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Research Article

Boundary Value Problems for Fractional Differential Equations with Fractional Multiterm Integral Conditions

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We discuss the existence and uniqueness of solutions for boundary value problems involving multiterm fractional integral boundary conditions. Our study relies on standard fixed point theorems. Illustrative examples are also presented.

1. Introduction

In this paper, we study the existence and uniqueness of solutions for the following fractional differential equation:

$$ {}^{c}D^{q}x(t) = f(t, x(t)), \quad 1 < q \le 2, \ 0 < t < T, \tag{1}$$

subject to nonlocal fractional integral boundary conditions

$$\sum_{i=1}^{m} \lambda_i I^{\alpha_i} x(\eta_i) = \omega_1, \qquad \sum_{j=1}^{n} \mu_j \left(I^{\beta_j} x(T) - I^{\beta_j} x(\xi_j) \right) = \omega_2, \tag{2}$$

where ${}^{c}D^{q}$ denotes the Caputo fractional derivative of order q, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\eta_i, \xi_j \in (0,T)$, $\lambda_i, \mu_j \in \mathbb{R}$, for all $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, $\omega_1, \omega_2 \in \mathbb{R}$, and I^{ϕ} is the Riemann-Liouville fractional integral of order $\phi > 0$ ($\phi = \alpha_i, \beta_j, i = 1,2,\ldots,m, j = 1,2,\ldots,n$).

The significance of studying problem (1)-(2) is that condition (2) is very general and includes many conditions as special cases. In particular, if $\alpha_i = \beta_j = 1$, for all $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, then condition (2) reduces to

$$\lambda_1 \int_0^{\eta_1} x(s)\,ds + \lambda_2 \int_0^{\eta_2} x(s)\,ds + \cdots + \lambda_m \int_0^{\eta_m} x(s)\,ds = \omega_1,$$
$$\mu_1 \int_{\xi_1}^{T} x(s)\,ds + \mu_2 \int_{\xi_2}^{T} x(s)\,ds + \cdots + \mu_n \int_{\xi_n}^{T} x(s)\,ds = \omega_2. \tag{3}$$

Note that condition (3) does not contain values of unknown function x at the left-side and right-side of boundary points $t = 0$ and $t = T$, respectively.

In recent years, the boundary value problems of fractional order differential equations have emerged as an important area of research, since these problems have applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, regular variation in thermodynamics, biophysics, aerodynamics, viscoelasticity and damping, electrodynamics of complex medium, wave propagation, and blood flow phenomena (see [1–5]). Many researchers have studied the existence theory for nonlinear fractional differential equations with a variety of boundary conditions; for instance, see the papers [6–16] and the references therein.

The main objective of this paper is to develop some existence and uniqueness results for the boundary value problem (1)-(2) by using standard fixed point theorems. The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel, and Section 3 contains our main results. Finally, illustrative examples are presented in Section 4.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [2, 3] and present preliminary results needed in our proofs later.

Definition 1. For an at least n-times differentiable function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$ {}^{c}D^{q}g(t) = \frac{1}{\Gamma(n-q)}\int_{0}^{t}(t-s)^{n-q-1}g^{(n)}(s)\,ds, \qquad n-1<q<n, \quad n=[q]+1, \tag{4}$$

where $[q]$ denotes the integer part of the real number q.

Definition 2. The Riemann-Liouville fractional integral of order q is defined as

$$I^{q}g(t) = \frac{1}{\Gamma(q)}\int_{0}^{t}\frac{g(s)}{(t-s)^{1-q}}\,ds, \quad q>0, \tag{5}$$

provided the integral exists.

Lemma 3. *For $q > 0$, the general solution of the fractional differential equation ${}^{c}D^{q}x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}, \tag{6}$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

In view of Lemma 3, it follows that

$$I^{q}\,{}^{c}D^{q}x(t) = x(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}, \tag{7}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

For convenience, we set

$$\begin{gathered}\Omega_1 = \sum_{i=1}^{m}\lambda_i\frac{\eta_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)}, \qquad \Omega_2 = \sum_{i=1}^{m}\lambda_i\frac{\eta_i^{\alpha_i}}{\Gamma(\alpha_i+1)},\\ \Omega_3 = \sum_{j=1}^{n}\mu_j\frac{T^{\beta_j+1}-\xi_j^{\beta_j+1}}{\Gamma(\beta_j+2)}, \qquad \Omega_4 = \sum_{j=1}^{n}\mu_j\frac{T^{\beta_j}-\xi_j^{\beta_j}}{\Gamma(\beta_j+1)},\\ \Delta = \Omega_1\Omega_4 - \Omega_2\Omega_3.\end{gathered} \tag{8}$$

Lemma 4. *Let $\Delta \neq 0$, $1 < q \leq 2$, $\alpha_i, \beta_j > 0$, $\eta_i, \xi_j \in (0,T)$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and $h \in C([0,T],\mathbb{R})$. Then, the problem*

$${}^{c}D^{q}x(t) = h(t), \quad t \in (0,T) \tag{9}$$

$$\sum_{i=1}^{m}\lambda_i I^{\alpha_i}x(\eta_i) = \omega_1, \qquad \sum_{j=1}^{n}\mu_j\left(I^{\beta_j}x(T) - I^{\beta_j}x(\xi_j)\right) = \omega_2, \tag{10}$$

has a unique solution given by

$$\begin{aligned} x(t) = I^{q}h(t) &+ \frac{\Omega_4 t - \Omega_3}{\Delta}\left(\omega_1 - \sum_{i=1}^{m}\lambda_i I^{\alpha_i+q}h(\eta_i)\right)\\ &+ \frac{\Omega_1 - \Omega_2 t}{\Delta}\left(\omega_2 - \sum_{j=1}^{n}\mu_j\left(I^{\beta_j+q}h(T) - I^{\beta_j+q}h(\xi_j)\right)\right).\end{aligned} \tag{11}$$

Proof. Using Lemma 3, (9) can be expressed as an equivalent integral equation:

$$x(t) = I^{q}h(t) + c_1 t + c_2. \tag{12}$$

Taking the Riemann-Liouville fractional integral of order $p > 0$ for (12), we have

$$I^{p}x(t) = I^{p+q}h(t) + c_1\frac{t^{p+1}}{\Gamma(p+2)} + c_2\frac{t^{p}}{\Gamma(p+1)}. \tag{13}$$

From the first condition of (10) and the equation (13) with $p = \alpha_i$, it follows that

$$c_1\sum_{i=1}^{m}\lambda_i\frac{\eta_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)} + c_2\sum_{i=1}^{m}\lambda_i\frac{\eta_i^{\alpha_i}}{\Gamma(\alpha_i+1)} = \omega_1 - \sum_{i=1}^{m}\lambda_i I^{\alpha_i+q}h(\eta_i). \tag{14}$$

According to the above process, the second conditions of (10) and (13) with $p = \beta_j$ imply that

$$\begin{aligned} c_1\sum_{j=1}^{n}\mu_j\frac{T^{\beta_j+1}-\xi_j^{\beta_j+1}}{\Gamma(\beta_j+2)} &+ c_2\sum_{j=1}^{n}\mu_j\frac{T^{\beta_j}-\xi_j^{\beta_j}}{\Gamma(\beta_j+1)}\\ &= \omega_2 - \sum_{j=1}^{n}\mu_j\left(I^{\beta_j+q}h(T) - I^{\beta_j+q}h(\xi_j)\right).\end{aligned} \tag{15}$$

Solving the system of linear equations for constants c_1 and c_2, we have

$$\begin{aligned} c_1 &= \frac{\Omega_4}{\Delta}\left(\omega_1 - \sum_{i=1}^{m}\lambda_i I^{\alpha_i+q}h(\eta_i)\right)\\ &\quad - \frac{\Omega_2}{\Delta}\left(\omega_2 - \sum_{j=1}^{n}\mu_j\left(I^{\beta_j+q}h(T) - I^{\beta_j+q}h(\xi_j)\right)\right),\\ c_2 &= \frac{\Omega_1}{\Delta}\left(\omega_2 - \sum_{j=1}^{n}\mu_j\left(I^{\beta_j+q}h(T) - I^{\beta_j+q}h(\xi_j)\right)\right)\\ &\quad - \frac{\Omega_3}{\Delta}\left(\omega_1 - \sum_{i=1}^{m}\lambda_i I^{\alpha_i+q}h(\eta_i)\right).\end{aligned} \tag{16}$$

Substituting constants c_1 and c_2 into (12), we obtain (11) as required. □

3. Main Results

Let $\mathscr{C} = C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t\in[0,T]}|x(t)|$. As in Lemma 4, we define an operator $\mathscr{Q}:\mathscr{C}\to\mathscr{C}$ by

$$\begin{aligned}
\mathscr{Q}x(t) &= I^q f(s,x(s))(t) \\
&+ \frac{\Omega_4 t-\Omega_3}{\Delta}\left(\omega_1-\sum_{i=1}^m \lambda_i I^{\alpha_i+q} f(s,x(s))(\eta_i)\right) \\
&+ \frac{\Omega_1-\Omega_2 t}{\Delta} \\
&\times\left(\omega_2-\sum_{j=1}^n \mu_j\left(I^{\beta_j+q} f(s,x(s))(T)\right.\right. \\
&\qquad \left.\left.-I^{\beta_j+q} f(s,x(s))(\xi_j)\right)\right).
\end{aligned} \tag{17}$$

It should be noticed that problem (1)-(2) has solutions if and only if the operator $\mathscr{Q}$ has fixed points.

We are in a position to establish our main results. In the following subsections, we prove existence as well as existence and uniqueness results for the BVP (1)-(2) by using a variety of fixed point theorems.

3.1. Existence and Uniqueness Result via Banach's Fixed Point Theorem

Theorem 5. *Assume that*

(H_1) *there exists a constant $L>0$ such that $|f(t,u)-f(t,v)|\le L|u-v|$, for each $t\in[0,T]$ and $u,v\in\mathbb{R}$.*

If

$$L\Lambda<1, \tag{18}$$

where Λ is defined by

$$\begin{aligned}
\Lambda &= \frac{T^q}{\Gamma(q+1)}+\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}\sum_{i=1}^m|\lambda_i|\frac{\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}\sum_{j=1}^n|\mu_j|\left(\frac{T^{\beta_j+q}+\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right),
\end{aligned} \tag{19}$$

then problem (1)-(2) *has a unique solution in* $[0,T]$.

Proof. We transform problem (1)-(2) into a fixed point problem, $x=\mathscr{Q}x$, where the operator $\mathscr{Q}$ is defined as in (17). Observe that the fixed points of the operator $\mathscr{Q}$ are solutions of problem (1)-(2). Applying the Banach contraction mapping principle, we will show that $\mathscr{Q}$ has a unique fixed point.

We let $\sup_{t\in[0,T]}|f(t,0)|=M<\infty$ and choose

$$r\ge\frac{\Lambda M+\Phi}{1-L\Lambda}, \tag{20}$$

where a constant Φ is defined by

$$\Phi=\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}|\omega_1|+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}|\omega_2|. \tag{21}$$

Now, we show that $\mathscr{Q}B_r\subset B_r$, where $B_r=\{x\in\mathscr{C}:\|x\|\le r\}$. For any $x\in B_r$, we have

$$\begin{aligned}
&|\mathscr{Q}x(t)| \\
&\le \sup_{t\in[0,T]}\left\{I^q|f(s,x(s))|(t)+\frac{|\Omega_3|+t|\Omega_4|}{|\Delta|}|\omega_1|\right. \\
&\quad+\frac{|\Omega_1|+t|\Omega_2|}{|\Delta|}|\omega_2| \\
&\quad+\frac{|\Omega_3|+t|\Omega_4|}{|\Delta|}\sum_{i=1}^m|\lambda_i|I^{\alpha_i+q}|f(s,x(s))|(\eta_i) \\
&\quad+\frac{|\Omega_1|+t|\Omega_2|}{|\Delta|} \\
&\quad\times\sum_{j=1}^n|\mu_j|\left(I^{\beta_j+q}|f(s,x(s))|(T)\right. \\
&\qquad\left.\left.+I^{\beta_j+q}|f(s,x(s))|(\xi_j)\right)\right\} \\
&\le I^q\left(|f(s,x(s))-f(s,0)|+|f(s,0)|\right)(T) \\
&\quad+\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}|\omega_1|+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}|\omega_2| \\
&\quad+\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|} \\
&\quad\times\sum_{i=1}^m|\lambda_i|I^{\alpha_i+q}\left(|f(s,x(s))-f(s,0)|+|f(s,0)|\right)(\eta_i) \\
&\quad+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|} \\
&\quad\times\sum_{j=1}^n|\mu_j|\left(I^{\beta_j+q}\left(|f(s,x(s))-f(s,0)|+|f(s,0)|\right)(T)\right. \\
&\qquad+I^{\beta_j+q}\left(|f(s,x(s))-f(s,0)|\right. \\
&\qquad\left.\left.+|f(s,0)|\right)(\xi_j)\right) \\
&\le (Lr+M)I^q(1)(T)+\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}|\omega_1|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}|\omega_2| \\
&+(Lr+M)\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}\sum_{i=1}^{m}|\lambda_i| I^{\alpha_i+q}(1)(\eta_i) \\
&+(Lr+M)\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|} \\
&\times\sum_{j=1}^{n}|\mu_j|\left(I^{\beta_j+q}(1)(T)+I^{\beta_j+q}(1)(\xi_j)\right) \\
\le{}& (Lr+M)\frac{T^q}{\Gamma(q+1)}+\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}|\omega_1| \\
&+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}|\omega_2| \\
&+(Lr+M)\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}\sum_{i=1}^{m}|\lambda_i|\frac{\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&+(Lr+M)\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}\sum_{j=1}^{n}|\mu_j|\left(\frac{T^{\beta_j+q}+\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right) \\
={}& (Lr+M)\Lambda+\Phi\le r,
\end{aligned}
\tag{22}
$$

which implies that $\mathcal{Q}B_r\subset B_r$.

Next, we let $x, y\in\mathscr{C}$. Then, for $t\in[0,T]$, we have

$$
\begin{aligned}
&|\mathcal{Q}x(t)-\mathcal{Q}y(t)| \\
&\le I^q|f(s,x(s))-f(s,y(s))|(t) \\
&\quad+\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|} \\
&\quad\times\sum_{i=1}^{m}|\lambda_i| I^{\alpha_i+q}|f(s,x(s))-f(s,y(s))|(\eta_i) \\
&\quad+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|} \\
&\quad\times\sum_{j=1}^{n}|\mu_j|\left(I^{\beta_j+q}|f(s,x(s))-f(s,y(s))|(T)\right. \\
&\qquad\left.+I^{\beta_j+q}|f(s,x(s))-f(s,y(s))|(\xi_j)\right) \\
&\le I^q|f(s,x(s))-f(s,y(s))|(T) \\
&\quad+\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|} \\
&\quad\times\sum_{i=1}^{m}|\lambda_i| I^{\alpha_i+q}|f(s,x(s))-f(s,y(s))|(\eta_i) \\
&\quad+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|} \\
&\quad\times\sum_{j=1}^{n}|\mu_j|\left(I^{\beta_j+q}|f(s,x(s))-f(s,y(s))|(T)\right. \\
&\qquad\left.+I^{\beta_j+q}|f(s,x(s))-f(s,y(s))|(\xi_j)\right) \\
&\le\frac{T^qL\|x-y\|}{\Gamma(q+1)}+\frac{(|\Omega_3|+T|\Omega_4|)L\|x-y\|}{|\Delta|} \\
&\quad\times\sum_{i=1}^{m}|\lambda_i|\frac{\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&\quad+\frac{(|\Omega_1|+T|\Omega_2|)L\|x-y\|}{|\Delta|} \\
&\quad\times\sum_{j=1}^{n}|\mu_j|\left(\frac{T^{\beta_j+q}+\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right) \\
&= L\Lambda\|x-y\|,
\end{aligned}
\tag{23}
$$

which implies that $\|\mathcal{Q}x-\mathcal{Q}y\|\le L\Lambda\|x-y\|$. As $L\Lambda<1$, $\mathcal{Q}$ is a contraction. Therefore, we deduce by Banach's contraction mapping principle that $\mathcal{Q}$ has a fixed point which is the unique solution of problem (1)-(2). The proof is completed. □

3.2. Existence Result via Krasnoselskii's Fixed Point Theorem

Lemma 6 (Krasnoselskii's fixed point theorem, [17]). *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A and B be the operators such that (a) $Ax+By\in M$ whenever $x, y\in M$; (b) A is compact and continuous; and (c) B is a contraction mapping. Then, there exists $z\in M$ such that $z=Az+Bz$.*

Theorem 7. *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying (H_1). Assume that*

(H_2) *$|f(t,u)|\le\theta(t)$, for all $(t,u)\in[0,T]\times\mathbb{R}$, and $\theta\in C([0,T],\mathbb{R}^+)$.*

Then, the boundary value problem (1)-(2) has at least one solution on $[0,T]$ provided

$$
\begin{aligned}
&\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}\sum_{i=1}^{m}|\lambda_i|\frac{\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&\quad+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}\sum_{j=1}^{n}|\mu_j|\left(\frac{T^{\beta_j+q}+\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right)<1.
\end{aligned}
\tag{24}
$$

Proof. Setting $\sup_{t\in[0,T]}|\theta(t)|=\|\theta\|$ and choosing

$$
\rho\ge\|\theta\|\Lambda+\Phi,
\tag{25}
$$

(Λ and Φ are defined in (19) and (21), resp.), we consider $B_\rho = \{x \in \mathscr{C}([0,T],\mathbb{R}) : \|x\| \le \rho\}$. We define the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on B_ρ by

$$\begin{aligned}\mathcal{Q}_1 x(t) &= I^q f(s,x(s))(t),\\ \mathcal{Q}_2 x(t) &= \frac{\Omega_3 - \Omega_4 t}{\Delta}\omega_1 + \frac{\Omega_2 t - \Omega_1}{\Delta}\omega_2\\ &\quad + \frac{\Omega_3 - \Omega_4 t}{\Delta}\sum_{i=1}^{m}\lambda_i I^{\alpha_i+q} f(s,x(s))(\eta_i)\\ &\quad + \frac{\Omega_2 t - \Omega_1}{\Delta}\sum_{j=1}^{n}\mu_j\left(I^{\beta_j+q} f(s,x(s))(T)\right.\\ &\quad\left. - I^{\beta_j+q} f(s,x(s))(\xi_j)\right).\end{aligned} \tag{26}$$

For any $x, y \in B_\rho$, we have

$$\begin{aligned}&\left|\mathcal{Q}_1 x(t) + \mathcal{Q}_2 y(t)\right|\\ &\le \sup_{t\in[0,T]}\left\{I^q\left|f(s,x(s))\right|(t) + \frac{|\Omega_3| + T|\Omega_4|}{|\Delta|}|\omega_1|\right.\\ &\quad + \frac{|\Omega_1| + T|\Omega_2|}{|\Delta|}|\omega_2|\\ &\quad + \frac{|\Omega_3| + T|\Omega_4|}{|\Delta|}\sum_{i=1}^{m}|\lambda_i| I^{\alpha_i+q}\left|f(s,y(s))\right|(\eta_i)\\ &\quad + \frac{|\Omega_1| + T|\Omega_2|}{|\Delta|}\\ &\quad \times \sum_{j=1}^{n}|\mu_j|\left(I^{\beta_j+q}\left|f(s,y(s))\right|(T)\right.\\ &\quad \left.\left. + I^{\beta_j+q}\left|f(s,y(s))\right|(\xi_j)\right)\right\}\\ &\le \|\theta\|\left[\frac{T^q}{\Gamma(q+1)} + \frac{|\Omega_3| + T|\Omega_4|}{|\Delta|}\sum_{i=1}^{m}|\lambda_i|\frac{\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\right.\\ &\quad\left. + \frac{|\Omega_1| + T|\Omega_2|}{|\Delta|}\sum_{j=1}^{n}|\mu_j|\left(\frac{T^{\beta_j+q} + \xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right)\right]\\ &\quad + \frac{|\Omega_3| + T|\Omega_4|}{|\Delta|}|\omega_1| + \frac{|\Omega_1| + T|\Omega_2|}{|\Delta|}|\omega_2|\\ &= \|\theta\|\Lambda + \Phi\\ &\le \rho.\end{aligned} \tag{27}$$

This shows that $\mathcal{Q}_1 x + \mathcal{Q}_2 y \in B_\rho$. It is easy to see using (24) that $\mathcal{Q}_2$ is a contraction mapping.

Continuity of f implies that the operator $\mathcal{Q}_1$ is continuous. Also, $\mathcal{Q}_1$ is uniformly bounded on B_ρ as

$$\left\|\mathcal{Q}_1 x\right\| \le \frac{T^q}{\Gamma(q+1)}\|\theta\|. \tag{28}$$

Now, we prove the compactness of the operator $\mathcal{Q}_1$.

We define $\sup_{(t,x)\in[0,T]\times B_\rho}|f(t,x)| = \overline{f} < \infty$, and consequently we have

$$\begin{aligned}&\left|(\mathcal{Q}_1 x)(t_2) - (\mathcal{Q}_1 x)(t_1)\right|\\ &= \frac{1}{\Gamma(q)}\left|\int_0^{t_1}\left[(t_2-s)^{q-1} - (t_1-s)^{q-1}\right] f(s,x(s))\,ds\right.\\ &\quad\left. + \int_{t_1}^{t_2}(t_2-s)^{q-1} f(s,x(s))\,ds\right|\\ &\le \frac{\overline{f}}{\Gamma(q+1)}\left|t_1^q - t_2^q\right|,\end{aligned} \tag{29}$$

which is independent of x and tends to zero as $t_2 - t_2 \to 0$. Thus, $\mathcal{Q}_1$ is equicontinuous. So $\mathcal{Q}_1$ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, $\mathcal{Q}_1$ is compact on B_ρ. Thus, all the assumptions of Lemma 6 are satisfied. So the conclusion of Lemma 6 implies that the boundary value problem (1)-(2) has at least one solution on $[0,T]$. □

3.3. Existence Result via Leray-Schauder's Nonlinear Alternative

Theorem 8 (nonlinear alternative for single valued maps, [18]). *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C, and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (i.e., $F(\overline{U})$ is a relatively compact subset of C) map. Then, either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Theorem 9. *Assume that*

(H_3) *there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,T],\mathbb{R}^+)$ such that*

$$\left|f(t,x)\right| \le p(t)\,\psi(\|x\|) \quad \text{for each } (t,x) \in [0,T]\times\mathbb{R}; \tag{30}$$

(H_4) *there exists a constant $M > 0$ such that*

$$\frac{M}{\psi(M)\left\|p\right\|\Lambda + \Phi} > 1, \tag{31}$$

where Λ and Φ are defined in (19) and (21), respectively.

Then, the boundary value problem (1)-(2) has at least one solution on $[0,T]$.

Proof. Let the operator $\mathcal{Q}$ be defined by (17). Firstly, we will show that *$\mathcal{Q}$ maps bounded sets (balls) into bounded sets in* $C([0,T],\mathbb{R})$. For a number $r>0$, let $B_r=\{x\in C([0,T],\mathbb{R}):\|x\|\le r\}$ be a bounded ball in $C([0,T],\mathbb{R})$. Then, for $t\in[0,T]$ we have

$$
\begin{aligned}
&|\mathcal{Q}x(t)|\\
&\le \sup_{t\in[0,T]}\Big\{I^q|f(s,x(s))|(t)+\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}|\omega_1|\\
&\quad+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}|\omega_2|\\
&\quad+\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}\sum_{i=1}^{m}|\lambda_i|I^{\alpha_i+q}|f(s,x(s))|(\eta_i)\\
&\quad+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}\\
&\quad\times\sum_{j=1}^{n}|\mu_j|\big(I^{\beta_j+q}|f(s,x(s))|(T)\\
&\qquad+I^{\beta_j+q}|f(s,x(s))|(\xi_j)\big)\Big\}\\
&\le \psi(\|x\|)I^q p(s)(T)+\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}|\omega_1|\\
&\quad+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}|\omega_2|\\
&\quad+\psi(\|x\|)\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}\sum_{i=1}^{m}|\lambda_i|I^{\alpha_i+q}p(s)(\eta_i)\\
&\quad+\psi(\|x\|)\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}\\
&\quad\times\sum_{j=1}^{n}|\mu_j|\big(I^{\beta_j+q}p(s)(T)+I^{\beta_j+q}p(s)(\xi_j)\big)\\
&\le \psi(\|x\|)\|p\|\frac{T^q}{\Gamma(q+1)}+\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}|\omega_1|\\
&\quad+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}|\omega_2|\\
&\quad+\psi(\|x\|)\|p\|\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}\sum_{i=1}^{m}|\lambda_i|\frac{\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\\
&\quad+\psi(\|x\|)\|p\|\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}\sum_{j=1}^{n}|\mu_j|\left(\frac{T^{\beta_j+q}-\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right)\\
&=\psi(\|x\|)\|p\|\Big\{\frac{T^q}{\Gamma(q+1)}+\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}\\
&\quad\times\sum_{i=1}^{m}|\lambda_i|\frac{\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\\
&\quad+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}\\
&\quad\times\sum_{j=1}^{n}|\mu_j|\left(\frac{T^{\beta_j+q}+\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right)\Big\}\\
&\quad+\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}|\omega_1|+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}|\omega_2|,
\end{aligned}\tag{32}
$$

and, consequently,

$$\|\mathcal{Q}x\|\le\psi(r)\|p\|\Lambda+\Phi,\tag{33}$$

where Λ and Φ are defined by (19) and (21), respectively.

Next, we will show that *$\mathcal{Q}$ maps bounded sets into equicontinuous sets of* $C([0,T],\mathbb{R})$. Let $\tau_1,\tau_2\in[0,T]$ with $\tau_1<\tau_2$ and $x\in B_r$. Then, we have

$$
\begin{aligned}
&|(\mathcal{Q}x)(\tau_2)-(\mathcal{Q}x)(\tau_1)|\\
&\le\frac{1}{\Gamma(q)}\Big|\int_0^{\tau_1}\big[(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}\big]f(s,x(s))\,ds\\
&\qquad+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{q-1}f(s,x(s))\,ds\Big|\\
&\quad+\frac{|\omega_1\Omega_4|+|\omega_2\Omega_2|}{|\Delta|}|\tau_2-\tau_1|\\
&\quad+\frac{|\Omega_4|}{|\Delta|}\sum_{i=1}^{m}|\lambda_i|I^{\alpha_i+q}|f(s,x(s))|(\eta_i)(\tau_2-\tau_1)\\
&\quad+\frac{|\Omega_2|}{|\Delta|}\sum_{j=1}^{n}|\mu_j|\big(I^{\beta_j+q}|f(s,x(s))|(T)\\
&\qquad+I^{\beta_j+q}|f(s,x(s))|(\xi_j)\big)(\tau_2-\tau_1)\\
&\le\frac{\psi(r)}{\Gamma(q)}\Big|\int_0^{\tau_1}\big[(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}\big]p(s)\,ds\\
&\qquad+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{q-1}p(s)\,ds\Big|\\
&\quad+\frac{|\omega_1\Omega_4|+|\omega_2\Omega_2|}{|\Delta|}(\tau_2-\tau_1)\\
&\quad+\frac{|\Omega_4|}{|\Delta|}\psi(r)\sum_{i=1}^{m}|\lambda_i|I^{\alpha_i+q}p(s)(\eta_i)(\tau_2-\tau_1)\\
&\quad+\frac{|\Omega_2|}{|\Delta|}\psi(r)\sum_{j=1}^{n}|\mu_j|\big(I^{\beta_j+q}p(s)(T)\\
&\qquad+I^{\beta_j+q}p(s)(\xi_j)\big)(\tau_2-\tau_1).
\end{aligned}\tag{34}
$$

As $\tau_2 - \tau_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore, by the Arzelá-Ascoli theorem, the operator $\mathcal{Q} : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ is completely continuous.

Let x be a solution. Then, for $t \in [0,T]$, following the similar computations as in the first step, we have

$$|x(t)| \le \psi(\|x\|)\|p\|\Lambda + \Phi, \tag{35}$$

which leads to

$$\frac{\|x\|}{\psi(\|x\|)\|p\|\Lambda + \Phi} \le 1. \tag{36}$$

In view of (H_4), there exists M such that $\|x\| \ne M$. Let us set

$$U = \{x \in C([0,T],\mathbb{R}) : \|x\| < M\}. \tag{37}$$

We see that the operator $\mathcal{Q} : \overline{U} \to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \nu \mathcal{Q}x$ for some $\nu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that $\mathcal{Q}$ has a fixed point $x \in \overline{U}$ which is a solution of problem (1)-(2). This completes the proof. □

3.4. Existence Result via Leray-Schauder Degree

Theorem 10. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(H_5) *there exist constants $0 \le \kappa < \Lambda^{-1}$, where Λ is given by* (19) *and $N > 0$ such that*

$$|f(t,x)| \le \kappa |x| + N \quad \text{for each } (t,x) \in [0,T] \times \mathbb{R}. \tag{38}$$

Then, the boundary value problem (1)-(2) *has at least one solution on* $[0,T]$.

Proof. Let us define an operator $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ as in (17) and consider the fixed point problem

$$x = \mathcal{Q}x. \tag{39}$$

In view of the fixed point problem (39), we are going to prove the existence of at least one solution $x \in \mathcal{C}$ satisfying (39). Define a ball $B_R \subset \mathcal{C}$ with a constant radius $R > 0$ given by

$$B_R = \left\{x \in \mathcal{C} : \max_{t\in[0,1]} |x(t)| < R, R > 0\right\}. \tag{40}$$

Hence, it is sufficient to show that $\mathcal{Q} : \overline{B}_R \to \mathcal{C}$ satisfies

$$x \ne \lambda \mathcal{Q}x, \quad \forall x \in \partial B_R, \forall \lambda \in [0,1]. \tag{41}$$

We define

$$H(\lambda, x) = \lambda \mathcal{Q}x, \quad x \in C,\ \lambda \in [0,1]. \tag{42}$$

As shown in Theorem 9, we deduce that the operator $\mathcal{Q}$ is continuous, uniformly bounded, and equicontinuous. Then, by the Arzelá-Ascoli theorem, a continuous map h_λ defined by $h_\lambda(x) = x - H(\lambda,x) = x - \lambda\mathcal{Q}x$ is completely continuous. If (41) holds, then the following Leray-Schauder degree is well defined, and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(h_\lambda, B_R, 0) &= \deg(I - \lambda\mathcal{Q}, B_R, 0) = \deg(h_1, B_R, 0) \\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \ne 0, \\ &\qquad 0 \in B_R,\end{aligned} \tag{43}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, $h_1(x) = x - \lambda\mathcal{Q}x = 0$ for at least one $x \in B_R$. In order to prove (41), we assume that $x = \lambda\mathcal{Q}x$ for some $\lambda \in [0,1]$ and for all $t \in [0,T]$. Then,

$$\begin{aligned}|x(t)| &= |\lambda(\mathcal{Q}x)(t)| \\ &\le \sup_{t\in[0,T]} \left\{ I^q |f(s,x(s))|(t) + \frac{|\Omega_3| + |\Omega_4| t}{|\Delta|} |\omega_1| \right. \\ &\quad + \frac{|\Omega_1| + |\Omega_2| t}{|\Delta|} |\omega_2| + \frac{|\Omega_3| + |\Omega_4| t}{|\Delta|} \\ &\quad \times \sum_{i=1}^m |\lambda_i| I^{\alpha_i+q} |f(s,x(s))|(\eta_i) \\ &\quad + \frac{|\Omega_1| + |\Omega_2| t}{|\Delta|} \\ &\quad \times \sum_{j=1}^n |\mu_j| \left(I^{\beta_j+q} |f(s,x(s))|(T) \right. \\ &\quad \left.\left. + I^{\beta_j+q} |f(s,x(s))|(\xi_j)\right) \right\} \\ &\le I^q |f(s,x(s))|(T) + \frac{|\Omega_3| + |\Omega_4| T}{|\Delta|} |\omega_1| \\ &\quad + \frac{|\Omega_1| + |\Omega_2| T}{|\Delta|} |\omega_2| \\ &\quad + \frac{|\Omega_3| + |\Omega_4| T}{|\Delta|} \sum_{i=1}^m |\lambda_i| I^{\alpha_i+q} |f(s,x(s))|(\eta_i) \\ &\quad + \frac{|\Omega_1| + |\Omega_2| T}{|\Delta|} \sum_{j=1}^n |\mu_j| \left(I^{\beta_j+q} |f(s,x(s))|(T) \right. \\ &\quad \left. + I^{\beta_j+q} |f(s,x(s))|(\xi_j)\right) \\ &\le (\kappa |x| + N) I^q(1)(T) + \frac{|\Omega_3| + |\Omega_4| T}{|\Delta|} |\omega_1| \\ &\quad + \frac{|\Omega_1| + |\Omega_2| T}{|\Delta|} |\omega_2| \\ &\quad + \frac{|\Omega_3| + |\Omega_4| T}{|\Delta|} (\kappa |x| + N) \sum_{i=1}^m |\lambda_i| I^{\alpha_i+q}(1)(\eta_i)\end{aligned}$$

$$\begin{aligned}
&+\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|}(\kappa|x|+N)\\
&\times\sum_{j=1}^{n}|\mu_j|\left(I^{\beta_j+q}(1)(T)+I^{\beta_j+q}(1)(\xi_j)\right)\\
&\leq(\kappa|x|+N)\left[\frac{T^q}{\Gamma(q+1)}+\frac{|\Omega_3|+|\Omega_4|T}{|\Delta|}\right.\\
&\qquad\times\sum_{i=1}^{m}\frac{|\lambda_i|\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\\
&\qquad+\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|}\\
&\qquad\left.\times\sum_{j=1}^{n}\frac{|\mu_j|\left(T^{\beta_j+q}+\xi_j^{\beta_j+q}\right)}{\Gamma(\beta_j+q+1)}\right]+\Phi\\
&=(\kappa|x|+N)\Lambda+\Phi.
\end{aligned}\tag{44}$$

Computing directly for $\|x\|=\sup_{t\in[0,T]}|x(t)|$, we have

$$\|x\|\leq\frac{N\Lambda+\Phi}{1-\kappa\Lambda}.\tag{45}$$

Let $R=(N\Lambda+\Phi)/(1-\kappa\Lambda)+1$; then (41) holds. This completes the proof. □

4. Examples

In this section, we present some examples to illustrate our results.

Example 11. Consider the following boundary value problem with multiterm fractional integral:

$$\begin{aligned}
&{}^cD^{3/2}x(t)=\frac{e^{-t}}{14(1+e^t)}\frac{|x|}{1+|x|},\quad t\in(0,1),\\
&\frac{1}{2}I^{2/3}x\left(\frac{3}{4}\right)-\frac{2}{3}I^{1/2}x\left(\frac{1}{4}\right)+\frac{3}{4}I^{4/3}x\left(\frac{2}{5}\right)=\frac{3}{2},\\
&\quad\frac{2}{5}\left(I^{3/2}x(1)-I^{3/2}x\left(\frac{1}{8}\right)\right)\\
&\qquad+\frac{1}{3}\left(I^{1/4}x(1)-I^{1/4}x\left(\frac{5}{8}\right)\right)=-\frac{2}{3}.
\end{aligned}\tag{46}$$

Here, $q=3/2$, $m=3$, $\lambda_1=1/2$, $\lambda_2=-2/3$, $\lambda_3=3/4$, $\alpha_1=2/3$, $\alpha_2=1/2$, $\alpha_3=4/3$, $\eta_1=3/4$, $\eta_2=1/4$, $\eta_3=2/5$, $n=2$, $\mu_1=2/5$, $\mu_2=1/3$, $\beta_1=3/2$, $\beta_2=1/4$, $\xi_1=1/8$, $\xi_2=5/8$, $T=1$, $\omega_1=3/2$, $\omega_2=-2/3$, and $f(t,x)=(e^{-t}/(14(1+e^t)))(|x|/(1+|x|))$. Since $|f(t,x)-f(t,y)|\leq(1/28)|x-y|$, then, (H_1) is satisfied with $L=1/28$. We can show that

$$\begin{aligned}
\Lambda&=\frac{T^q}{\Gamma(q+1)}+\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}\sum_{i=1}^{m}|\lambda_i|\frac{\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\\
&\quad+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}\sum_{j=1}^{n}|\mu_j|\left(\frac{T^{\beta_j+q}+\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right)\\
&\approx 27.028626,
\end{aligned}\tag{47}$$

$$L\Lambda\approx 0.965308<1.$$

Hence, by Theorem 5, the boundary value problem (46) has a unique solution on $[0,1]$.

Example 12. Consider the following boundary value problem with multiterm fractional integral:

$$\begin{aligned}
&{}^cD^{7/5}x(t)=\frac{1}{12}\left(1+t^2\right)\frac{x^2}{1+|x|}+2\frac{\sqrt{|x|}}{1+\sqrt{|x|}},\\
&\qquad t\in\left(0,\frac{9}{10}\right),\\
&2I^{2/3}x\left(\frac{1}{2}\right)+\frac{10}{3}I^{3/2}x\left(\frac{5}{6}\right)-\frac{7}{9}I^{1/4}x\left(\frac{2}{3}\right)\\
&\quad-\frac{1}{3}I^{2/9}x\left(\frac{1}{9}\right)=-\frac{1}{2},\\
&\frac{2}{5}\left(I^{5/2}x\left(\frac{9}{10}\right)-I^{5/2}x\left(\frac{1}{7}\right)\right)\\
&\quad-\frac{3}{4}\left(I^{1/2}x\left(\frac{9}{10}\right)-I^{1/2}x\left(\frac{1}{5}\right)\right)=-\frac{3}{4}.
\end{aligned}\tag{48}$$

Here, $q=7/5$, $m=4$, $\lambda_1=2$, $\lambda_2=10/3$, $\lambda_3=-7/9$, $\lambda_4=-1/3$, $\alpha_1=2/3$, $\alpha_2=3/2$, $\alpha_3=1/4$, $\alpha_4=2/9$, $\eta_1=1/2$, $\eta_2=5/6$, $\eta_3=2/3$, $\eta_4=1/9$, $n=2$, $\mu_1=2/5$, $\mu_2=-3/4$, $\beta_1=5/2$, $\beta_2=1/2$, $\xi_1=1/7$, $\xi_2=1/5$, $T=9/10$, $\omega_1=-1/2$, $\omega_2=-3/4$, and $f(t,x)=(1/12)(1+t^2)(x^2/(1+|x|))+2(\sqrt{|x|}/(1+\sqrt{|x|}))$. It is easy to verify that

$$\begin{aligned}
\Lambda&=\frac{T^q}{\Gamma(q+1)}+\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}\sum_{i=1}^{m}|\lambda_i|\frac{\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\\
&\quad+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}\sum_{j=1}^{n}|\mu_j|\left(\frac{T^{\beta_j+q}+\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right)\\
&\approx 1.641942,\\
\Phi&=\frac{|\Omega_3|+T|\Omega_4|}{|\Delta|}|\omega_1|+\frac{|\Omega_1|+T|\Omega_2|}{|\Delta|}|\omega_2|\\
&\approx 2.582519.
\end{aligned}\tag{49}$$

Clearly,

$$\begin{aligned} |f(t,x)| &= \left| \frac{1}{12}\left(1+t^2\right)\frac{x^2}{1+|x|} + 2\frac{\sqrt{|x|}}{1+\sqrt{|x|}} \right| \\ &\le \frac{1}{12}\left(1+t^2\right)(|x|+24). \end{aligned} \tag{50}$$

Choosing $p(t) = (1/12)(t^2+1)$ and $\psi(|x|) = |x| + 24$, we can show that

$$\frac{M}{\psi(M)\|p\|\Lambda + \Phi} > 1, \tag{51}$$

which implies that $M > 16.736885$. Hence, by Theorem 9, the boundary value problem (48) has at least one solution on $[0, 9/10]$.

Example 13. Consider the following boundary value problem with multiterm fractional integral:

$$\begin{aligned} {}^cD^{3/2}x(t) &= \frac{1}{5\pi}\sin\frac{\pi}{2}x + \frac{|x|}{1+|x|}, \quad t \in (0,1), \\ &-\frac{1}{2}I^{1/3}x\left(\frac{3}{5}\right) + \frac{4}{5}I^{3/2}x\left(\frac{2}{5}\right) + \frac{3}{2}I^{2/5}x\left(\frac{1}{4}\right) \\ &-\frac{2}{5}I^{1/4}x\left(\frac{2}{3}\right) = \frac{1}{2}, \\ -\frac{2}{7}&\left(I^{3/2}x(1) - I^{3/2}x\left(\frac{2}{3}\right)\right) + \frac{1}{3}\left(I^{1/2}x(1) - I^{1/2}x\left(\frac{3}{4}\right)\right) \\ &-\frac{3}{4}\left(I^{2/3}x(1) - I^{2/3}x\left(\frac{1}{4}\right)\right) = -\frac{2}{3}. \end{aligned} \tag{52}$$

Here, $q = 3/2$, $m = 4$, $\lambda_1 = -1/2$, $\lambda_2 = 4/5$, $\lambda_3 = 3/2$, $\lambda_4 = -2/5$, $\alpha_1 = 1/3$, $\alpha_2 = 3/2$, $\alpha_3 = 2/5$, $\alpha_4 = 1/4$, $\eta_1 = 3/5$, $\eta_2 = 2/5$, $\eta_3 = 1/4$, $\eta_4 = 2/3$, $n = 3$, $\mu_1 = -2/7$, $\mu_2 = 1/3$, $\mu_3 = -3/4$, $\beta_1 = 3/2$, $\beta_2 = 1/2$, $\beta_3 = 2/3$, $\xi_1 = 2/3$, $\xi_2 = 3/4$, $\xi_3 = 1/4$, $T = 1$, $\omega_1 = 1/2$, $\omega_2 = -2/3$, and $f(t,x) = (1/5\pi)(\sin(\pi x/2)) + (|x|/(1+|x|))$. We can show that

$$\begin{aligned} \Lambda &= \frac{T^q}{\Gamma(q+1)} + \frac{|\Omega_3| + T|\Omega_4|}{|\Delta|}\sum_{i=1}^{m}|\lambda_i|\frac{\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\ &\quad + \frac{|\Omega_1| + T|\Omega_2|}{|\Delta|}\sum_{j=1}^{n}|\mu_j|\left(\frac{T^{\beta_j+q} + \xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right) \\ &\approx 3.400339. \end{aligned} \tag{53}$$

Since

$$|f(t,x)| = \left|\frac{1}{5\pi}\sin\frac{\pi}{2}x + \frac{|x|}{1+|x|}\right| \le \frac{1}{10}|x| + 1, \tag{54}$$

then, (H_5) is satisfied with $\kappa = 1/10$ and $N = 1$ such that

$$\kappa = \frac{1}{10} < \frac{1}{\Lambda} \approx 0.294084. \tag{55}$$

Hence, by Theorem 10, the boundary value problem (52) has at least one solution on $[0, 1]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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