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Research Article

Meromorphic Solutions of Some Algebraic Differential Equations

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By means of the normal family theory, we estimate the growth order of meromorphic solutions of some algebraic differential equations and improve the related results of Barsegian et al. (2002). We also give some examples to show that our results occur in some special cases.

1. Introduction and Main Results

Let $f(z)$ be a function meromorphic or holomorphic in the complex plane. We use the standard notations of Nevanlinna theory and denote the order of $f(z)$ by $\rho(f)$ (see Hayman [1], He and Xiao [2], and Laine [3] and Yang [4]).

Let D be a domain in the complex plane. A family $\mathcal{F}$ of meromorphic functions in D is normal, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges locally uniformly by spherical distance to a function $g(z)$ meromorphic in D ($g(z)$ is permitted to be identically infinite).

We define spherical derivative of the meromorphic function $f(z)$ as follows:

$$f^{\sharp}(z) := \frac{\left|f'(z)\right|}{1 + \left|f(z)\right|^2}. \tag{1}$$

An algebraic differential equation for $w(z)$ is of the form

$$P\left(z, w, w', \dots, w^{(k)}\right) = 0, \tag{2}$$

where P is a polynomial in each of its variables.

It is one of the important and interesting subjects to research the growth of meromorphic solution $w(z)$ of differential equation (2) in the complex plane.

In 1956, Gol'dberg [5] proved that the meromorphic solutions have finite growth order when $k = 1$. Some alternative proofs of this result have been given by Bank and Kaufman [6] and by Barsegian [7].

In 1998, Barsegian [8, 9] introduced an essentially new type of weight for differential monomial below and gave the estimate for the first time for the growth order of meromorphic solutions of large classes of complex differential equations of higher degrees by using his initial method [10]. Later Bergweiler [11] reproved Barsegian's result by using Zalcman's lemma.

In order to state the result, we first introduce some notations [8]: $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, $t_j \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$ for $j = 1, 2, \dots, n$, and put $t = (t_1, t_2, \dots, t_n)$. Define $M_t[w]$ by

$$M_t[w](z) := \left[w'(z)\right]^{t_1} \left[w''(z)\right]^{t_2} \cdots \left[w^{(n)}\right]^{t_n}, \tag{3}$$

with the convention that $M_{\{0\}}[w] = 1$. We call $p(t) := t_1 + 2t_2 + \cdots + nt_n$ the weight of $M_t[w]$. A differential polynomial $P[w]$ is an expression of the form

$$P[w](z) := \sum_{t \in I} a_t(z, w(z)) M_t[w](z), \tag{4}$$

where the a_t are rational in two variables and I is a finite index set. The total weight $W(P)$ of $P[w]$ is given by $(P) := \max_{t \in I} p(t)$.

Definition 1. $\deg_{z,\infty} a_t$ denotes the degree at infinity in variable z concerning $a_t(z,w)$. $\deg_{z,\infty} a := \max_{t\in I} \max\{\deg_{z,\infty} a_t, 0\}$:

$$\alpha_{m,P} := \max_{t\in I, m>p(t)} \frac{\max\left\{\deg_{z,\infty} a_t, 0\right\}}{m-p(t)},$$
$$\beta_{m,P} := \max_{t\in I, m=p(t)} \deg_{z,\infty} a_t. \tag{5}$$

When all $p(t)=m$, $t\in I$, we set $\alpha_{m,P}=0$.

In 2002, the following general estimate of growth order of meromorphic solutions $w(z)$ of the equation $[w'(z)]^m = P[w]$ was obtained, which depends on the degrees at infinity of coefficients of differential polynomial in z, by Barsegian et al. [9].

Theorem A (see [9]). *Let $w(z)$ be a meromorphic solution to the differential equation $[w'(z)]^m = P[w]$, where $m \in \mathbb{N}$. If $m > W(P)$ or $m = W(P)$ and $\beta_{m,P} < 0$, then the growth order $\rho := \rho(w)$ of $w(z)$ satisfies $\rho \le 2 + 2\alpha_{m,P}$.*

Remark 2. Barsegian [8, 12], Bergweiler [11], Frank and Wang [13], and Yuan et al. [14, 15] proved $\rho < \infty$ or the conditions hold for all $t \in I$.

In this paper, by the normal family method of Bergweiler in [11], we extend Theorem A and obtain the following result.

Theorem 3. *Let $k, m, n, q \in \mathbb{N}$, and let $P[w]$ be a differential polynomial. If any meromorphic function $w(z)$ whose all zeros have multiplicities at least k satisfies the differential equations $[([Q(w^{(k-1)}(z))]^n)']^m = P[w]$ and $(nqk-nq+1)m > W(P)$ or $(nqk-nq+1)m = W(P)$ and $\beta_{(nqk-nq+1)m,P} < 0$, then the growth order $\rho := \rho(w)$ of $w(z)$ satisfies*

$$\rho \le 2 + 2\alpha_{(nqk-nq+1)m,P}, \tag{6}$$

where $Q(z)$ is a polynomial of degree q.

The following examples are to show that Theorem 3 is an extending result of Theorem A.

Example 4 (see [2]). For $n > 0$, let $w(z) = \cos z^{n/2}$; then $\rho(w) = n/2$ and w satisfies the following algebraic differential equation:

$$\left[\left(w^2\right)'\right]^2 = n^2 z^{n-2} w^2 \left(1 - w^2\right) = 0. \tag{7}$$

When $n = 1$ or 2, $\alpha_{2,P} = 0$, and the growth order $\rho(w)$ of any entire solution $w(z)$ of (7) satisfies $\rho(w) \le 2$. When $n \ge 3$, $\alpha_{2,P} = n/2 - 1$, and the growth order $\rho(w)$ of any entire solution $w(z)$ of the above equation satisfies $\rho(w) \le n$.

Example 5. For $n = 2$, the entire function $w(z) = e^{z^2}$ satisfies the following algebraic differential equation:

$$\left[\left(w'\right)^2\right]' = 8zw^2 + 8z^2 w' w. \tag{8}$$

We know that $k=2$, $m=1$, $n=2$, $q=1$, $\alpha_{3,P}=1$, $W(P)=1$, and then $\rho = 2 \le 2 + 2\alpha_{3,P} = 4$.

Example 6. The entire function $w(z) = ze^z$ satisfies the following algebraic differential equation:

$$\left(w'\right)^2 = \frac{2z+1}{z(z+2)} ww'' + \frac{z^2}{z(z+1)} ww'. \tag{9}$$

We know that $p(0) = 2 = W(P)$, $p(1) = 1$, $\alpha_{2,P} = 0$, $\beta_{2,P} = -1 < 0$, and the growth order $\rho(w)$ of any entire solution $w(z)$ of (9) satisfies $\rho \le 2 + 2\alpha_{2,P} = 2$.

Obviously, Example 6 shows that the result in Theorem 3 occurs.

Now we consider the similar result to Theorem 3 for the system of the algebraic differential equations:

$$\left[\left(\left[Q\left(w_2^{(k-1)}\right)\right]^{m_3}\right)'\right]^{m_1} = a(z) R\left(w_1^{(n)}\right),$$
$$\left(R\left(w_1^{(n)}\right)\right)^{m_2} = P[w_2], \tag{10}$$

where $a(z)$ and $R(z)$ are two rational functions.

Qi et al. [16] obtained the following result.

Theorem B. *Let $k, n, q, m_1, m_2, m_3 \in \mathbb{N}$, and let $w = (w_1, w_2)$ be a pair of meromorphic solutions of system (10). If $(m_3qk - m_3q+1)m_1m_2 > W(P)$, and all zeros of w_2 have multiplicity at least k, then the growth orders $\rho(w_i)$ of $w_i(z)$ for $i = 1, 2$ satisfy*

$$\rho(w_1) = \rho(w_2) \le 2 + 2\frac{\deg_{z,\infty} a + m_2 \deg a}{(m_3qk - m_3q + 1)m_1m_2 - W(P)}. \tag{11}$$

Remark 7. In 2009, Gu et al. [17] obtained Theorem B where $R(z) = z$, $k = 1$, $q = 1$, $m_3 = 1$, and $a(z)$ is a polynomial.

We obtain the following result.

Theorem 8. *Let $k, n, q, m_1, m_2, m_3 \in \mathbb{N}$, and let $w = (w_1, w_2)$ be a pair of meromorphic solutions of system (10). If $(m_3qk - m_3q+1)m_1m_2 > W(P)$ or $(m_3qk-m_3q+1)m_1m_2 = W(P)$ and $\beta_{(m_3qk-m_3q+1)m_1m_2,P} + m_2 \deg_{z,\infty} a < 0$, and all zeros of w_2 have multiplicity at least k, then the growth orders $\rho(w_i)$ of $w_i(z)$ for $i = 1, 2$ satisfy*

$$\rho(w_1) = \rho(w_2) \le 2 + 2\alpha^*, \tag{12}$$

where

$$\alpha^* = \max_{t\in I, (m_3qk-m_3q+1)m_1m_2 > p(t)} \frac{\max\left\{\deg_{z,\infty} a_t + m_2 \deg_{z,\infty} a, 0\right\}}{(m_3qk - m_3q + 1)m_1m_2 - p(t)}, \tag{13}$$

and $\alpha^* = 0$, if all $p(t) = (m_3qk - m_3q + 1)m_1m_2$, $t \in I$.

Example 9. The entire functions $w_1(z) = e^z + c$, $w_2(z) = e^z$ satisfy an algebraic differential equation system:

$$\left[\left(w_2^{(k-1)}\right)^2\right]' = 2\left(w_1^{(n)}\right)^2,$$
$$\left\{\left(w_1^{(n)}\right)^2\right\}^3 = \left(w_2\right)^3 \left(w_2'\right)^2, \tag{14}$$

where c is a constant, $m_1 = 1$, $m_2 = 3$, $m_3 = 2$, $q = 1$, $\deg_{z,\infty} a = 0$, $W(P) = 2$, $\alpha^* = 0$, and $(m_3qk - m_3q + 1)m_1m_2 = 3(2k-1) > 2 = W(P)$. So $\rho(w_1) = \rho(w_2) = 1 \le 2$. It shows that the conclusion of Theorem 8 may occur.

Example 10. The entire functions $w_1(z) = e^z$, $w_2(z) = ze^z$ satisfy the following algebraic differential equation system:

$$\begin{gathered}\left(w_2'\right)^2 = (z+1)^2\left(w_1^{(n)}\right)^2, \\ \left(w_1^{(n)}\right)^2 = \frac{2z+1}{z(z+2)(z+1)^2}ww'' \\ + \frac{z^2}{z(z+1)^3}ww'.\end{gathered} \tag{15}$$

We know that $m_1 = 2$, $m_2 = 1$, $R(z) = z^2$, $a(z) = (z+1)^2$, $p(0) = 2 = W(P)$, $p(1) = 1$, $\alpha^* = 0$, $\beta_{2,P} + m_2\deg_{z,\infty} a = -1 < 0$, and the growth order $\rho(w_i)$ of any meromorphic solution (w_1, w_2) of (15) satisfies $\rho(w_1) = \rho(w_2) \le 2 + 2\alpha^* = 2$. It shows that Theorem 8 is an improvement result of Theorem B.

2. Main Lemmas

In order to prove our result, we need the following lemmas. Lemma 11 is an extending result of Zalcman [18] concerning normal family.

Lemma 11 (see [19]). *Let $\mathcal{F}$ be a family of meromorphic (or analytic) functions on the unit disc. Then $\mathcal{F}$ is not normal on the unit disc if and only if there exist*

(a) *a number $r \in (0,1)$;*

(b) *points z_n with $|z_n| < r$;*

(c) *functions $f_n \in \mathcal{F}$;*

(d) *positive numbers $\rho_n \to 0$,*

such that $g_n(\zeta) := f_n(z_n + \rho_n\zeta)$ converges locally uniformly to a nonconstant meromorphic (or entire) function $g(\zeta)$, and its order is at most 2. In particular, we may choose w_n and ρ_n, such that

$$\rho_n \le \frac{2}{f_n^\sharp(w_n)}, \quad f_n^\sharp(w_n) \ge f_n^\sharp(0). \tag{16}$$

Lemma 12 (see [14]). *Let $f(z)$ be meromorphic in whole complex plane with growth order $\rho := \rho(f) > 2$; then for each $0 \le \mu < (\rho-2)/2$, there exists a sequence $a_n \to \infty$, such that*

$$\lim_{n\to\infty} \frac{f^\sharp(a_n)}{|a_n|^\mu} = +\infty. \tag{17}$$

3. Proofs of Theorems

Proof of Theorem 3. Suppose that the conclusion of theorem is not true; then there exists a meromorphic entire solution $w(z)$ that satisfies the equation $[([Q(w^{(k-1)}(z))]^n)']^m = P[w]$, such that

$$\rho > 2 + 2\alpha_{(nqk-nq+1)m,P}. \tag{18}$$

By Lemma 12 we know that for each $0 < \mu < (\rho-2)/2$, there exists a sequence of points $a_j \to \infty$ $(j \to \infty)$, such that (17) is right. This implies that the family $\{w_j(z) := w(a_j + z)\}_{j\in\mathrm{N}}$ is not normal at $z = 0$. By Lemma 11, there exist sequences $\{b_j\}$ and $\{\rho_j\}$ such that

$$\left|a_j - b_j\right| < 1, \quad \rho_j \longrightarrow 0, \tag{19}$$

and $g_j(\zeta) := w_j(b_j - a_j + \rho_j\zeta) = w(b_j + \rho_j\zeta)$ converges locally uniformly to a nonconstant meromorphic function $g(\zeta)$, whose order is at most 2, and all zeros of $g(\zeta)$ have multiplicity at least k. In particular, we may choose b_j and ρ_j, such that

$$\rho_j \le \frac{2}{w^\sharp(b_j)}, \quad w^\sharp(b_j) \ge w^\sharp(a_j). \tag{20}$$

According to (17) and (18)–(20), we can get the following conclusion.

For any fixed constant $0 \le \mu < (\rho-2)/2$, we have

$$\lim_{j\to\infty} b_j^\mu \rho_j = 0. \tag{21}$$

In the differential equation $[([Q(w^{(k-1)}(z))]^n)']^m = P[w]$, we now replace z by $b_j + \rho_j\zeta$. Assuming that $P[w]$ has the form (4), then we obtain

$$\begin{aligned}&\left[\left(\left[Q\left(w^{(k-1)}\left(b_j+\rho_j\zeta\right)\right)\right]^n\right)'\right]^m \\ &\quad = \sum_{r\in I} a_r\left(b_j+\rho_j\zeta, g_j(\zeta)\right)\rho_j^{-p(r)} M_r\left[g_j\right](\zeta).\end{aligned} \tag{22}$$

From

$$\begin{aligned}&\left(\left[Q\left(w^{(k-1)}\left(b_j+\rho_j\zeta\right)\right)\right]^n\right)' \\ &\quad = n\left[Q\left(w^{(k-1)}\left(b_j+\rho_j\zeta\right)\right)\right]^{n-1} \\ &\qquad \times Q'\left(w^{(k-1)}\left(b_j+\rho_j\zeta\right)\right)\left(w^{(k)}\left(b_j+\rho_j\zeta\right)\right),\end{aligned} \tag{23}$$

we have

$$\begin{aligned}&\left(\left[Q\left(w^{(k-1)}\left(b_j+\rho_j\zeta\right)\right)\right]^n\right)' \\ &\quad = \rho_j^{-(nqk-nq+1)} g_j^{(k)}(\zeta) \\ &\qquad \times \left[nq\left(g_j^{(k-1)}\right)^{nq-1}(\zeta) + H\left(\rho_j, g_j^{(k-1)}(\zeta)\right)\right],\end{aligned} \tag{24}$$

where $H(s,t)$ is a polynomial in two variables, whose degree $\deg_s H$ in s satisfies $\deg_s H \ge 1$.

Hence we deduce that

$$\begin{aligned}&\left\{g_j^{(k)}(\zeta)\left[nq\left(g_j^{(k-1)}\right)^{nq-1}(\zeta) + H\left(\rho_j, g_j^{(k-1)}(\zeta)\right)\right]\right\}^m \\ &\quad = \sum_{r\in I} a_r\left(b_j+\rho_j\zeta, g_j(\zeta)\right)\ \rho_j^{(nqk-nq+1)m-p(r)} M_r\left[g_j\right](\zeta).\end{aligned} \tag{25}$$

Therefore, for every fixed $r \in I$, $\zeta \in \mathrm{C}$, and ζ is not the zero and pole of $g(\zeta)$, both (18) and $p(r) < (nqk - nq + 1)m$ imply that $0 \le \mu = \max\{\deg_{z,\infty} a_r, 0\}/((nqk-nq+1)m-p(r)) \le \alpha_{(nqk-nq+1)m,P} < (\rho - 2)/2$, and then

$$
\begin{aligned}
& a_r\left(b_j + \rho_j\zeta, g_j(\zeta)\right)\rho_j^{(nqk-nq+1)m-p(r)} M_r\left[g_j\right](\zeta) \\
& \quad = \frac{a_r\left(b_j + \rho_j\zeta, g_j(\zeta)\right)}{b_j^{\deg_{z,\infty} a_r}} \\
& \qquad \times \left[b_j^{\deg_{z,\infty} a_r/((nqk-nq+1)m-p(r))}\rho_j\right]^{(nqk-nq+1)m-p(r)} \\
& \qquad \times M_r\left[g_j\right](\zeta) \\
& \quad \Longrightarrow 0,
\end{aligned} \tag{26}
$$

by (21), converges 0 local uniformly as $j \to \infty$. Both $p(r) = (nqk - nq + 1)m$ and $\beta_{(nqk-nq+1)m,P} < 0$ give that

$$
\begin{aligned}
& a_r\left(b_j + \rho_j\zeta, g_j(\zeta)\right)\rho_j^{(nqk-nq+1)m-p(r)} M_r\left[g_j\right](\zeta) \\
& \quad = a_r\left(b_j + \rho_j\zeta, g_j(\zeta)\right) M_r\left[g_j\right](\zeta) \\
& \quad \Longrightarrow 0,
\end{aligned} \tag{27}
$$

by (19), converges 0 local uniformly as $j \to \infty$. Both (26) and (27) deduce that $g^{(k)}(\zeta) = 0$ from (25) as $j \to \infty$. Since all zeros of $g(\zeta)$ have multiplicity at least k, this is a contradiction.

The proof of Theorem 3 is complete. □

Proof of Theorem 8. By the first equation of the systems of algebraic differential equations (7), we know

$$
R\left(w_1^{(n)}\right) = \frac{\left[\left(\left[Q\left(w_2^{(k-1)}\right)\right]^{m_3}\right)'\right]^{m_1}}{a(z)}. \tag{28}
$$

Therefore we have

$$
\rho(w_1) = \rho(w_2). \tag{29}
$$

If w_2 is a rational function, then w_1 must be a rational function, so that the conclusion of Theorem 8 is right. If w_2 is a transcendental meromorphic function, by the system of algebraic differential equations (7), then we have

$$
\left[\left(\left[Q\left(w_2^{(k-1)}\right)\right]^{m_3}\right)'\right]^{m_1 m_2} = (a(z))^{m_2} P\left[w_2\right]. \tag{30}
$$

By applying Theorem 3 to (30), we know that the conclusions of Theorem 8 hold.

The proof of Theorem 8 is complete. □

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

Authors' Contribution

Wenjun Yuan and Jianming Lin carried out the design of the study and performed the analysis. Weiling Xiong participated in its design and coordination. All authors read and approved the final paper.

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