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Research Article

The Lattice of Intuitionistic Fuzzy Filters in Residuated Lattices

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The notion of tip-extended pair of intuitionistic fuzzy filters is introduced by which it is proved that the set of all intuitionistic fuzzy filters in a residuated lattice forms a bounded distributive lattice.

1. Introduction

Nowadays, it is generally accepted that in fuzzy logic the algebraic structure should be a residuated lattice which was introduced by Ward and Dilworth [1]. Some other logical algebras such as MTL-algebras [2], BL-algebras [3], MV-algebras [4], G-algebras, Π-algebras, and NM-algebras [2], which are also called R_0-algebras [5], are all able to be considered particular classes of residuated lattices. (For details, see, e.g., [6].)

Filters are an important tool to study these logical algebras and the completeness of the corresponding nonclassical logics. On the one hand, filters are closely related to congruence relations with which one can associate quotient algebras [7]; on the other hand, various filters correspond to various sets of provable formula [3, 4]. A filter is also called a deductive system in BL-algebras [8]. It has been widely investigated in residuated lattices [7, 9–11] and particular residuated lattices [2, 3, 6, 8, 12–15].

Since Rosenfeld [16] applied the notion of fuzzy sets [17] to abstract algebra and introduced the notion of fuzzy subgroups, the literature of various fuzzy algebraic concepts has been growing very rapidly [18]. In particular, in [19–21], the notion of tip-extended pair of fuzzy sets was introduced to investigate the lattices of all fuzzy normal subgroups and L-ideals.

The notion of fuzzy filters was introduced, and some properties of them were obtained [22–24]. Moreover, based on the notion of intuitionistic fuzzy sets (IFS) proposed by Atanassov [25], the concept of the intuitionistic fuzzy filter in BL-algebras was introduced in [26]. However, the study of residuated lattices from the point of lattice theory is less frequent.

In this paper, the intuitionistic fuzzy filter theory in residuated lattices is developed. This paper is organized as follows: in Section 2, some basic concepts and properties of intuitionistic fuzzy sets and intuitionistic fuzzy filters in residuated lattices are recalled. In Section 3, by introducing the notion of tip-extended pair of intuitionistic fuzzy filters, it is proved that the set of all intuitionistic fuzzy filters forms a bounded distributive lattice. The last section concludes this paper.

2. Preliminaries

The concepts of residuated lattices and intuitionistic fuzzy filters will be extensively used in the sequel. Therefore, we recall their definitions and summarize their main properties.

Let $U \neq \emptyset$. A mapping $f : U \to [0,1]$ is called a fuzzy set [17]. Let f and g be fuzzy sets on U. Then tip-extended pair of f and g [19, 20] can be defined by

$$f^{g}(x) = \begin{cases} f(x), & x \neq 1, \\ f(1) \vee g(1), & x = 1, \end{cases}$$
$$g^{f}(x) = \begin{cases} g(x), & x \neq 1, \\ g(1) \vee f(1), & x = 1. \end{cases} \quad (1)$$

Let $u_A, \nu_A : U \to [0,1]$ be two fuzzy sets satisfying $0 \leq u_A(x) + \nu_A(x) \leq 1$ for all $x \in U$. Then $A = (u_A, \nu_A)$ is called an intuitionistic fuzzy set [25] (or equivalently denoted by

$A = \{\langle x, u_A(x), v_A(x)\rangle \mid x \in U\}$). The family of all intuitionistic fuzzy sets on U will be denoted by IFS(U).

Basic operations on intuitionistic fuzzy sets are defined in the following way.

Let $A, B \in \mathrm{IFS}(U)$. One has

$$\begin{aligned} A \cap B &= (u_A \wedge u_B, v_A \vee v_B), \\ A \cup B &= (u_A \vee u_B, v_A \wedge v_B), \\ A \subseteq B &\quad \text{iff } u_A \le u_B,\ v_A \ge v_B, \\ A \supseteq B &\quad \text{iff } A \subseteq B, \\ A = B &\quad \text{iff } A \supseteq B,\ A \subseteq B. \end{aligned} \tag{2}$$

Definition 1 (see [3]). A *residuated lattice* is an algebra $L = (L, \wedge, \vee, \otimes, \to, 0, 1)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded lattice with the least element 0 and the greatest element 1, $(L, \otimes, 1)$ is a commutative monoid, and $(\otimes, \to)$ forms an adjoint pair; that is, $x \otimes y \le z$ if and only if $x \le y \to z$ for all $x, y, z \in L$.

A nonempty subset F of L is called a *filter* of L if (i) $\forall x, y \in F, x \otimes y \in F$; (ii) $\forall x \in F, y \in L, x \le y$ implies $y \in F$ or, equivalently, (iii) $1 \in F$; and (iv) $\forall x \in F, y \in L, x \to y \in F$ implies $y \in F$.

The following alternative definitions of intuitionistic fuzzy filters were proved in [26], but they can be similarly verified in residuated lattices.

Definition 2. Let $A \in \mathrm{IFS}(L)$. Then A is called an intuitionistic fuzzy filter if

(1) $u_A(x) \le u_A(1)$, $v_A(x) \ge v_A(1)$ for all $x \in L$;

(2) $u_A(x) \wedge u_A(x \to y) \le u_A(y)$ for all $x, y \in L$;

(3) $v_A(x) \vee v_A(x \to y) \ge v_A(y)$ for all $x, y \in L$.

The set of all intuitionistic fuzzy filters on a residuated lattice L will be denoted by IFF(L).

Theorem 3. *Let $A \in IFS(L)$. Then A is an intuitionistic fuzzy filter if and only if $x \otimes y \le z$ implies $u_A(x) \wedge u_A(y) \le u_A(z)$ and $v_A(x) \vee v_A(y) \ge v_A(z)$ for all $x, y, z \in L$.*

Theorem 4. *Let $A \in IFS(L)$. Then A is an intuitionistic fuzzy filter if and only if the following assertions hold:*

(1) *$x \le y$ implies $u_A(x) \le u_A(y)$ and $v_A(x) \ge v_A(y)$ for all $x, y \in L$;*

(2) *$u_A(x) \wedge u_A(y) \le u_A(x \otimes y)$ and $v_A(x) \vee v_A(y) \ge v_A(x \otimes y)$ for all $x, y \in L$.*

3. Lattice of Intuitionistic Fuzzy Filters

In this section, we mainly investigate the lattice of all intuitionistic fuzzy filters by introducing the notion of tip-extended pair of intuitionistic fuzzy sets.

The following lemma is obvious but necessary.

Lemma 5. *Let A, B be intuitionistic fuzzy filters of L. Then so is $A \cap B$.*

For $A \in \mathrm{IFS}(L)$, the intersection of all intuitionistic fuzzy filters containing A is called the generated intuitionistic fuzzy filter by A, denoted as $\langle A\rangle$.

Theorem 6. *Let $A \in IFS(L)$. Define a new intuitionistic fuzzy set B by $B = (u_B, v_B)$ where*

$$\begin{aligned} u_B(x) &= \bigvee_{\substack{a_1 \otimes \cdots \otimes a_n \le x \\ a_i \in L, n \in N}} \{u_A(a_1) \wedge \cdots \wedge u_A(a_n)\}, \\ v_B(x) &= \bigwedge_{\substack{a_1 \otimes \cdots \otimes a_n \le x \\ a_i \in L, n \in N}} \{v_A(a_1) \vee \cdots \vee u_A(a_n)\} \end{aligned} \tag{3}$$

for all $x \in L$. Then $B = \langle A\rangle$.

Proof. We complete the proof by two steps. Firstly, we verify that B is an intuitionistic fuzzy filter. For all $x, y \in L$, such that $x \le y$, the definition of B yields that $u_B(x) \le u_B(y)$ and $v_B(x) \ge v_B(y)$. For all $x, y \in L$, we have

$$\begin{aligned} & u_B(x) \wedge u_B(y) \\ &\quad = \bigvee_{\substack{a_1 \otimes \cdots \otimes a_n \le x \\ a_i \in L, n \in N}} \{u_A(a_1) \wedge \cdots \wedge u_A(a_n)\} \\ &\qquad \wedge \bigvee_{\substack{b_1 \otimes \cdots \otimes b_m \le y \\ b_i \in L, m \in N}} \{u_A(b_1) \wedge \cdots \wedge u_A(b_m)\} \\ &\quad = \bigvee_{\substack{a_1 \otimes \cdots \otimes a_n \le x \\ a_i \in L, n \in N}} \bigvee_{\substack{b_1 \otimes \cdots \otimes b_m \le y \\ b_i \in L, m \in N}} \{u_A(a_1) \wedge \cdots \wedge u_A(a_n) \\ &\qquad\qquad \wedge u_A(b_1) \wedge \cdots \wedge u_A(b_m)\} \\ &\quad \le \bigvee_{\substack{c_1 \otimes \cdots \otimes c_n \le x \otimes y \\ c_i \in L, k \in N}} \{u_A(c_1) \wedge \cdots \wedge u_A(c_k)\} \\ &\quad = u_B(x \otimes y), \\ & v_B(x) \vee v_B(y) \\ &\quad = \bigwedge_{\substack{a_1 \otimes \cdots \otimes a_n \le x \\ a_i \in L, n \in N}} \{u_A(a_1) \vee \cdots \vee u_A(a_n)\} \\ &\qquad \vee \bigwedge_{\substack{b_1 \otimes \cdots \otimes b_m \le y \\ b_i \in L, m \in N}} \{u_A(b_1) \vee \cdots \vee u_A(b_m)\} \\ &\quad = \bigwedge_{\substack{a_1 \otimes \cdots \otimes a_n \le x \\ a_i \in L, n \in N}} \bigwedge_{\substack{b_1 \otimes \cdots \otimes b_m \le y \\ b_i \in L, m \in N}} \{u_A(a_1) \vee \cdots \vee u_A(a_n) \\ &\qquad\qquad \vee u_A(b_1) \vee \cdots \vee u_A(b_m)\} \\ &\quad \ge \bigwedge_{\substack{c_1 \otimes \cdots \otimes c_n \le x \otimes y \\ c_i \in L, k \in N}} \{u_A(c_1) \vee \cdots \vee u_A(c_k)\} \\ &\quad = u_B(x \otimes y). \end{aligned} \tag{4}$$

Thus B is an intuitionistic fuzzy filter.

Secondly, let C be an intuitionistic fuzzy filter such that $C \supseteq A$. By the definition of intuitionistic fuzzy filter, it holds that

$$\begin{aligned} u_B(x) &= \bigvee_{\substack{a_1\otimes\cdots\otimes a_n\le x\\ a_i\in L, n\in N}} \{u_A(a_1)\wedge\cdots\wedge u_A(a_n)\} \\ &\le \bigvee_{\substack{a_1\otimes\cdots\otimes a_n\le x\\ a_i\in L, n\in N}} \{u_C(a_1)\wedge\cdots\wedge u_C(a_n)\} \\ &\le \bigvee_{\substack{a_1\otimes\cdots\otimes a_n\le x\\ a_i\in L, n\in N}} \{u_C(a_1\otimes\cdots\otimes a_n)\} \\ &\le u_C(x), \\ v_B(x) &= \bigwedge_{\substack{a_1\otimes\cdots\otimes a_n\le x\\ a_i\in L, n\in N}} \{u_A(a_1)\vee\cdots\vee u_A(a_n)\} \\ &\ge \bigwedge_{\substack{a_1\otimes\cdots\otimes a_n\le x\\ a_i\in L, n\in N}} \{u_C(a_1)\vee\cdots\vee u_C(a_n)\} \\ &\ge \bigwedge_{\substack{a_1\otimes\cdots\otimes a_n\le x\\ a_i\in L, n\in N}} \{u_C(a_1\otimes\cdots\otimes a_n)\} \\ &\ge u_C(x), \end{aligned} \tag{5}$$

and hence $B \subseteq C$. Thus $B = \langle A\rangle$. □

Example 7. Let $L = \{0, a, b, 1\}$ with $0 < a < b < 1$. The operations $\otimes$ and $\rightarrow$ are defined as

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

(6)

Define $u_A, v_A : L \rightarrow [0,1]$ as $u_A(0) = 0.2, v_A(0) = 0.7, u_A(a) = 0.5, v_A(a) = 0.45, u_A(b) = 0.6, v_A(b) = 0.4, u_A(1) = 0.8$, and $v_A(1) = 0.15$. Since $0.5 = u_A(a) \wedge u_A(a) \nleq u_A(a^2) = 0.2$, f is not a fuzzy filter. It is routine to verify that $\langle A\rangle$ is an intuitionistic fuzzy filter, where $u_{\langle A\rangle}(0) = u_{\langle A\rangle}(a) = 0.5, v_{\langle A\rangle}(0) = v_{\langle A\rangle}(a) = 0.4, u_{\langle A\rangle}(b) = 0.6, v_{\langle A\rangle}(b) = 0.3, u_{\langle A\rangle}(1) = 0.8$, and $v_{\langle A\rangle}(1) = 0.15$ from the above theorem.

Lemma 8. *Let $a, b, s, t \in [0,1]$ such that $0 \le a + b \le 1$ and $0 \le s + t \le 1$. Then $0 \le a \vee s + b \wedge t \le 1$.*

Proof. Not losing the generality, we assume that $a \le s$. Then $a \vee s + b \wedge t \le s + t \le 1$. It is obvious that $0 \le a \vee s + b \wedge t$. Thus it holds that $0 \le a \vee s + b \wedge t \le 1$. □

Theorem 9. *Let A be an intuitionistic fuzzy filter of L and for all $t, s \in [0,1]$ such that $0 \le t + s \le 1$. Then $A^{t,s} = (u_A^t, v_A^s)$ is an intuitionistic fuzzy filter, where*

$$\begin{aligned} u_A^t(x) &= \begin{cases} u_A(x), & x \ne 1, \\ u_A(1)\vee t, & x = 1, \end{cases} \\ v_A^s(x) &= \begin{cases} v_A(x), & x \ne 1, \\ v_A(1)\wedge s, & x = 1. \end{cases} \end{aligned} \tag{7}$$

Proof. It follows from Lemma 8 that $A^{t,s} \in \mathrm{IFS}(L)$. Now we prove that $A^{t,s}$ is an intuitionistic fuzzy filter.

If $x \le y$, we consider the following two cases:

Case 1 ($y = 1$). It is obvious that $u_A^t(x) \le u_A^t(1) = u_A^t(y)$, $v_A^t(x) \ge v_A^t(1) = v_A^t(y)$.

Case 2 ($y \ne 1$). The definition of $A^{t,s}$ leads that $u_A^t(x) = u_A(x) \le u_A(y) = u_A^t(y), v_A^t(x) = v_A(x) \ge v_A(y) = v_A^t(y)$.

Thus $u_A^t(x) \le u_A^t(y), v_A^t(x) \ge v_A^t(y)$.

Let $x, y \in L$. We consider the following two cases:

Case 1 ($x \otimes y = 1$). If $x = y = 1$, it is obvious that $u_A^t(x) \wedge u_A^t(y) \le u_A^t(x\otimes y), v_A^t(x) \vee y_A^t(x) \ge v_A^t(x \otimes y)$.

If $x = 1, y \ne 1$ or $x \ne 1, y = 1$, it is a contradiction.

If $x \ne 1, y \ne 1$, it holds that $u_A^t(x) \wedge u_A^t(y) = u_A(x) \wedge u_A(y) \le u_A(x\otimes y) = u_A^t(x\otimes y), v_A^t(x) \vee y_A^t(x) = v_A(x) \vee y_A(x) \ge v_A(x\otimes y) = v_A^t(x\otimes y)$.

Case 2 ($x \otimes y \ne 1$). If $x = y = 1$, it is a contradiction.

If $x = 1, y \ne 1$ or $x \ne 1, y = 1$, it is obvious that $u_A^t(x) \wedge u_A^t(y) \le u_A^t(x\otimes y), v_A^t(x) \vee y_A^t(x) \ge v_A^t(x\otimes y)$.

If $x \ne 1, y \ne 1$,we have $u_A^t(x) \wedge u_A^t(y) = u_A(x) \wedge u_A(y) \le u_A(x\otimes y) = u_A^t(x\otimes y), v_A^t(x) \vee y_A^t(x) = v_A(x) \vee y_A(x) \ge v_A(x\otimes y) = v_A^t(x\otimes y)$.

All in all, it yields that $u_A^t(x)\wedge u_A^t(y) = u_A(x)\wedge u_A(y) \le u_A(x\otimes y) = u_A^t(x\otimes y), v_A^t(x) \vee y_A^t(x) = v_A(x) \vee y_A(x) \ge v_A(x\otimes y) = v_A^t(x\otimes y)$.

Thus $A^{t,s}$ is an intuitionistic fuzzy filter. □

For given $A, B \in \mathrm{IFS}(L)$, the operation $\widetilde{\star}$ is defined by

$$A \widetilde{\star} B = (u_A \widetilde{\otimes} u_B, v_A \widetilde{\odot} v_B), \tag{8}$$

where

$$\begin{aligned} u_A \widetilde{\otimes} u_B(x) &= \bigvee_{y\otimes z\le x} \{u_A(y)\wedge u_B(z)\}, \\ v_A \widetilde{\odot} v_B(x) &= \bigwedge_{y\otimes z\le x} \{v_A(y)\vee v_B(z)\}. \end{aligned} \tag{9}$$

Furthermore, the tip-extended pair for intuitionistic fuzzy sets A and B are defined by

$$A^B = (u_A^{u_B}, v_A^{v_B}), \tag{10}$$

where

$$u_A^{u_B}(x)=\begin{cases}u_A(x), & x\neq 1,\\ u_A(1)\vee u_B(1), & x=1,\end{cases}$$
$$v_A^{v_B}(x)=\begin{cases}v_A(x), & x\neq 1,\\ v_A(1)\wedge v_B(1), & x=1,\end{cases}\tag{11}$$
$$B^A=\left(u_B^{u_A},v_B^{v_A}\right),$$

where

$$u_B^{u_A}(x)=\begin{cases}u_B(x), & x\neq 1,\\ u_B(1)\vee u_A(1), & x=1,\end{cases}$$
$$v_B^{v_A}(x)=\begin{cases}v_B(x), & x\neq 1,\\ v_B(1)\wedge v_A(1), & x=1.\end{cases}\tag{12}$$

Theorem 10. *Let $A,B\in IFF(L)$. Then $A^B\tilde{\barwedge}B^A\in IFF(L)$.*

Proof. It is obvious that $u_A^{u_B}\tilde{\otimes}u_B^{u_A}$ is order-preserving, and $v_A^{v_B}\tilde{\odot}v_B^{v_A}$ is order-reserving. For all $x,y\in L$, we have

$$\begin{aligned}
&u_A^{u_B}\tilde{\otimes}u_B^{u_A}(x\otimes y)\\
&\quad=\bigvee_{u\otimes v\leq x\otimes y}\left\{u_A^{u_B}(u)\wedge u_B^{u_A}(v)\right\}\\
&\quad\geq\bigvee_{\substack{p\otimes q\leq x\\ r\otimes s\leq y}}\left\{u_A^{u_B}(p\otimes r)\wedge u_B^{u_A}(q\otimes s)\right\}\\
&\quad\geq\bigvee_{\substack{p\otimes q\leq x\\ r\otimes s\leq y}}\left\{u_A^{u_B}(p)\wedge u_A^{u_B}(r)\wedge u_B^{u_A}(q)\wedge u_B^{u_A}(s)\right\}\\
&\quad=\bigvee_{p\otimes q\leq x}\left\{u_A^{u_B}(p)\wedge u_B^{u_A}(q)\right\}\wedge\bigvee_{r\otimes s\leq y}\left\{u_A^{u_B}(r)\wedge u_B^{u_A}(s)\right\}\\
&\quad=u_A^{u_B}\tilde{\otimes}u_B^{u_A}(x)\wedge u_A^{u_B}\tilde{\otimes}u_B^{u_A}(y),\\
&v_A^{v_B}\tilde{\odot}v_B^{v_A}(x\otimes y)\\
&\quad=\bigwedge_{u\otimes v\leq x\otimes y}\left\{v_A^{v_B}(u)\vee v_B^{v_A}(v)\right\}\\
&\quad\leq\bigwedge_{\substack{p\otimes q\leq x\\ r\otimes s\leq y}}\left\{v_A^{v_B}(p\otimes r)\vee v_B^{v_A}(q\otimes s)\right\}\\
&\quad\leq\bigwedge_{\substack{p\otimes q\leq x\\ r\otimes s\leq y}}\left\{v_A^{v_B}(p)\vee v_A^{v_B}(r)\vee v_B^{v_A}(q)\vee v_B^{v_A}(s)\right\}\\
&\quad=\bigwedge_{p\otimes q\leq x}\left\{v_A^{v_B}(p)\vee v_B^{v_A}(q)\right\}\vee\bigwedge_{r\otimes s\leq y}\left\{v_A^{v_B}(r)\vee v_B^{v_A}(s)\right\}\\
&\quad=v_A^{v_B}\tilde{\odot}v_B^{v_A}(x)\vee v_A^{v_B}\tilde{\odot}v_B^{v_A}(y),
\end{aligned}\tag{13}$$

and hence $u_A^{u_B}\tilde{\otimes}u_B^{u_A}(x\otimes y)\geq u_A^{u_B}\tilde{\otimes}u_B^{u_A}(x)\wedge u_A^{u_B}\tilde{\otimes}u_B^{u_A}(y)$, $v_A^{v_B}\tilde{\odot}v_B^{v_A}(x\otimes y)\leq v_A^{v_B}\tilde{\odot}v_B^{v_A}(x)\vee v_A^{v_B}\tilde{\odot}v_B^{v_A}(y)$.

Thus $A^B\tilde{\barwedge}B^A\in \mathrm{IFF}(L)$. □

Theorem 11. *Let $A,B\in IFF(L)$. Then $A^B\tilde{\barwedge}B^A=\langle A\cup B\rangle$.*

Proof. It is easy to prove that $A,B\subseteq A^B\tilde{\barwedge}B^A$, and hence $A\cup B\subseteq A^B\tilde{\barwedge}B^A$. Thus $\langle A\cup B\rangle\subseteq A^B\tilde{\barwedge}B^A$.

Assume that $C\in\mathrm{IFF}(L)$ such that $A\cup B\subseteq C$. If $x=1$, we have $u_A^{u_B}\tilde{\otimes}u_B^{u_A}(1)=u_A(1)\vee u_B(1)\leq u_C(1)$, $v_A^{v_B}\tilde{\odot}v_B^{v_A}(1)=v_A(1)\wedge v_B(1)\geq v_C(1)$. If $x<1$, it holds that

$$\begin{aligned}
&u_A^{u_B}\tilde{\otimes}u_B^{u_A}(x)\\
&\quad=\bigvee_{y\otimes z\leq x}\left\{u_A^{u_B}(y)\wedge u_B^{u_A}(z)\right\}\\
&\quad=\bigvee_{\substack{y\otimes z\leq x\\ y\neq 1,z\neq 1}}\left\{u_A^{u_B}(y)\wedge u_B^{u_A}(z)\right\}\vee\bigvee_{y\leq x}\left\{u_A(y)\right\}\vee\bigvee_{z\leq x}\left\{u_B(z)\right\}\\
&\quad=\bigvee_{\substack{y\otimes z\leq x\\ y\neq 1,z\neq 1}}\left\{u_A(y)\wedge u_B(z)\right\}\vee\bigvee_{y\leq x}\left\{u_A(y)\right\}\vee\bigvee_{z\leq x}\left\{u_B(z)\right\}\\
&\quad\leq\bigvee_{\substack{y\otimes z\leq x\\ y\neq 1,z\neq 1}}\left\{u_C(y)\wedge u_C(z)\right\}\vee\bigvee_{y\leq x}\left\{u_C(y)\right\}\vee\bigvee_{z\leq x}\left\{u_C(z)\right\}\\
&\quad=\bigvee_{y\otimes z\leq x}\left\{u_A(y)\wedge u_B(z)\right\}\\
&\quad\leq u_C(x),\\
&v_A^{v_B}\tilde{\odot}v_B^{v_A}(x)\\
&\quad=\bigwedge_{y\otimes z\leq x}\left\{v_A^{v_B}(y)\vee v_B^{v_A}(z)\right\}\\
&\quad=\bigwedge_{\substack{y\otimes z\leq x\\ y\neq 1,z\neq 1}}\left\{v_A^{v_B}(y)\vee v_B^{v_A}(z)\right\}\wedge\bigwedge_{y\leq x}\left\{v_A(y)\right\}\wedge\bigwedge_{z\leq x}\left\{v_B(z)\right\}\\
&\quad=\bigwedge_{\substack{y\otimes z\leq x\\ y\neq 1,z\neq 1}}\left\{v_A(y)\vee v_B(z)\right\}\wedge\bigwedge_{y\leq x}\left\{v_A(y)\right\}\wedge\bigwedge_{z\leq x}\left\{v_B(z)\right\}\\
&\quad\geq\bigwedge_{\substack{y\otimes z\leq x\\ y\neq 1,z\neq 1}}\left\{v_C(y)\vee v_C(z)\right\}\wedge\bigwedge_{y\leq x}\left\{v_C(y)\right\}\wedge\bigwedge_{z\leq x}\left\{v_C(z)\right\}\\
&\quad=\bigwedge_{y\otimes z\leq x}\left\{v_A(y)\vee v_B(z)\right\}\\
&\quad\geq v_C(x).
\end{aligned}\tag{14}$$

It follows from Theorem 10 that $A^B\tilde{\barwedge}B^A=\langle A\cup B\rangle$. □

Associating with the above results, we prove the main theorem here. For $A,B\in\mathrm{IFF}(L)$, the operations $\sqcap$ and $\sqcup$ on $\mathrm{IFF}(L)$ are defined by

$$A\sqcap B=A\cap B,\qquad A\sqcup B=A^B\tilde{\barwedge}B^A.\tag{15}$$

Theorem 12. *$(IFF(L),\sqcap,\sqcup,\emptyset,L)$ is a bounded distributive lattice.*

Proof. We only verify the distributivity. Obviously, it holds that $C\sqcap(A\sqcup B)\supseteq(C\sqcap A)\sqcup(C\sqcap B)$, so we only prove that $C \sqcap (A \sqcup B) \subseteq (C \sqcap A) \sqcup (C \sqcap B)$. Assume that $x \in L$ for $u_C \wedge u_A^{u_B} \widetilde{\otimes} u_B^{u_A}(x) \leq (u_C \wedge u_A)^{u_C \wedge u_B} \widetilde{\otimes} (u_C \wedge u_B)^{u_C \wedge u_A}(x)$, we consider the following two cases.

Case 1 ($x = 1$). We have

$$
\begin{aligned}
u_C \wedge u_{A \sqcup B}(1) &= u_C(1) \wedge u_A^{u_B} \widetilde{\otimes} u_B^{u_A}(1) \\
&= u_C(1) \wedge (u_A(1) \vee u_B(1)) \\
&= (u_C(1) \wedge u_A(1)) \vee (u_C(1) \wedge u_B(1)) \\
&= (u_C \wedge u_A)^{u_C \wedge u_B} \widetilde{\otimes} (u_C \wedge u_B)^{u_C \wedge u_A}(1), \\
v_C \vee v_{A \sqcup B}(1) &= v_C(1) \vee v_A^{v_B} \widetilde{\odot} v_B^{v_A}(1) \\
&= v_C(1) \vee (v_A(1) \wedge v_B(1)) \\
&= (v_C(1) \vee v_A(1)) \wedge (v_C(1) \vee v_B(1)) \\
&= (v_C \vee v_A)^{v_C \vee v_B} \widetilde{\odot} (v_C \vee v_B)^{v_C \vee v_A}(1);
\end{aligned}
\tag{16}
$$

Case 2 ($x \neq 1$). It holds that

$$
\begin{aligned}
& u_C(x) \wedge u_A^{u_B} \widetilde{\otimes} u_B^{u_A}(x) \\
&= u_C(x) \wedge \bigvee_{y \otimes z \leq x} \{u_A^{u_B}(y) \wedge u_B^{u_A}(z)\} \\
&= \bigvee_{y \otimes z \leq x} \{u_C(x) \wedge u_A^{u_B}(y) \wedge u_B^{u_A}(z)\} \\
&= \bigvee_{\substack{y \otimes z \leq x \\ y \neq 1, z \neq 1}} \{u_C(x) \wedge u_A(y) \wedge u_C(x) \wedge u_B(z)\} \\
&\quad \vee \{u_C(x) \wedge u_A^{u_B}(1) \wedge u_B(x)\} \\
&\quad \vee \{u_C(x) \wedge u_A(x) \wedge u_B^{u_A}(1)\} \\
&= \bigvee_{\substack{y \otimes z \leq x \\ y \neq 1, z \neq 1}} \{u_C(x) \wedge u_A(y) \wedge u_C(x) \wedge u_B(z)\} \\
&\quad \vee \{u_C(x) \wedge u_C(1) \wedge u_A^{u_B}(1) \wedge u_B(x)\} \\
&\quad \vee \{u_C(x) \wedge u_C(1) \wedge u_A(x) \wedge u_B^{u_A}(1)\} \\
&= \bigvee_{\substack{y \otimes z \leq x \\ y \neq 1, z \neq 1}} \{u_C(x) \wedge u_A(y) \wedge u_C(x) \wedge u_B(z)\} \\
&\quad \vee \{u_C(1) \wedge (u_A(1) \vee u_B(1)) \\
&\qquad \wedge [(u_C(x) \wedge u_B(x)) \vee (u_C(x) \wedge u_A(x))]\} \\
&= \bigvee_{\substack{y \otimes z \leq x \\ y \neq 1, z \neq 1}} \{u_C(x) \wedge u_A(y) \wedge u_C(x) \wedge u_B(z)\} \\
&\quad \vee \{[(u_C(1) \wedge u_A(1)) \vee (u_C(1) \wedge u_B(1))] \\
&\qquad \wedge [(u_C(x) \wedge u_B(x)) \vee (u_C(x) \wedge u_A(x))]\} \\
&\leq \bigvee_{\substack{y \otimes z \leq x \\ y \neq 1, z \neq 1}} \{(u_C \wedge u_A)^{u_C \wedge u_B}(y \vee x) \\
&\qquad \wedge (u_C \wedge u_B)^{u_C \wedge u_A}(z \vee x)\} \\
&\quad \vee [(u_C \wedge u_A)^{u_C \wedge u_B}(1) \wedge (u_C \wedge u_A)(y \vee x)] \\
&\quad \vee [(u_C \wedge u_B)^{u_C \wedge u_A}(1) \wedge (u_C \wedge u_B)(z \vee x)] \\
&= \bigvee_{y \otimes z \leq x} \{(u_C \wedge u_A)^{u_C \wedge u_B}(y \vee x) \\
&\qquad \wedge (u_C \wedge u_B)^{u_C \wedge u_A}(z \vee x)\}.
\end{aligned}
\tag{17}
$$

Let $y \vee x = y'$ and $z \vee x = z'$; it is easy to verify that $y' \otimes z' \leq x$, and then the above can be written as

$$
\begin{aligned}
&= \bigvee_{y' \otimes z' \leq x} \{(u_C \wedge u_A)^{u_C \wedge u_B}(y') \wedge (u_C \wedge u_B)^{u_C \wedge u_A}(z')\} \\
&= (u_C \wedge u_A)^{u_C \wedge u_B} \widetilde{\otimes} (u_C \wedge u_B)^{u_C \wedge u_A}(x), \\
& v_C(x) \vee v_A^{v_B} \widetilde{\odot} v_B^{v_A}(x) \\
&\quad = v_C(x) \vee \bigwedge_{y \otimes z \leq x} \{v_A^{v_B}(y) \vee v_B^{v_A}(z)\} \\
&\quad = \bigwedge_{y \otimes z \leq x} \{v_C(x) \vee v_A^{v_B}(y) \vee v_B^{v_A}(z)\} \\
&\quad = \bigwedge_{\substack{y \otimes z \leq x \\ y \neq 1, z \neq 1}} \{v_C(x) \vee v_A(y) \vee v_C(x) \vee v_B(z)\} \\
&\qquad \wedge \{v_C(x) \vee v_A^{v_B}(1) \vee v_B(x)\} \\
&\qquad \wedge \{v_C(x) \wedge v_A(x) \wedge v_B^{v_A}(1)\} \\
&\quad = \bigwedge_{\substack{y \otimes z \leq x \\ y \neq 1, z \neq 1}} \{v_C(x) \vee v_A(y) \vee v_C(x) \vee v_B(z)\} \\
&\qquad \wedge \{v_C(x) \vee v_C(1) \vee v_A^{v_B}(1) \vee v_B(x)\} \\
&\qquad \wedge \{v_C(x) \vee v_C(1) \vee v_A(x) \vee v_B^{v_A}(1)\} \\
&\quad = \bigwedge_{\substack{y \otimes z \leq x \\ y \neq 1, z \neq 1}} \{v_C(x) \vee v_A(y) \vee v_C(x) \vee v_B(z)\} \\
&\qquad \wedge \{v_C(1) \vee (v_A(1) \wedge v_B(1)) \\
&\qquad\quad \vee [(v_C(x) \vee v_B(x)) \wedge (v_C(x) \vee v_A(x))]\} \\
&\quad = \bigwedge_{\substack{y \otimes z \leq x \\ y \neq 1, z \neq 1}} \{v_C(x) \vee v_A(y) \vee v_C(x) \vee v_B(z)\} \\
&\qquad \wedge \{[(v_C(1) \vee v_A(1)) \wedge (v_C(1) \vee v_B(1))] \\
&\qquad\quad \vee [(v_C(x) \vee v_B(x)) \wedge (v_C(x) \vee v_A(x))]\}
\end{aligned}
$$

$$\geq \bigwedge_{\substack{y\otimes z\leq x\\ y\neq 1, z\neq 1}} \left\{(\nu_C \vee \nu_A)^{\nu_C\vee\nu_B}(y\vee x) \vee(\nu_C\vee\nu_B)^{\nu_C\vee\nu_A}(z\vee x)\right\}$$

$$\wedge\left[(\nu_C\vee\nu_A)^{\nu_C\vee\nu_B}(1)\vee(\nu_C\vee\nu_A)(y\vee x)\right]$$

$$\wedge\left[(\nu_C\vee\nu_B)^{\nu_C\vee\nu_A}(1)\wedge(\nu_C\vee\nu_B)(z\vee x)\right]$$

$$= \bigwedge_{y\otimes z\leq x} \left\{(\nu_C \vee \nu_A)^{\nu_C\vee\nu_B}(y\vee x) \vee(\nu_C\vee\nu_B)^{\nu_C\vee\nu_A}(z\vee x)\right\}. \tag{18}$$

Let $y\vee x = y'$ and $z\vee x = z'$; it is easy to verify that $y'\otimes z' \leq x$, and then the above can be written as

$$= \bigwedge_{y'\otimes z'\leq x} \left\{(\nu_C \vee \nu_A)^{\nu_C\vee\nu_B}(y') \vee(\nu_C\vee\nu_B)^{\nu_C\vee\nu_A}(z')\right\}$$
$$= (\nu_C \vee \nu_A)^{\nu_C\vee\nu_B}\widetilde{\odot}(\nu_C\vee\nu_B)^{\nu_C\vee\nu_A}(x). \tag{19}$$

Thus $u_C \wedge u_A^{u_B}\widetilde{\otimes}u_B^{u_A} \leq (u_C\wedge u_A)^{u_C\wedge u_B}\widetilde{\otimes}(u_C\wedge u_B)^{u_C\wedge u_A}$ and $\nu_C(x)\vee \nu_A^{\nu_B}\widetilde{\odot}\nu_B^{\nu_A}(x) \geq (\nu_C\vee\nu_A)^{\nu_C\vee\nu_B}\widetilde{\odot}(\nu_C\vee\nu_B)^{\nu_C\vee\nu_A}$; that is, the distributivity holds. □

4. Conclusions

In this paper, by the notion of tip-extended pair of intuitionistic fuzzy sets, it is verified that the set of all intuitionistic fuzzy filters forms a bounded distributive lattice.

Future research will focus on the lattice of fuzzy notions in other algebras by tip-extended pair.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] M. Ward and R. P. Dilworth, "Residuated lattices," *Transactions of the American Mathematical Society*, vol. 45, no. 3, pp. 335–354, 1939.

[2] F. Esteva and L. Godo, "Monoidal t-norm based logic: towards a logic for left-continuous t-norms," *Fuzzy Sets and Systems*, vol. 124, no. 3, pp. 271–288, 2001.

[3] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic, Dordrecht, The Netherlands, 1998.

[4] C. C. Chang, "Algebraic analysis of many valued logics," *Transactions of the American Mathematical Society*, vol. 88, pp. 467–490, 1958.

[5] G. J. Wang, *An Introduction to Mathematical Logic and Resolution Principle*, Science in China Press, Beijing, China, 2003.

[6] E. Turunen, *Mathematics Behind Fuzzy Logic*, Physica, Heidelberg, Germany, 1999.

[7] B. van Gasse, G. Deschrijver, C. Cornelis, and E. E. Kerre, "Filters of residuated lattices and triangle algebras," *Information Sciences*, vol. 180, no. 16, pp. 3006–3020, 2010.

[8] E. Turunen, "Boolean deductive systems of BL-algebras," *Archive for Mathematical Logic*, vol. 40, no. 6, pp. 467–473, 2001.

[9] M. Kondo, "Filters on commutative residuated lattices," in *Integrated Uncertainty Management and Applications*, vol. 68, pp. 343–347, Springer, Berlin, Germany, 2010.

[10] L. Liu and K. Li, "Boolean filters and positive implicative filters of residuated lattices," *Information Sciences*, vol. 177, no. 24, pp. 5725–5738, 2007.

[11] Y. Zhu and Y. Xu, "On filter theory of residuated lattices," *Information Sciences*, vol. 180, no. 19, pp. 3614–3632, 2010.

[12] M. Haveshki, A. B. Saeid, and E. Eslami, "Some types of filters in *BL* algebras," *Soft Computing*, vol. 10, no. 8, pp. 657–664, 2006.

[13] M. Kondo and W. A. Dudek, "Filter theory of BL algebras," *Soft Computing*, vol. 12, no. 5, pp. 419–423, 2008.

[14] A. B. Saeid and S. Motamed, "Normal filters in BL-algebras," *World Applied Sciences Journal*, vol. 7, pp. 70–76, 2009.

[15] E. Turunen, "BL-algebras of basic fuzzy logic," *Mathware & Soft Computing*, vol. 6, no. 1, pp. 49–61, 1999.

[16] A. Rosenfeld, "Fuzzy groups," *Journal of Mathematical Analysis and Applications*, vol. 35, no. 3, pp. 512–517, 1971.

[17] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[18] J. N. Mordeson and D. S. Malik, *Fuzzy Commutative Algebra*, World Scientific, London, UK, 1998.

[19] T. Head, "A metatheorem for deriving fuzzy theorems from crisp versions," *Fuzzy Sets and Systems*, vol. 73, no. 3, pp. 349–358, 1995.

[20] T. Head, "Erratum to "A metatheorem for deriving fuzzy theorems from crisp versions"," *Fuzzy Sets and Systems*, vol. 79, no. 2, pp. 277–278, 1996.

[21] I. Jahan, "The lattice of L-ideals of a ring is modular," *Fuzzy Sets and Systems*, vol. 199, pp. 121–129, 2012.

[22] Y. B. Jun, Y. Xu, and X. H. Zhang, "Fuzzy filters of MTL-algebras," *Information Sciences*, vol. 175, no. 1-2, pp. 120–138, 2005.

[23] L. Liu and K. Li, "Fuzzy filters of *BL*-algebras," *Information Sciences*, vol. 173, no. 1–3, pp. 141–154, 2005.

[24] J. L. Zhang and H. J. Zhou, "Fuzzy filters on the residuated lattices," *New Mathematics and Natural Compution*, vol. 2, no. 1, pp. 11–28, 2006.

[25] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[26] Z. Xue, Y. Xiao, W. Liu, H. Cheng, and Y. Li, "Intuitionistic fuzzy filter theory of *BL*-algebras," *International Journal of Machine Learning and Cybernetics*, vol. 4, no. 6, pp. 659–669, 2013.