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Research Article

The Hybrid Projection Methods for Pseudocontractive, Nonexpansive Semigroup, and Monotone Mapping

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We modify the three-step iterative schemes to prove the strong convergence theorems by using the hybrid projection methods for finding a common element of the set of solutions of fixed points for a pseudocontractive mapping and a nonexpansive semigroup mapping and the set of solutions of a variational inequality problem for a monotone mapping in a Hilbert space under some appropriate control conditions. Our theorems extend and unify most of the results that have been proved for this class of nonlinear mappings.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Recall that a mapping $T : C \to C$ is said to be a *k-strict pseudocontraction* if there exists $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C, \tag{1}$$

where I denotes the identity operator on C. When $k = 0$, $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{2}$$

And when $k = 1$, $T : C \to C$ is said to be *pseudocontraction* if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{3}$$

Clearly, the class of k-strict pseudocontraction falls into the one between classes of nonexpansive mappings and pseudocontraction mapping. We denote the set of fixed points of T by $F(T)$.

A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \ge 0, \quad \forall u, v \in C. \tag{4}$$

The classical variational inequality is used for finding $u \in C$ such that

$$\langle Au, v - u\rangle \ge 0, \quad \forall v \in C. \tag{5}$$

The set of solutions of variational inequality problems is denoted by $\mathrm{VI}(C, A)$. See, for example, [1–7] and the references therein.

In 1953, Mann [8] introduced the iteration as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S x_n, \tag{6}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. The Mann iteration has been extensively investigated for nonexpansive mappings. In an infinite-dimensional Hilbert space, the Mann iteration can conclude only weak convergence [9]. Attempts to modify the Mann iteration method (6) so that strong convergence is guaranteed have recently been made. In 1974, The Ishikawa's iteration process is defined by Ishikawa [10] as the following:

$$y_n = \beta_n x_n + (1 - \beta_n) T x_n,$$
$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \quad n \geq 0, \tag{7}$$

where the initial guess x_0 is taken in C arbitrarily and the sequences $\alpha_n, \beta_n \in [0,1]$. This is called Ishikawa Iteration. This has been studied in strong convergence theorem for lipschitzian pseudocontractive mapping in Hilbert spaces. Several years later, inspired by the idea of one and two step iterative scheme, Noor [11, 12] introduced a three-step iterative scheme and studied the approximate solution of variational inclusion in Hilbert spaces by using the techniques of updating the solution and the auxiliary principle. It has been shown in [13] by Goebel and Kirk that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations.

A family $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ of mappings of C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $S(0)x = x$ for all $x \in C$;

(ii) $S(s+t) = S(s)S(t)$ for all $s, t \geq 0$;

(iii) $\|S(s)x - S(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;

(iv) for all $x \in C$, $s \mapsto S(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of all common *fixed points of* $\mathcal{S}$; that is, $F(\mathcal{S}) = \{x \in C : S(s)x = x, 0 \leq s < \infty\}$. It is known that $F(\mathcal{S})$ is closed and convex. In the sense of nonexpansive semigroup mapping, we also see [14–24].

In 2003, Nakajo and Takahashi [25] proposed the following modification of Mann iteration method for a nonexpansive mapping T from C into itself in a Hilbert space

$$x_0 \in C \text{ is arbitrary},$$
$$y_n = \alpha_n x_n + (1 - \alpha_n) T x_n,$$
$$C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \tag{8}$$
$$Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\},$$
$$x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0,$$

where P_K denotes the metric projection from a Hilbert space H onto a close convex subset K of H and proves that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$. A projection onto intersection of two halfspaces is computed by solving a linear system of two equations with two unknowns. In 2008, Takahashi et al. [26] proved the following strong convergence theorem by the new *hybrid method* in a Hilbert space. They assume $x_0 \in H$, $C_1 = C$, $u_1 = P_{C_1} x_0$, and defined the sequence by (8) where $0 \leq \alpha_n < a < 1$

$$y_n = \alpha_n u_n + (1 - \alpha_n) T u_n,$$
$$C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \tag{9}$$
$$u_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 0,$$

where $0 \leq \alpha_n < a < 1$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Recently, Zegeye and Shahzad [27] defined the mappings as follows:

$$F_{r_n}(x) = \Big\{ z \in C : \langle y - z, Az \rangle + \frac{1}{r_n} \langle y - z, z - x \rangle \geq 0, \ \forall y \in C \Big\}, \tag{10}$$

$$T_{r_n}(x) = \Big\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r_n} \langle y - z, (1 + r_n) z - x \rangle \leq 0, \ \forall y \in C \Big\}, \tag{11}$$

for all $x \in H$ and $r_n \in (0, \infty)$, where $T : C \to C$ is a continuous pseudocontractive mapping and $A : C \to H$ is a continuous monotone mapping. In the following year, Tang [28] introduced a viscosity iterative process, which converges strongly to a common element of the set of fixed points of a pseudocontractive mapping and the set of solutions of a monotone mapping as the following:

$$y_n = \lambda_n x_n + (1 - \lambda_n) F_{r_n} x_n,$$
$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{r_n} y_n, \tag{12}$$

where F_{r_n} and T_{r_n} are defined by (10) and (11), respectively.

In this paper, we modify the three-step iterative schemes to prove the strong convergence theorems by using the hybrid projection methods for finding a common element of the set of solutions of fixed points for a pseudocontractive mapping and a nonexpansive semigroup mapping and the set of solutions of a variational inequality problem for a monotone mapping in a Hilbert space under some appropriate control conditions. The results that are presented in this paper extend and improve the corresponding ones announced by Nakajo and Takahashi [25], Takahashi et al. [26], Zegeye and Shahzad [27], Tang [28], and many authors.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ and let C be a closed convex subset of H. Then

$$\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda (1 - \lambda) \|x - y\|^2, \tag{13}$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

Recall that, the metric projection P_C from a Hilbert space H to a closed convex subset C of H is defined as the following: given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf \{\|x - y\| : y \in C\}. \tag{14}$$

$P_C x$ is characterized as follows:

$$\begin{gathered}\|x-P_C x\| \leq \|x-y\|,\\ \langle x-P_C x, y-P_C x\rangle \leq 0,\\ \|x-y\|^2 \geq \|x-P_C x\|^2+\|y-P_C x\|^2,\end{gathered} \tag{15}$$

for all $x \in H$, $y \in C$.

Hilbert space H satisfies the *Kadec-Klee property* [28, 29]; that is, for any sequence $\{x_n\}$. $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.

A normed space X is said to satisfy Opial's condition [30], if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty}\|x_n-x\| < \liminf_{n\to\infty}\|x_n-y\|, \quad \forall y \in X,\ y \neq x. \tag{16}$$

Lemma 1 (see [27]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T : C \to C$ be a continuous pseudocontractive mapping and let $A : C \to H$ be a continuous monotone mapping and define mappings F_r and T_r as follows: $x \in H$, $r \in (0,\infty)$*

$$\begin{aligned}F_r(x) = \Big\{z \in C : &\langle y-z, Az\rangle\\ &+\frac{1}{r}\langle y-z, z-x\rangle \geq 0,\ \forall y \in C\Big\},\\ T_r(x) = \Big\{z \in C : &\langle y-z, Tz\rangle\\ &-\frac{1}{r}\langle y-z, (1+r)z-x\rangle \leq 0,\ \forall y \in C\Big\}.\end{aligned} \tag{17}$$

Then, the following hold:

(1) *F_r and T_r are single-valued;*

(2) *F_r and T_r are firmly nonexpansive; that is, for any $x, y \in H$, $\|F_r x - F_r y\|^2 \leq \langle F_r x - F_r y, x - y\rangle$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x-y\rangle$;*

(3) *$F(T_r) = F(T)$, $F(F_r) = VI(C,A)$;*

(4) *$F(T)$ and $VI(C,A)$ are closed and convex.*

3. Main Results

3.1. The Hybrid Method

Theorem 2. *Let C be a nonempty bounded closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudocontractive mapping and let $A : C \to H$ be a continuous monotone mapping. Let $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that* $\liminf_{n\to 0}\mu_n = 0$, $\limsup_{n\to 0}\mu_n > 0$, *and* $\lim_{n\to 0}(\mu_{n+1} - \mu_n) = 0$. *Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be the sequences in $[0,a)$ for some $a \in [0,1)$, $\{r_n\} \subset (0,\infty)$ such that* $\liminf_{n\to\infty} r_n > 0$ *and suppose $F = F(\mathcal{S}) \cap F(T) \cap VI(C,A) \neq \emptyset$. The mappings T_{r_n} and F_{r_n} are defined by* (10) *and* (11). *Let $\{x_n\}$ be sequences generated by $x_0 \in C$ and*

$$\begin{gathered}w_n = \gamma_n x_n + (1-\gamma_n) F_{r_n} x_n,\\ z_n = \beta_n w_n + (1-\beta_n) T_{r_n} w_n,\\ y_n = \alpha_n z_n + (1-\alpha_n) S(\mu_n) z_n,\\ C_n = \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\},\\ Q_n = \{z \in C \mid \langle x_n - z, x_0 - x_n\rangle \geq 0\},\\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0.\end{gathered} \tag{18}$$

Then the sequence $\{x_n\}$ converges strongly to $P_F x_0$.

Proof. Consider that

$$\begin{aligned}\|y_n - x_n\|^2 &= \|(y_n - z) - (x_n - z)\|^2\\ &= \|y_n - z\|^2 - 2\langle y_n - z, x_n - z\rangle + \|x_n - z\|^2\\ &= \|y_n - z\|^2 - 2\langle y_n - x_n + x_n - z, x_n - z\rangle\\ &\quad + \|x_n - z\|^2\\ &= \|y_n - z\|^2 - 2\langle y_n - x_n, x_n - z\rangle\\ &\quad - 2\langle x_n - z, x_n - z\rangle + \|x_n - z\|^2\\ &= \|y_n - z\|^2 - 2\langle y_n - x_n, x_n - z\rangle - \|x_n - z\|^2.\end{aligned} \tag{19}$$

On the other hand, we get that C_n is closed and Q_n is closed and convex for all $n \geq 0$. From (19), $\|y_n - z\| \leq \|x_n - z\|$ is equivalent to $\|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z\rangle \leq 0$ for $z \in C$. Thus, we have C_n is convex for all $n \geq 0$. Therefore, $C_n \cap Q_n$ is closed and convex for all $n \geq 0$. Let $x^* \in F$, we have

$$\begin{aligned}\|w_n - x^*\| &\leq \gamma_n\|x_n - x^*\| + (1-\gamma_n)\|F_{r_n}x_n - x^*\|\\ &\leq \gamma_n\|x_n - x^*\| + \|F_{r_n}x_n - x^*\| - \gamma_n\|x_n - x^*\|\\ &= \|F_{r_n}x_n - x^*\|\\ &\leq \|x_n - x^*\|,\\ \|z_n - x^*\| &\leq \beta_n\|w_n - x^*\| + (1-\beta_n)\|T_{r_n}w_n - x^*\|\\ &\leq \beta_n\|x_n - x^*\| + \|T_{r_n}w_n - x^*\| - \beta_n\|x_n - x^*\|\\ &= \|T_{r_n}w_n - x^*\|\\ &\leq \|w_n - x^*\|\\ &\leq \|F_{r_n}x_n - x^*\|\\ &\leq \|x_n - x^*\|.\end{aligned} \tag{20}$$

It follows that

$$\begin{aligned}\|y_n - x^*\| &\leq \alpha_n \|z_n - x^*\| + (1-\alpha_n)\|S(\mu_n) z_n - x^*\| \\ &\leq \alpha_n \|x_n - x^*\| + \|z_n - x^*\| - \alpha_n \|x_n - x^*\| \\ &= \|z_n - x^*\| \\ &\leq \|w_n - x^*\| \\ &\leq \|x_n - x^*\|.\end{aligned} \tag{21}$$

Therefore, $x^* \in C_n$ for all $n \geq 0$. Thus, we have $F \subset C_n$ for all $n \geq 0$.

Next, we use mathematical induction. Start with $n = 0$, we have $x_0 \in C$ and $Q_0 = C$ then $F \subset C_0 \cap Q_0$. Assume that x_k is given and $F \subset C_k \cap Q_k$ for some $k \geq 0$. There exists a unique $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k} x_0$, then we get $\langle x_{k+1} - z, x_0 - x_{k+1}\rangle \geq 0$ for $z \in C_k \cap Q_k$. From $F \subset C_k \cap Q_k$, we have $F \subset Q_{k+1}$. Therefore, $F \subset C_{k+1} \cap Q_{k+1}$. Thus $\{x_n\}$ is well defined and $F \subset C_n \cap Q_n$ for all $n \geq 0$.

Since F is a nonempty closed convex subset of C, there exists a unique $\overline{x} \in F$ such that $\overline{x} = P_F x_0$. From $x_{n+1} = P_{C_n \cap Q_n} x_0$ and the metric projection property, we have

$$\begin{aligned}0 &\leq \langle x_{n+1} - x_0, \overline{x} - x_{n+1}\rangle \\ &= \langle x_{n+1} - x_0, \overline{x} - x_0 + x_0 - x_{n+1}\rangle \\ &= \langle x_{n+1} - x_0, \overline{x} - x_0\rangle + \langle x_{n+1} - x_0, x_0 - x_{n+1}\rangle \\ &= \|x_{n+1} - x_0\| \|\overline{x} - x_0\| - \|x_{n+1} - x_0\|^2.\end{aligned} \tag{22}$$

It follows that $\|x_{n+1} - x_0\| \leq \|\overline{x} - x_0\|$ for all $\overline{x} \in F \subset C_n \cap Q_n$ and $n \geq 0$. This implied $\{x_n\}$ is bounded. So, $\{F_{r_n} x_n\}$, $\{T_{r_n} w_n\}$, $\{S(\mu_n) z_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are bounded.

Next, we show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{23}$$

Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$, $x_n = P_{Q_n} x_0$, and $\langle x_n - x_0, x_{n+1} - x_n\rangle \geq 0$, as same as the prove of (22), we get $\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$ for all $n \geq 0$. Thus, $\{\|x_n - x_0\|\}$ is nondecreasing. By $\{\|x_n - x_0\|\}$ is bounded and nondecreasing, there exists the limit of $\{\|x_n - x_0\|\}$. Since $x_{n+1} \in Q_n$, we have

$$\langle x_n - x_{n+1}, x_0 - x_n\rangle \geq 0, \quad \forall n \geq 0, \tag{24}$$

$$\begin{aligned}\|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle \\ &\quad + \|x_{n+1} - x_0\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle \\ &\quad + \|x_{n+1} - x_0\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n\rangle \\ &\quad + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_{n+1} - x_0\|^2 \\ &\leq \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n\rangle \\ &\quad + \|x_{n+1} - x_0\|^2 \\ &= \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_{n+1} - x_0\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.\end{aligned} \tag{25}$$

Since $\lim_{n\to\infty}\|x_n - x_0\|$ exists, therefore (23) holds.

Next, we show that

$$\lim_{n\to\infty} \|S(\mu_n) x_n - x_n\| = 0. \tag{26}$$

Since $x_{n+1} \in C_n$, we have

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \leq 2\|x_{n+1} - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{27}$$

Let $x^* \in F$, $v_n = F_{r_n} x_n$, and $u_n = T_{r_n} w_n$; it follows from Lemma 1, we get

$$\begin{aligned}\|v_n - x^*\|^2 &= \|F_{r_n} x_n - F_{r_n} x^*\|^2 \\ &\leq \langle F_{r_n} x_n - F_{r_n} x^*, x_n - x^*\rangle \\ &= \langle v_n - x^*, x_n - x^*\rangle \\ &= \frac{1}{2}\left(\|v_n - x^*\|^2 + \|x_n - x^*\|^2 - \|v_n - x_n\|^2\right), \\ \|u_n - x^*\|^2 &= \|T_{r_n} w_n - T_{r_n} x^*\|^2 \\ &\leq \langle T_{r_n} w_n - T_{r_n} x^*, w_n - x^*\rangle \\ &= \langle u_n - x^*, w_n - x^*\rangle \\ &= \frac{1}{2}\left(\|u_n - x^*\|^2 + \|w_n - x^*\|^2 - \|u_n - w_n\|^2\right).\end{aligned} \tag{28}$$

Hence

$$\begin{aligned}\|v_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|v_n - x_n\|^2, \\ \|u_n - x^*\|^2 &\leq \|w_n - x^*\|^2 - \|u_n - w_n\|^2.\end{aligned} \tag{29}$$

It follows that

$$\begin{aligned}&\|y_n - x^*\|^2 \\ &\quad\leq \alpha_n\|z_n - x^*\|^2 + (1-\alpha_n)\|S(\mu_n) z_n - x^*\|^2 \\ &\quad\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\ &\quad\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\left(\|x_n - x^*\|^2 - \|v_n - x_n\|^2\right)\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - \|v_n - x_n\|^2 \\
&= \|x_n - x^*\|^2 - \|v_n - x_n\|^2, \\
&\|y_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\left(\|w_n - x^*\|^2 - \|u_n - w_n\|^2\right) \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - \|u_n - w_n\|^2 \\
&= \|x_n - x^*\|^2 - \|u_n - w_n\|^2.
\end{aligned} \tag{30}$$

Consequently, we have that

$$\begin{aligned}
\|v_n - x_n\|^2 &\leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 \\
&\leq (\|x_n - x^*\| + \|y_n - x^*\|)\|x_n - y_n\|, \\
\|u_n - w_n\|^2 &\leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 \\
&\leq (\|x_n - x^*\| + \|y_n - x^*\|)\|x_n - y_n\|.
\end{aligned} \tag{31}$$

Equation (27) implies that

$$\begin{aligned}
&\lim_{n\to\infty}\|v_n - x_n\| = 0, \\
&\lim_{n\to\infty}\|u_n - w_n\| = 0.
\end{aligned} \tag{32}$$

On the other hand, from (18) and (32), we also have

$$\begin{aligned}
&\|w_n - v_n\| = \gamma_n\|x_n - v_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\
&\|z_n - u_n\| = \beta_n\|w_n - u_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{33}$$

It follows from (32)-(33) we get

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{34}$$

Since $y_n - x_n = \alpha_n(z_n - x_n) + (1-\alpha_n)(S(\mu_n)z_n - x_n)$ and from (27) and (34), we get

$$\|x_n - S(\mu_n)z_n\| \leq \frac{\alpha_n}{1-\alpha_n}\|z_n - x_n\| + \frac{1}{1-\alpha_n}\|x_n - y_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{35}$$

It follows that

$$\begin{aligned}
\|x_n - S(\mu_n)x_n\| &\leq \|x_n - S(\mu_n)z_n\| + \|S(\mu_n)z_n - S(\mu_n)x_n\| \\
&\leq \|x_n - S(\mu_n)z_n\| + \|z_n - x_n\|.
\end{aligned} \tag{36}$$

Therefore (26) holds.

Next, we show that $x' \in F = F(\mathcal{S}) \cap F(T) \cap \mathrm{VI}(C, A)$. First, we show that x' is the unique solution in $F(\mathcal{S})$. Since $\{x_n\}$ is bounded, we choose subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and assume that $x_{n_i} \rightharpoonup x'$. Suppose that $x' \notin F(\mathcal{S}) = \cap_{0\leq s<\infty}F(T(s))$; that is, $x' \neq S(s)x'$. From Opial's condition and (26), we have

$$\begin{aligned}
&\liminf_{i\to\infty}\|x_{n_i} - x'\| \\
&\quad < \liminf_{i\to\infty}\|x_{n_i} - S(s)x'\| \\
&\quad \leq \liminf_{i\to\infty}\left(\|x_{n_i} - S(s)x_{n_i}\| + \|S(s)x_{n_i} - S(s)x'\|\right) \\
&\quad \leq \liminf_{i\to\infty}\|x_{n_i} - x'\|.
\end{aligned} \tag{37}$$

This is a contradiction. Thus, we obtain $x' \in F(\mathcal{S})$.

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \rightharpoonup x'$, $x' \in C$. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup x'$. From the setting $v_n = F_{r_n}x_n$ and (10), we have

$$\left\langle y - v_{n_i}, Av_{n_i}\right\rangle + \frac{1}{r_{n_i}}\left\langle y - v_{n_i}, v_{n_i} - x_{n_i}\right\rangle \geq 0, \quad \forall y \in C. \tag{38}$$

For $t \in (0,1)$ and $v \in C$, let $v_t = tv + (1-t)x'$. Since $v \in C$ and $x' \in C$, we have $v_t \in C$ and

$$\begin{aligned}
\left\langle v_t - v_{n_i}, Av_t\right\rangle &\geq \left\langle v_t - v_{n_i}, Av_t - Av_{n_i}\right\rangle \\
&\quad - \left\langle v_t - v_{n_i}, \frac{v_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle.
\end{aligned} \tag{39}$$

Since A is a monotone and $v_{n_i} - x_{n_i} \to 0$, we obtain

$$\langle v_t - x', Av_t\rangle = \lim_{i\to\infty}\langle v_t - v_{n_i}, Av_t\rangle \geq 0. \tag{40}$$

By the continuity of A, if $t \to 0$ then $\langle v - x', Av\rangle \geq 0, \forall v \in C$. Therefore, $x' \in \mathrm{VI}(C, A)$.

On the other hand, since $u_n = T_{r_n}w_n$, from (11) we have

$$\left\langle y - u_{n_i}, Tu_{n_i}\right\rangle - \frac{1}{r_{n_i}}\left\langle y - u_{n_i}, \left(1 + r_{n_i}\right)u_{n_i} - w_{n_i}\right\rangle \leq 0, \quad \forall y \in C. \tag{41}$$

For $t \in (0,1)$ and $v \in C$, let $v_t = tv + (1-t)x'$. Since $v \in C$ and $x' \in C$, we have $v_t \in C$ and

$$\begin{aligned}
\left\langle u_{n_i} - v_t, Tv_t\right\rangle &\geq \left\langle u_{n_i} - v_t, Tv_t\right\rangle + \left\langle v_t - u_{n_i}, Tu_{n_i}\right\rangle \\
&\quad - \frac{1}{r_{n_i}}\left\langle v_t - u_{n_i}, \left(1 + r_{n_i}\right)u_{n_i} - w_{n_i}\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= -\left\langle v_t - u_{n_i}, Tv_t - Tu_{n_i}\right\rangle \\
&\quad - \frac{1}{r_{n_i}}\left\langle v_t - u_{n_i}, u_{n_i} - w_{n_i}\right\rangle - \left\langle v_t - u_{n_i}, u_{n_i}\right\rangle \\
&\geq -\left\|v_t - u_{n_i}\right\|^2 - \frac{1}{r_{n_i}}\left\langle v_t - u_{n_i}, u_{n_i} - w_{n_i}\right\rangle \\
&\quad - \left\langle v_t - u_{n_i}, u_{n_i}\right\rangle \\
&= -\frac{1}{r_{n_i}}\left\langle v_t - u_{n_i}, u_{n_i} - w_{n_i}\right\rangle - \left\langle v_t - u_{n_i}, v_t\right\rangle .
\end{aligned} \tag{42}
$$

It follows from $u_n - w_n \to 0$ as $n \to \infty$, we get $\langle x' - v_t, Tv_t\rangle \geq -\langle v_t - x', v_t\rangle$ and hence $-\langle v - x', Tv_t\rangle \geq -\langle v - x', v_t\rangle, \forall v \in C$. By the continuity of T, if $t \to 0$ then $-\langle v - x', Tx'\rangle \geq -\langle v - x', x'\rangle, \forall v \in C$. Let $v = Tx'$, we have $x' = Tx'$; therefore, $x' \in F(T)$. Consequently, we conclude that $x' \in F = F(\mathcal{S}) \cap F(T) \cap \mathrm{VI}(C, A)$.

Finally, we show that $x_n \to x'$, where $x' = P_F x_0$. Since $x_{n+1} = P_{C_n \cap Q_n} x_0$ and $x' \in F \subset C_n \cap Q_n$, we get

$$\left\|x_{n+1} - x_0\right\| \leq \left\|x' - x_0\right\|, \quad \forall n \geq 0. \tag{43}$$

If $\overline{x} = P_F x_0$, it follows from (43), and the lower semicontinuity of the norm that

$$
\begin{aligned}
\left\|x' - x_0\right\| &\leq \left\|\overline{x} - x_0\right\| \leq \liminf_{i \to \infty} \left\|x_{n_i} - x_0\right\| \\
&\leq \limsup_{i \to \infty} \left\|x_{n_i} - x_0\right\| \leq \left\|x' - x_0\right\| .
\end{aligned} \tag{44}
$$

Thus, we obtain that $\lim_{i \to \infty}\|x_{n_i} - x_0\| = \|x' - x_0\| = \|\overline{x} - x_0\|$. Using the Kadec-Klee property of H, we obtain that

$$\lim_{i \to \infty} x_{n_i} = x' = \overline{x}. \tag{45}$$

Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to x', where $x' = P_F x_0$. This completes the proof. □

Corollary 3. *Let C be a nonempty bounded closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudocontractive mapping and let $A : C \to H$ be a continuous monotone mapping. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $\liminf_{n \to 0}\mu_n = 0$, $\limsup_{n \to 0}\mu_n > 0$, and $\lim_{n \to 0}(\mu_{n+1} - \mu_n) = 0$. Let $\{r_n\} \subset (0, \infty)$ such that $\liminf_{n \to \infty} r_n > 0$ and suppose $F = F(\mathcal{S}) \cap F(T) \cap VI(C, A) \neq \emptyset$. The mappings T_{r_n} and F_{r_n} are defined by* (10) *and* (11)*. Let $\{x_n\}$ be a sequences generated by $x_0 \in C$ and*

$$
\begin{aligned}
y_n &= S(\mu_n) T_{r_n} F_{r_n}, \\
C_n &= \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C \mid \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0.
\end{aligned} \tag{46}
$$

Then the sequence $\{x_n\}$ converges strongly to $P_F x_0$.

Proof. Putting $\alpha_n = \beta_n = \gamma_n = 0, \forall n \geq 0$ in Theorem 2, we can obtain the result. □

3.2. The Shrinking Projection Method

Theorem 4. *Let C be a nonempty bounded closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudocontractive mapping and let $A : C \to H$ be a continuous monotone mapping. Let $\mathcal{S} = \{S(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $\liminf_{n \to 0}\mu_n = 0$, $\limsup_{n \to 0}\mu_n > 0$, and $\lim_{n \to 0}(\mu_{n+1} - \mu_n) = 0$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be the sequences in $[0, a)$ for some $a \in [0, 1)$, $\{r_n\} \subset (0, \infty)$ such that $\liminf_{n \to \infty} r_n > 0$ and suppose $F = F(\mathcal{S}) \cap F(T) \cap VI(C, A) \neq \emptyset$. The mappings T_{r_n} and F_{r_n} are defined by* (10) *and* (11)*. Let $\{x_n\}$ be sequences generated by $x_0 \in C$ and*

$$
\begin{aligned}
w_n &= \gamma_n x_n + (1 - \gamma_n) F_{r_n} x_n, \\
z_n &= \beta_n w_n + (1 - \beta_n) T_{r_n} w_n, \\
y_n &= \alpha_n z_n + (1 - \alpha_n) S(\mu_n) z_n, \\
C_{n+1} &= \{z \in C_n \mid \|y_n - z\| \leq \|x_n - z\|\}, \\
x_{n+1} &= P_{C_{n+1}} x_0, \quad \forall n \geq 0.
\end{aligned} \tag{47}
$$

Then the sequence $\{x_n\}$ converges strongly to $P_F x_0$.

Proof. First, we show that $F \subset C_n$. By induction, it is obvious that $F \subset C_1$. Suppose that $F \subset C_k$ for some $k \geq 0$, so we have $x^* \in F \subset C_k$ such that $\|y_k - x^*\| \leq \|x_k - x^*\|$. Then, we get $x^* \in C_{n+1}$. Therefore $F \subset C_n$ for all $n \geq 0$.

On the other hand, we show that C_n is closed and convex for all $n \geq 0$. By mathematical induction, it is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \geq 0$. For $z \in C_k$, we have that $\|y_k - z\| \leq \|x_k - z\|$ is equivalent to $\|y_k - x_k\|^2 + 2\langle y_k - x_k, x_k - z\rangle \leq 0$. Thus, we have C_{k+1} is closed and convex for all $n \geq 0$. Therefore, C_n is closed and convex for all $n \geq 0$. This implies that $\{x_n\}$ is well defined.

Next, we show that $\{x_n\}$ is bounded. From the metric projection property and (47), we have $x_n = P_{C_n} x_0$ and $\langle x_n - x_0, \overline{x} - x_n\rangle \geq 0$ for all $\overline{x} \in F \subset C_n$ and $n \geq 0$. Consider

$$
\begin{aligned}
0 &\leq \left\langle x_n - x_0, \overline{x} - x_n\right\rangle \\
&= \left\langle x_n - x_0, \overline{x} - x_0 + x_0 - x_n\right\rangle \\
&= \left\langle x_n - x_0, \overline{x} - x_0\right\rangle + \left\langle x_n - x_0, x_0 - x_n\right\rangle \\
&= \left\|x_n - x_0\right\| \left\|\overline{x} - x_0\right\| - \left\|x_n - x_0\right\|^2.
\end{aligned} \tag{48}
$$

It follows that $\|x_n - x_0\| \leq \|\overline{x} - x_0\|$ for all $\overline{x} \in F$ and $n \geq 0$. This implied $\{x_n\}$ is bounded. So, $\{F_{r_n} x_n\}$, $\{T_{r_n} w_n\}$, $\{S(\mu_n) z_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are bounded.

Next, we show that

$$\lim_{n \to \infty} \left\|x_{n+1} - x_n\right\| = 0. \tag{49}$$

From the metric projection property and (47), we have $x_n = P_{C_n} x_0$, $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, and $\langle x_n - x_0, x_{n+1} - x_n\rangle \geq$

0. As same as the prove of (48), we get $\|x_n - x_0\| \le \|x_{n+1} - x_0\|$ for all $n \ge 0$. Thus, $\{\|x_n - x_0\|\}$ is nondecreasing. Since $\{\|x_n - x_0\|\}$ is bounded and nondecreasing, there exists the limit of $\{\|x_n - x_0\|\}$. Similar to the proved of (25), we get (49) is hold.

Since sequence $\{x_n\}$ is bounded, we can choose subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and assume that $x_{n_i} \rightharpoonup x'$. Similar to the proof of Theorem 2, we also have $x' \in F$.

Finally, we show that $x_n \to x'$, where $x' = P_F x_0$. Since $x_n = P_{C_n} x_0$ and $x' \in F \subset C_n$, we have

$$\|x_n - x_0\| \le \|\overline{x} - x_0\| . \tag{50}$$

It follows from (50), if $\overline{x} = P_F x_0$ and the lower semicontinuity of the norm

$$\begin{aligned} \|x' - x_0\| &\le \|\overline{x} - x_0\| \le \liminf_{i \to \infty} \|x_{n_i} - x_0\| \\ &\le \limsup_{i \to \infty} \|x_{n_i} - x_0\| \le \|x' - x_0\| , \end{aligned} \tag{51}$$

thus, we obtain that $\lim_{i \to \infty} \|x_{n_i} - x_0\| = \|\overline{x} - x_0\| = \|x' - x_0\|$. Using the Kadec-Klee property of H, we obtain that

$$\lim_{i \to \infty} x_{n_i} = \overline{x} = x'. \tag{52}$$

Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to x', where $x' = P_F x_0$. □

Corollary 5. *Let C be a nonempty bounded closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudocontractive mapping and let $A : C \to H$ be a continuous monotone mapping. Let $\mathcal{S} = \{T(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on C such that $\liminf_{n \to 0} \mu_n = 0$, $\limsup_{n \to 0} \mu_n > 0$, and $\lim_{n \to 0} (\mu_{n+1} - \mu_n) = 0$. Let $\{r_n\} \subset (0, \infty)$ such that $\liminf_{n \to \infty} r_n > 0$ and suppose $F = F(\mathcal{S}) \cap F(T) \cap VI(C, A) \neq \emptyset$. The mappings T_{r_n} and F_{r_n} are defined by* (10) *and* (11)*. Let $\{x_n\}$ be a sequences generated by $x_0 \in C$ and*

$$\begin{aligned} y_n &= S(\mu_n) T_{r_n} F_{r_n}, \\ C_{n+1} &= \{z \in C_n \mid \|y_n - z\| \le \|x_n - z\|\} , \\ x_{n+1} &= P_{C_{n+1}} x_0, \quad \forall n \ge 0. \end{aligned} \tag{53}$$

Then the sequence $\{x_n\}$ converges strongly to $P_F x_0$.

Proof. Putting $\alpha_n = \beta_n = \gamma_n = 0, \forall n \ge 0$ in Theorem 4, we can obtain the result. □

Remark 6. According to nonexpansive semigroup mapping, it will be interesting if we replace the semigroup $S = (N, +)$ by an additive positive real numbers of a commutative semigroup or a left amenable semigroup or a left reversible semigroup with using an asymptotically invariant sequence. See [31, 32].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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