leq C |B| \operatorname{diam}(B) \|u\|_{\varphi,\sigma B} \tag{26}$$

for some $\sigma > 1$ and all balls B with $\sigma B \subset \Omega$.

Theorem 10. *Let φ be a Young function in the class $G(p,q,C)$, $1 \leq p < q < \infty$, $C \geq 1$, and let Ω be a bounded convex domain. Assume that $\varphi(|u|) \in L^1_{loc}(\Omega)$ and u is a solution of the non-homogeneous A-harmonic* (1) *in Ω, $\varphi(|du|) \in L^1_{loc}(\Omega)$. Let H be the projection operator, and let $T : C^\infty(\Omega, \Lambda^l) \to C^\infty(\Omega, \Lambda^{l-1})$ be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\left\|T(d(H(u))) - (T(d(H(u))))_{B_0}\right\|_{\varphi,\Omega} \leq C |B| \operatorname{diam}(B) \|du\|_{\varphi,\Omega}, \tag{27}$$

where $B_0 \subset \Omega$ is some fixed ball.

Proof. From definition of the $L^\varphi(\Omega)$ and (12), we have

$$\begin{aligned}\left\|T(d(H(u))) - (T(d(H(u))))_{B_0}\right\|_{\varphi,\Omega} &\leq C_1 \sup_{B\subset\Omega} \left\|T(d(H(u))) - (T(d(H(u))))_B\right\|_{\varphi,B} \\ &\leq C_1 \sup_{B\subset\Omega} \left(C_2 |B| \operatorname{diam}(B) \|du\|_{\varphi,\sigma B}\right) \\ &\leq C_1 \sup_{B\subset\Omega} \left(C_2 |B| \operatorname{diam}(B) \|du\|_{\varphi,\Omega}\right) \\ &\leq C_3 |B| \operatorname{diam}(B) \|du\|_{\varphi,\Omega}.\end{aligned} \tag{28}$$

We have completed the proof of Theorem 10. □

Using a similar method to the proof of Theorem 8, we obtain Theorem 11.

Theorem 11. *Let φ be a Young function in the class $G(p,q,C)$, $1 \leq p < q < \infty, C \geq 1$, and let Ω be a bounded convex domain. Assume that $\varphi(|u|) \in L^1_{loc}(\Omega)$ and u is a solution of the non-homogeneous A-harmonic* (1) *in $\Omega, \varphi(|du|) \in L^1_{loc}(\Omega)$. Let H be the projection operator, and let $T : C^\infty(\Omega, \Lambda^l) \to C^\infty(\Omega, \Lambda^{l-1})$ be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\left\|T(d(H(u))) - (T(d(H(u))))_{B_0}\right\|_{\varphi,\Omega} \leq C |B| \operatorname{diam}(B) \|u\|_{\varphi,\Omega}, \tag{29}$$

where $B_0 \subset \Omega$ is some fixed ball.

It has been proved in [5] that any John domain is special $L^\varphi(\Omega)$-averaging domain. Hence, we have the following results.

Corollary 12. *Let φ be a Young function in the class $G(p,q,C)$, $1 \leq p < q < \infty, C \geq 1$, and let Ω be a bounded John domain. Assume that $\varphi(|u|) \in L^1_{loc}(\Omega)$ and u is a solution of the non-homogeneous A-harmonic* (3) *in $\Omega, \varphi(|du|) \in L^1_{loc}(\Omega)$. Let H be the projection operator, and let $T : C^\infty(\Omega, \Lambda^l) \to C^\infty(\Omega, \Lambda^{l-1})$ be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\left\|T(d(H(u))) - (T(d(H(u))))_{B_0}\right\|_{\varphi,\Omega} \leq C |B| \operatorname{diam}(B) \|du\|_{\varphi,\Omega}, \tag{30}$$

where $B_0 \subset \Omega$ is some fixed ball.

For some special convex function, we have the following theorems.

Theorem 13. *Let $\varphi = t^p$ or $\varphi = t^p \log^\alpha(e+t) \in G(p,q,C)$, $1 \leq p < q < \infty, C \geq 1, \alpha \in R$ a Young function, and Ω a bounded convex domain. Assume that $\varphi(|u|) \in L^1_{loc}(\Omega)$ and u is a solution of the nonhomogeneous A-harmonic* (1) *in $\Omega, \varphi(|du|) \in L^1_{loc}(\Omega)$. Let H be the projection operator, and let $T : C^\infty(\Omega, \Lambda^l) \to C^\infty(\Omega, \Lambda^{l-1})$ be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\left\|T(d(H(u))) - (T(d(H(u))))_B\right\|_{\varphi,B} \leq C |B| \operatorname{diam}(B) \|du\|_{\varphi,\sigma B} \tag{31}$$

for some $\sigma > 1$ and all balls B with $\sigma B \subset \Omega$.

Theorem 14. *Let $\varphi = t^p$ or $\varphi = t^p \log^{\alpha}(e+t) \in G(p,q,C)$, $1 \leq p < q < \infty, C \geq 1, \alpha \in R$ a Young function, and Ω a bounded convex domain. Assume that $\varphi(|u|) \in L^1_{loc}(\Omega)$ and u is a solution of the nonhomogeneous A-harmonic (1) in $\Omega, \varphi(|du|) \in L^1_{loc}(\Omega)$. Let H be the projection operator, and let $T : C^{\infty}(\Omega, \Lambda^l) \rightarrow C^{\infty}(\Omega, \Lambda^{l-1})$ be the homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\left\| T(d(H(u))) - (T(d(H(u))))_{B_0} \right\|_{\varphi,\Omega} \leq C|B| \operatorname{diam}(B) \|u\|_{\varphi,\Omega}, \quad (32)$$

where $B_0 \subset \Omega$ is some fixed ball.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of the paper.

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