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Research Article

On Weakly Singular Versions of Discrete Nonlinear Inequalities and Applications

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Some new weakly singular versions of discrete nonlinear inequalities are established, which generalize some existing weakly singular inequalities and can be used in the analysis of nonlinear Volterra type difference equations with weakly singular kernels. A few applications to the upper bound and the uniqueness of solutions of nonlinear difference equations are also involved.

1. Introduction

Recently, along with the development of the theory of integral inequalities and difference equations, many authors have researched some discrete versions of Gronwall-Bellman type inequalities [1–5]. Starting from the basic form,

$$u(n) \le a(n) + \sum_{s=0}^{n-1} f(s)\, u(s) \tag{1}$$

discussed originally by Pachpatte in [4], various such new inequalities have been established, which can be used as a powerful tool in the analysis of certain classes of finite difference equations. Among these results, discrete weakly singular integral inequalities also play an important role in the study of the behavior and numerical solutions for singular integral equations [6, 7] and the theory for parabolic equations [8–10]. For example, Dixon and McKee [7] investigated the convergence of discretization methods for the Volterra integral and integrodifferential equations using the following inequality:

$$x_i \le \psi_i + Mh^{1-\alpha} \sum_{j=0}^{i-1} \frac{x_j}{(i-j)^{\alpha}}, \quad i = 1, 2, \ldots, N, \qquad n > 0, \quad Nh = T, \tag{2}$$

and Beesack [6] also discussed the inequality,

$$x_i \le \psi_i + Mh^{1+\sigma-\alpha\beta} \sum_{j=0}^{i-1} \frac{j^{\sigma} x_j}{(i^{\beta} - j^{\beta})^{\alpha}}, \tag{3}$$

for the second kind Abel-Volterra singular integral equations. Henry [9] presented a linear inequality

$$x_n \le a_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k b_k x_k, \tag{4}$$

to investigate some qualitative properties for a parabolic equation. In particular, to avoid the shortcoming of analysis, Medved [11–13] used a new method to discuss some nonlinear weakly singular integral inequalities and difference inequalities. Following Medved's work, Ma and Yang [14] improved his method to discuss a more general nonlinear weakly singular integral inequality,

$$\begin{aligned} u(t) \le{}& a(t) + b(t) \int_0^t (t^{\sigma} - s^{\sigma})^{\mu-1} s^{\tau-1} g(s)\, u(s)\, ds \\ &+ c(t) \int_0^t (t^{\alpha} - s^{\alpha})^{\beta-1} s^{\gamma-1} g(s)\, w(u(s))\, ds, \end{aligned} \tag{5}$$

and a nonlinear difference inequality [15],

$$x_n^{\alpha} \le a_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k b_k x_k^{\lambda}. \tag{6}$$

As for other new weakly singular inequalities, recent work can be found, for example, in [16–25] and references therein.

In this paper, we investigate some new nonlinear discrete weakly singular inequalities

$$x_n \le a_n + \sum_{k=0}^{n-1} (t_n^{\alpha} - t_k^{\alpha})^{\beta-1} t_k^{\gamma-1} \tau_k b_k \omega(x_k), \tag{7}$$

$$\begin{aligned} x_n \le a_n &+ \sum_{k=0}^{n-1} (t_n^{\sigma} - t_k^{\sigma})^{\mu-1} t_k^{\lambda-1} \tau_k g_k x_k \\ &+ \sum_{k=0}^{n-1} (t_n^{\alpha} - t_k^{\alpha})^{\beta-1} t_k^{\gamma-1} \tau_k b_k \omega(x_k). \end{aligned} \tag{8}$$

Compared to the existing result, our result is more concise and can be used to obtain pointwise explicit bounds on solutions of a class of more general weakly singular difference equations of Volterra type. Finally, to illustrate the usefulness of the result, we give some applications to Volterra type difference equation with weakly singular kernels.

For convenience, before giving our main results, we first cite some useful lemmas here.

Lemma 1 (discrete inequality, see [15]). *Let $A_1, A_2, \ldots, A_n$ be nonnegative real numbers and $r > 1$ a real number. Then*

$$(A_1 + A_2 + \cdots + A_n)^r \le n^{r-1} (A_1^r + A_2^r + \cdots + A_n^r). \tag{9}$$

Lemma 2 (discrete Hölder inequality, see [15]). *Let a_i, b_i ($i = 1, 2, \ldots, n$) be nonnegative real numbers, and p, q positive numbers such that $1/p + 1/q = 1$ (or $p = 1, q = \infty$). Then*

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}. \tag{10}$$

Lemma 3 (see [14]). *Let α, β, γ, and p be positive constants. Then*

$$\begin{aligned} &\int_0^t (t^{\alpha} - s^{\alpha})^{p(\beta-1)} s^{p(\gamma-1)} ds \\ &\quad = \frac{t^{\theta}}{\alpha} B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right], \quad t \in R_+, \end{aligned} \tag{11}$$

where $B[\xi, \eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1} ds$ ($\operatorname{Re}\xi > 0, \operatorname{Re}\eta > 0$) is the well-known B-function and $\theta = p[\alpha(\beta-1) + \gamma - 1] + 1$.

In what follows, denote $\mathbb{R}$ to be the set of real numbers. Let $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$. $C(X, Y)$ denotes the collection of continuous functions from the set X to the set Y. As usual, the empty sum is taken to be 0.

2. Some New Nonlinear Weakly Singular Difference Inequalities

Lemma 4. *Suppose that $\omega(u) \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing with $w(u) > 0$ for $u > 0$. Let a_n, c_n be nonnegative and nondecreasing in n. If y_n is nonnegative such that*

$$y_n \le a_n + c_n \sum_{k=0}^{n-1} b_k \omega(y_k), \quad n \in \mathbb{N}, \tag{12}$$

then

$$y_n \le \Omega^{-1}\left[\Omega(a_n) + c_n \sum_{k=0}^{n-1} b_k\right], \quad 0 \le n \le M, \tag{13}$$

where $\Omega(v) = \int_{v_0}^{v} (1/\omega(s)) ds$, $v \ge v_0$, Ω^{-1} is the inverse function of Ω, and M is defined by

$$M = \sup\left\{i : \Omega(a_i) + c_i \sum_{k=0}^{i-1} b_k \in \operatorname{Dom}(\Omega^{-1})\right\}. \tag{14}$$

Assume that

(S_1) $\alpha \in (0, 1]$, $\beta \in (0, 1)$ and $1 > (p(\gamma - 1) + 1)/\alpha \ge p(\beta - 1) + 1 > 0$ such that $1/p + \alpha(\beta - 1) + \gamma - 1 \ge 0$;

(S_2) a_n, b_n are nonnegative functions for $n \in \mathbb{N}$, respectively;

(S_3) $\omega(u) \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing and $\omega(0) = 0$.

Lemma 5. *Suppose that $[\alpha, \beta, \gamma]$ satisfies assumption (S_1); then for sufficiently small τ_k, one has*

$$\begin{aligned} &\sum_{k=0}^{n-1} (t_n^{\alpha} - t_k^{\alpha})^{p_i(\beta-1)} t_k^{p_i(\gamma-1)} \tau_k \\ &\quad \le \int_0^{t_n} (t_n^{\alpha} - s^{\alpha})^{p_i(\beta-1)} s^{p_i(\gamma-1)} ds \\ &\quad = \frac{t_n^{\theta_i}}{\alpha} B\left[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1\right]. \end{aligned} \tag{15}$$

Proof. Consider the B-function in (15). Consider

$$\begin{aligned} &B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right] \\ &\quad = \int_0^1 (1-s)^{p(\beta-1)+1-1} s^{(p(\gamma-1)+1)/\alpha-1} ds \\ &\quad := \int_0^1 (1-s)^{n_2-1} s^{n_1-1} ds, \end{aligned} \tag{16}$$

and denote $f(s) := (1-s)^{n_2-1} s^{n_1-1}$ for $s \in (0, 1)$, where $n_1 = (p(\gamma - 1) + 1)/\alpha$ and $n_2 = p(\beta - 1) + 1$. If $n_2 = n_1$, then $f(s)$ is symmetric about $s = 1/2$. According to assumption (S_1),

$$1 > n_1 = \frac{p(\gamma-1)+1}{\alpha} > p(\beta-1)+1 = n_2 > 0; \tag{17}$$

that is,

$$0 > n_1 - 1 > n_2 - 1 > -1. \tag{18}$$

On the other hand, the zero-point of $f'(s)$ can be obtained as follows:

$$s_0 = \frac{n_1 - 1}{n_1 + n_2 - 2} < \frac{1}{2}. \tag{19}$$

Therefore, the function $f(s)$ is decreasing on the interval $(0, s_0]$ while increasing sharply on the interval $[s_0, 1)$. Consequently, for some given sufficiently small τ_k, by the properties of the left-rectangle integral formula, we have

$$\begin{aligned}\sum_{k=0}^{n-1} & (1^\alpha - t_k^\alpha)^{n_2-1} t_k^{n_1-1} \tau_k \\ & \leq \int_0^1 (1 - s^\alpha)^{n_2-1} s^{n_1-1} ds \\ & = B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right],\end{aligned} \tag{20}$$

where $0 = t_0 < t_1 < \cdots < t_n = 1$. For the general interval $(0, t_n]$, we can easily obtain the corresponding result (15) by the similar method and omit the details here.

Denote $\tilde{a}_n = \max_{0\leq k\leq n, k\in\mathbb{N}} a_k$ and $\tau = \max_{0\leq k\leq n-1, k\in\mathbb{N}} \tau_k$, where τ_k is the variable time step. Next, we first discuss inequality (7) and obtain the following result. □

Theorem 6. *Under assumptions* (S_1), (S_2), *and* (S_3), *if* x_n *is nonnegative such that* (7), *then*

$$\begin{aligned}x_n \leq \Bigg[\Omega^{-1}\Bigg(\Omega\left(2^{q-1}\tilde{a}_n^q\right) + 2^{q-1}\tau\left(\frac{t_n^\theta}{\alpha}\right)^{q/p} \\ \times \left(B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right]\right)^{q/p} \sum_{k=0}^{n-1} b_k^q\Bigg)\Bigg]^{1/q},\end{aligned} \tag{21}$$

for $0 \leq n \leq N_1$, *where* $\Omega(u) = \int_{u_0}^u (1/\omega^q(s^{1/q}))ds$, $u \geq u_0$, Ω^{-1} *is the inverse function of* Ω, $\theta = p[\alpha(\beta-1)+\gamma-1]+1$, *and* N_1 *is the largest integer number such that*

$$\begin{aligned}\Omega & \left(2^{q-1}\tilde{a}_n^q\right) + 2^{q-1}\tau\left(\frac{t_n^\theta}{\alpha}\right)^{q/p} \\ & \times \left(B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right]\right)^{q/p} \\ & \times \sum_{k=0}^{n-1} b_k^q \in \operatorname{Dom}\left(\Omega^{-1}\right).\end{aligned} \tag{22}$$

Proof. Since $\tilde{a}_n = \max_{0\leq k\leq n, k\in\mathbb{N}} a_k$, according to assumption (S_2), $\tilde{a}_n$ is nonnegative and nondecreasing, and $\tilde{a}_n \geq a_n$. From (7), we have

$$x_n \leq \tilde{a}_n + \sum_{k=0}^{n-1} (t_n^\alpha - t_k^\alpha)^{\beta-1} t_k^{\gamma-1} \tau_k b_k \omega(x_k). \tag{23}$$

Due to assumption (S_1), take suitable indices p, q such that $1/p + 1/q = 1$. An application of Lemma 2 yields

$$\begin{aligned}x_n & \leq \tilde{a}_n + \sum_{k=0}^{n-1} (t_n^\alpha - t_k^\alpha)^{\beta-1} t_k^{\gamma-1} \tau_k^{1/p} \tau_k^{1/q} b_k \omega(x_k) \\ & \leq \tilde{a}_n + \tau^{1/q} \sum_{k=0}^{n-1} (t_n^\alpha - t_k^\alpha)^{\beta-1} t_k^{\gamma-1} \tau_k^{1/p} b_k \omega(x_k) \\ & \leq \tilde{a}_n + \tau^{1/q}\left[\sum_{k=0}^{n-1} (t_n^\alpha - t_k^\alpha)^{p(\beta-1)} t_k^{p(\gamma-1)} \tau_k\right]^{1/p} \\ & \quad \times \left[\sum_{k=0}^{n-1} b_k^q \omega^q(x_k)\right]^{1/q}.\end{aligned} \tag{24}$$

By Lemma 1, it follows from the inequality above that

$$\begin{aligned}x_n^q \leq 2^{q-1}\tilde{a}_n^q + 2^{q-1}\tau & \left[\sum_{k=0}^{n-1} (t_n^\alpha - t_k^\alpha)^{p(\beta-1)} t_k^{p(\gamma-1)} \tau_k\right]^{q/p} \\ & \times \left[\sum_{k=0}^{n-1} b_k^q \omega^q(x_k)\right].\end{aligned} \tag{25}$$

Considering

$$\begin{aligned}\sum_{k=0}^{n-1} & (t_n^\alpha - t_k^\alpha)^{p(\beta-1)} t_k^{p(\gamma-1)} \tau_k \\ & \leq \int_0^{t_n} (t_n^\alpha - s^\alpha)^{p(\beta-1)} s^{p(\gamma-1)} ds \\ & = \frac{t_n^\theta}{\alpha} B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right],\end{aligned} \tag{26}$$

in which we apply Lemma 5, we have

$$\begin{aligned}x_n^q \leq 2^{q-1}\tilde{a}_n^q + 2^{q-1}\tau & \left(\frac{t_n^\theta}{\alpha}\right)^{q/p} \\ & \times \left(B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right]\right)^{q/p} \\ & \times \left[\sum_{k=0}^{n-1} b_k^q \omega^q(x_k)\right].\end{aligned} \tag{27}$$

Let $v_n = x_n^q$, $A_n = 2^{q-1}\tilde{a}_n^q$, and $C_n = 2^{q-1}\tau(t_n^\theta/\alpha)^{q/p} (B[(p(\gamma-1)+1)/\alpha, p(\beta-1)+1])^{q/p}$. Obviously, A_n, C_n are nondecreasing for $n \in \mathbb{N}$ and $\omega^q(v_k^{1/q})$ satisfies assumption (S_3). Equation (27) can be rewritten as

$$v_n \leq A_n + C_n\left(\sum_{k=0}^{n-1} b_k^q \omega^q\left(v_k^{1/q}\right)\right), \tag{28}$$

which is similar to inequality (12). Using Lemma 4 to (28), we have

$$v_n \leq \left[\Omega^{-1}\left(\Omega(A_n) + C_n \sum_{k=0}^{n-1} b_k^q\right)\right], \tag{29}$$

for $0 \le n \le N_1$, where $\Omega(u) = \int_{u_0}^{u}(1/\omega^q(s^{1/q}))ds$, $u \ge u_0$, Ω^{-1} is the inverse function of Ω, and N_1 is the largest integer number such that

$$\Omega\left(A_n\right) + C_n\sum_{k=0}^{n-1} b_k^q \in \operatorname{Dom}\left(\Omega^{-1}\right). \tag{30}$$

Therefore, by $x_n = v_n^{1/q}$, (21) holds for $0 \le n \le N_1$. □

Remark 7. When $\alpha = 1$ and $\gamma = 1$, the inequality was discussed by Medved [12] which is the special case of our result. Moreover, his result holds under the assumption "$\omega(u)$ satisfies the condition (q);" that is, "$e^{-qt}[\omega(u)]^q \le R(t)\omega(e^{-qt}u^q)$, where $R(t)$ is a continuous, nonnegative function." In our result, the condition (q) is eliminated.

Corollary 8. *Under assumptions (S_1) and (S_2), let $\nu > 0$, $\mu > 0 (\nu > \mu)$. If x_n is nonnegative such that*

$$x_n^\nu \le a_n + \sum_{k=0}^{n-1}(t_n^\alpha - t_k^\alpha)^{\beta-1} t_k^{\gamma-1}\tau_k b_k x_k^\mu, \tag{31}$$

then

$$x_n \le \left[2^{q-1}\tilde{a}_n^{(\nu-\mu)/\nu} + \frac{\nu-\mu}{\nu}2^{q-1}\tau\left(\frac{t_n^\theta}{\alpha}\right)^{q/p} \times\left(B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right]\right)^{q/p}\sum_{k=0}^{n-1} b_k^q\right]^{1/(\nu-\mu)q}, \tag{32}$$

for $n \ge 0$.

Proof. Let $y_n = x_n^\nu$; then $x_n = y_n^{1/\nu}$ or $x_n^\mu = y_n^{\mu/\nu}$. From (31) we have

$$y_n \le a_n + \sum_{k=0}^{n-1}(t_n^\alpha - t_k^\alpha)^{\beta-1} t_k^{\gamma-1}\tau_k b_k y_k^{\mu/\nu}. \tag{33}$$

Denote $\omega(y_k) = y_k^{\mu/\nu}$. Clearly, ω satisfies assumption (S_3). With the definition of Ω in Theorem 6, letting $u_0 = 0$, we have

$$\Omega(u) = \int_0^u \frac{ds}{s^{\mu/\nu}} = \frac{\nu}{\nu-\mu}u^{(\nu-\mu)/\nu}, \tag{34}$$

$$\Omega^{-1}(u) = \left(\frac{\nu-\mu}{\nu}u\right)^{\nu/(\nu-\mu)}, \quad \operatorname{Dom}\left(\Omega^{-1}\right) = [0,\infty). \tag{35}$$

Substituting (34) and (35) into (29), we get

$$y_n \le \left[2^{q-1}\tilde{a}_n^{(\nu-\mu)/\nu} + \frac{\nu-\mu}{\nu}2^{q-1}\tau\left(\frac{t_n^\theta}{\alpha}\right)^{q/p} \times\left(B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right]\right)^{q/p}\sum_{k=0}^{n-1} b_k^q\right]^{\nu/(\nu-\mu)q}. \tag{36}$$

In view of $x_n = y_n^{1/\nu}$, we can obtain (32). □

Remark 9. In [15], Yang et al. investigated inequality (6). Clearly, let $\alpha = 1$ and $\gamma = 1$ in (31), and we can get the same formula.

Remark 10. Let $\nu = 2$ and $\mu = 1$; we can get the interesting Henry's version of the Ou-Iang-Pachpatte type difference inequality [26]. Thus, our results are a more general discrete analogue for such inequality.

Remark 11. Ma and Pečarić discussed the continuous case of (2.15) in [27] and here we present the discrete version of their result. Furthermore, the result in [27] is established for the cases when the ordered parameter group $[\alpha,\beta,\gamma]$ obeys distribution I or II (for details, see [27]) which makes the application of inequality more inconvenient. Clearly, our result is based on the concise assumption to overcome this weakness.

Corollary 12. *Under assumptions (S_1) and (S_2), if x_n is nonnegative such that*

$$x_n \le a_n + \sum_{k=0}^{n-1}(t_n^\alpha - t_k^\alpha)^{\beta-1} t_k^{\gamma-1}\tau_k b_k x_k, \tag{37}$$

then

$$x_n \le 2^{(q-1)/q}\tilde{a}_n \times\exp\left(\frac{2^{q-1}}{q}\tau\left(\frac{t_n^\theta}{\alpha}\right)^{q/p} \times\left(B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right]\right)^{q/p}\sum_{k=0}^{n-1} b_k^q\right), \tag{38}$$

for $n \ge 0$.

Proof. In (7), $\omega(u) = u$ also satisfies assumption (S_3). Thus, we have

$$\begin{gathered}\Omega(u) = \int_{u_0}^u \frac{ds}{s} = \ln\frac{u}{u_0},\\ \Omega^{-1}(u) = u_0\exp(u),\\ \operatorname{Dom}\left(\Omega^{-1}\right) = [0,\infty).\end{gathered} \tag{39}$$

Similar to the computation in Corollary 8, estimate (38) holds.

Now, we discuss inequality (8)

$$\begin{aligned}x_n \le a_n &+ \sum_{k=0}^{n-1}(t_n^\sigma - t_k^\sigma)^{\mu-1} t_k^{\lambda-1}\tau_k g_k x_k\\ &+ \sum_{k=0}^{n-1}(t_n^\alpha - t_k^\alpha)^{\beta-1} t_k^{\gamma-1}\tau_k b_k \omega(x_k).\end{aligned} \tag{40}$$

Since there are two different parameter groups $[\sigma, \mu, \lambda]$ and $[\alpha, \beta, \gamma]$, assumption (S_1) is revised as follows:

(S_4) $\sigma \in (0,1]$, $\mu \in (0,1)$, $1 > (p(\lambda-1)+1)/\sigma \geq p(\mu-1)+1 > 0$ and $\alpha \in (0,1]$, $\beta \in (0,1)$, $1 > (p(\gamma-1)+1)/\alpha \geq p(\beta-1)+1 > 0$ such that $1/p + \sigma(\mu-1)+\lambda-1 \geq 0$ and $1/p+\alpha(\beta-1)+\gamma-1 \geq 0 (p > 1)$.

□

Theorem 13. *Under assumptions (S_3) and (S_4), suppose that g_k, b_k are nonnegative for $n \in \mathbb{N}$. If x_n is nonnegative such that* (8), *then*

$$x_n \leq \left[\Omega^{-1}\left(\Omega\left(2^{q-1}\tilde{a}_n^q Q_n^q\right) + 2^{q-1}Q_n^q \tau\left(\frac{t_n^{\theta_2}}{\alpha}\right)^{q/p} \times \left(B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right]\right)^{q/p} \times \sum_{k=0}^{n-1} b_k^q\right)\right]^{1/q}, \tag{41}$$

for $0 \leq n \leq N_2$, where Ω and Ω^{-1} are defined as in Theorem 6,

$$\begin{aligned} Q_n &= 2^{(q-1)/q} \times \exp\left(\frac{2^{q-1}}{q}\tau\left(\frac{t_n^{\theta_1}}{\sigma}\right)^{q/p} \times \left(B\left[\frac{p(\lambda-1)+1}{\sigma}, p(\mu-1)+1\right]\right)^{q/p} \sum_{k=0}^{n-1} g_k^q\right), \\ \theta_1 &= p[\sigma(\mu-1)+\lambda-1]+1, \\ \theta_2 &= p[\alpha(\beta-1)+\gamma-1]+1, \end{aligned} \tag{42}$$

and N_2 is the largest integer number such that

$$\Omega\left(2^{q-1}\tilde{a}_n^q Q_n^q\right) + 2^{q-1}Q_n^q \tau\left(\frac{t_n^{\theta_2}}{\alpha}\right)^{q/p} \times \left(B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right]\right)^{q/p} \times \sum_{k=0}^{n-1} b_k^q \in \operatorname{Dom}\left(\Omega^{-1}\right). \tag{43}$$

Proof. By the definition of $\tilde{a}_n$, we have

$$x_n \leq \tilde{a}_n + \sum_{k=0}^{n-1}\left(t_n^\sigma - t_k^\sigma\right)^{\mu-1} t_k^{\lambda-1}\tau_k g_k x_k + \sum_{k=0}^{n-1}\left(t_n^\alpha - t_k^\alpha\right)^{\beta-1} t_k^{\gamma-1}\tau_k b_k \omega\left(x_k\right). \tag{44}$$

Let

$$P_n = \tilde{a}_n + \sum_{k=0}^{n-1}\left(t_n^\alpha - t_k^\alpha\right)^{\beta-1} t_k^{\gamma-1}\tau_k b_k \omega\left(x_k\right), \tag{45}$$

which yields directly

$$x_n \leq P_n + \sum_{k=0}^{n-1}\left(t_n^\sigma - t_k^\sigma\right)^{\mu-1} t_k^{\lambda-1}\tau_k g_k x_k. \tag{46}$$

Using Corollary 12, from the inequality above, we get

$$x_n \leq 2^{(q-1)/q} P_n \times \exp\left(\frac{2^{q-1}}{q}\tau\left(\frac{t_n^{\theta_1}}{\sigma}\right)^{q/p} \times \left(B\left[\frac{p(\lambda-1)+1}{\sigma}, p(\mu-1)+1\right]\right)^{q/p} \sum_{k=0}^{n-1} g_k^q\right), \tag{47}$$

where $\theta_1 = p[\sigma(\mu-1)+\lambda-1]+1$. Letting

$$Q_n = 2^{(q-1)/q} \exp\left(\frac{2^{q-1}}{q}\tau\left(\frac{t_n^{\theta_1}}{\sigma}\right)^{q/p} \times \left(B\left[\frac{p(\lambda-1)+1}{\sigma}, p(\mu-1)+1\right]\right)^{q/p} \times \sum_{k=0}^{n-1} g_k^q\right), \tag{48}$$

from (47), we get

$$\begin{aligned} u_n &\leq P_n Q_n \\ &\leq \tilde{a}_n Q_n + Q_n \sum_{k=0}^{n-1}\left(t_n^\alpha - t_k^\alpha\right)^{\beta-1} t_k^{\gamma-1}\tau_k b_k \omega\left(x_k\right). \end{aligned} \tag{49}$$

Clearly, inequality (49) is similar to (7). According to Theorem 6, we obtain

$$x_n \leq \left[\Omega^{-1}\left(\Omega\left(2^{q-1}\tilde{a}_n^q Q_n^q\right) + 2^{q-1}Q_n^q \tau\left(\frac{t_n^{\theta_2}}{\alpha}\right)^{q/p} \times \left(B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right]\right)^{q/p} \times \sum_{k=0}^{n-1} b_k^q\right)\right]^{1/q}, \tag{50}$$

for $0 \leq n \leq N_2$. □

Remark 14. Our result for inequality (8) is also the discrete analogue of inequality (5). In fact, with the different choice of the parameter groups $[\alpha, \beta, \gamma]$ and $[\sigma, \mu, \lambda]$ in [14], the complicate results must be presented by four cases, respectively. Apparently, compared to their results, our result is quite simple.

3. Applications

In this section, we apply our results to discuss the upper bound and the uniqueness of solutions of a Volterra type difference equation with certain weakly singular kernels.

Example 15. Consider the following inequality:

$$\begin{aligned} x_n \leq \frac{1}{2} &+ \sum_{k=0}^{n-1} (t_n - t_k)^{-1/3} t_k^{-1/4} \tau_k x_k \\ &+ \sum_{k=0}^{n-1} (t_n - t_k)^{-1/3} t_k^{-1/3} \tau_k \sqrt{x_k}. \end{aligned} \tag{51}$$

Obviously, (51) is the special case of inequality (8), and we get

$$\begin{gathered} a_n = \frac{1}{2}, \qquad \sigma = 1, \qquad \mu = \frac{2}{3}, \qquad \lambda = \frac{3}{4}, \\ \alpha = 1, \qquad \beta = \frac{2}{3}, \qquad \gamma = \frac{2}{3}, \\ g_k = 1, \qquad b_k = 1. \end{gathered} \tag{52}$$

Next, we discuss the choice of parameter p. By assumption (S_4), from the conditions $1 > (p(\lambda-1)+1)/\sigma \geq p(\mu-1)+1 > 0$ and $1/p + \sigma(\mu - 1) + \lambda - 1 \geq 0$, we have $1 < p < 12/7$. From the conditions $1 > (p(\gamma - 1) + 1)/\alpha \geq p(\beta - 1) + 1 > 0$ and $1/p + \alpha(\beta - 1) + \gamma - 1 \geq 0 (p > 1)$, we have $1 < p < 3/2$. Thus, we can take $p = 4/3$; then $q = 4$, $q/p = 3$. According to Theorem 13, we obtain

$$\begin{gathered} \tilde{a}_n = \frac{1}{2}, \qquad \theta_1 = \frac{2}{9}, \qquad \theta_2 = \frac{1}{9}, \\ B\left[\frac{p(\lambda - 1) + 1}{\sigma}, p(\mu - 1) + 1\right] = B\left[\frac{2}{3}, \frac{5}{9}\right], \\ B\left[\frac{p(\gamma - 1) + 1}{\alpha}, p(\beta - 1) + 1\right] = B\left[\frac{5}{9}, \frac{5}{9}\right], \\ \frac{q-1}{q} = \frac{3}{4}, \qquad \frac{2^{q-1}}{q} = 2; \\ Q_n = 2^{3/4} \exp\left(2\tau \left(t_n^{2/9}\right)^3 \left(B\left[\frac{2}{3}, \frac{5}{9}\right]\right)^3 n\right), \\ \Omega(u) = \int_0^u \frac{ds}{\sqrt{s}} = 2\sqrt{u}, \qquad \Omega^{-1}(u) = \frac{u^2}{4}, \\ 2^{q-1} \tilde{a}_n^q Q_n^q = 4 \exp\left(8\tau t_n^{1/5} B^3 \left[\frac{2}{3}, \frac{5}{9}\right] n\right). \end{gathered} \tag{53}$$

Substituting the results above into (41), we can get the upper bound of x_n and omit the details for its complicated formula.

Example 16. Consider the linear weakly singular difference equation

$$x_n = a_n + \sum_{k=0}^{n-1} (t_n^\alpha - t_k^\alpha)^{\beta-1} t_k^{\gamma-1} \tau_k b_k x_k, \tag{54}$$

$$y_n = c_n + \sum_{k=0}^{n-1} (t_n^\alpha - t_k^\alpha)^{\beta-1} t_k^{\gamma-1} \tau_k b_k y_k, \tag{55}$$

where $|a_n - c_n| < \epsilon$, ϵ is an arbitrary positive number, and the parameter group $[\alpha, \beta, \gamma]$ satisfies assumption (S_1). From (54) and (55), we get

$$|x_n - y_n| \leq |a_n - c_n| + \sum_{k=0}^{n-1} (t_n^\alpha - t_k^\alpha)^{\beta-1} t_k^{\gamma-1} \tau_k b_k |x_k - y_k|, \tag{56}$$

which is the form of inequality (37). Applying Corollary 12, we have

$$\begin{aligned} |x_n - y_n| \leq{}& 2^{(q-1)/q} \epsilon \\ & \times \exp\left(\frac{2^{q-1}}{q} \tau \left(\frac{t_n^\theta}{\alpha}\right)^{q/p}\right. \\ & \qquad \times \left(B\left[\frac{p(\gamma - 1) + 1}{\alpha}, p(\beta - 1) + 1\right]\right)^{q/p} \\ & \qquad \left. \times \sum_{k=0}^{n-1} b_k^q \right), \end{aligned} \tag{57}$$

for $n \in \mathbb{N}$. If $a_n = c_n$, let $\epsilon \to 0$ and we obtain the uniqueness of the solution of (54).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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