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Research Article

Uniform Exponential Stability of Discrete Evolution Families on Space of p-Periodic Sequences

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We prove that the discrete system $\zeta_{n+1} = \mathscr{A}_n\zeta_n$ is uniformly exponentially stable if and only if the unique solution of the Cauchy problem $\zeta_{n+1} = \mathscr{A}_n\zeta_n + e^{i\theta(n+1)}z(n+1)$, $n \in \mathbb{Z}_+$, $\zeta_0 = 0$, is bounded for any real number θ and any p-periodic sequence $z(n)$ with $z(0) = 0$. Here, $\mathscr{A}_n$ is a sequence of bounded linear operators on Banach space X.

1. Introduction

The investigation of difference equations $\zeta_{n+1} = \mathscr{A}_n\zeta_n$ or $\zeta_{n+1} = \mathscr{A}_n\zeta_n + f_n$ leads to the idea of discrete evolution family. The main interest is the asymptotic behavior of the solutions and different types of stabilities in the study of such systems. There are a number of spectral criteria for the characterizations of stability of such systems.

New difficulties appear in the study of nonautonomous systems, especially because the part of the solution generated by the forced term (f_n), that is, $\sum_{k=\nu}^{n}\mathbb{U}(n,k)f_k$, is not a convolution in the classical sense. These difficulties may be passed by using the so-called evolution semigroups. The evolution semigroups were exhaustively studied in [1]. Clark et al. [2] developed this efficient method to the study of continuous case. So far, there are few related results regarding the investigation of discrete systems. Recently, the discrete versions of [3, 4] were obtained in [5, 6], respectively.

Buşe et al. [7] considered the uniform exponential stability of discrete nonautonomous systems on the space of sequences denoted by $C_{00}(\mathbb{Z}_+, X)$. The objective of this paper is to extend results obtained in [7] to the space of p-periodic sequences denoted by $\mathbb{P}_0^p(\mathbb{Z}_+, X)$. The results reported in this paper develop the theory of discrete evolution families on a space of bounded sequences. Similar results of this type for the continuous case may be found in the paper by Buşe and Jitianu [8] and the references cited therein.

2. Notation and Preliminaries

Let X be a real or complex Banach space and let $\mathscr{B}(X)$ be the Banach algebra of all linear and bounded operators acting on X.

We denote by $\|\cdot\|$ the norms of operators and vectors. Denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{Z}_+$ the set of all nonnegative integers.

Let $\mathscr{B}(\mathbb{Z}_+, X)$ be the space of X-valued bounded sequences with the supremum norm, and let $\mathbb{P}_0^p(\mathbb{Z}_+, X)$ be the space of p-periodic (with $p \in \{2, 3, \ldots\}$) sequences $z(n)$ with $z(0) = 0$. Then, $\mathbb{P}_0^p(\mathbb{Z}_+, X)$ is a closed subspace of $\mathscr{B}(\mathbb{Z}_+, X)$.

Throughout this paper, $\mathscr{A} \in \mathscr{B}(X)$, $\sigma(\mathscr{A})$ denotes the spectrum of $\mathscr{A}$, and $r(\mathscr{A}) := \sup\{|\lambda| : \lambda \in \sigma(\mathscr{A})\}$ denotes the spectral radius of $\mathscr{A}$. It is well known that $r(\mathscr{A}) = \lim_{n\to\infty}\|\mathscr{A}^n\|^{1/n}$. The resolvent set of $\mathscr{A}$ is defined as $\rho(\mathscr{A}) := \mathbb{C} \setminus \sigma(\mathscr{A})$, that is, the set of all $\lambda \in \mathbb{C}$ for which $\mathscr{A} - \lambda I$ is an invertible operator in $\mathscr{B}(X)$.

We give some results in the framework of general Banach space and spaces of sequences as defined above.

Recall that $\mathscr{A}$ is power bounded if there exists a positive constant M such that $\|\mathscr{A}^n\| \leq M$ for all $n \in \mathbb{Z}_+$.

We need the following auxiliary lemma.

Lemma 1 (Buşe et al. [3]). *Assume $\mathscr{V} \in \mathscr{B}(X)$ and let*

$$\sup_{\theta\in\mathbb{R}} \sup_{n\in\mathbb{Z}_+} \left\| \sum_{k=0}^{n} e^{i\theta k}\mathscr{V}^k \right\| < \infty. \tag{1}$$

Then, $r(\mathscr{V}) < 1$.

3. Uniform Exponential Stability of Discrete Evolution Family on Space $\mathbb{P}_0^p(\mathbb{Z}_+,X)$

The family $\mathscr{U} := \{\mathbb{U}(n,m) : n,m \in \mathbb{Z}_+, n \geq m\}$ of bounded linear operators is called p-periodic discrete evolution family, for a fixed integer $p \in \{2,3,\ldots\}$, if it satisfies the following properties:

(i) $\mathbb{U}(n,n) = I$, for all $n \in \mathbb{Z}_+$;

(ii) $\mathbb{U}(n,m)\mathbb{U}(m,r) = \mathbb{U}(n,r)$, for all $n \geq m \geq r, n,m,r \in \mathbb{Z}_+$;

(iii) $\mathbb{U}(n+p,m+p) = \mathbb{U}(n,m)$, for all $n \geq m, n,m \in \mathbb{Z}_+$;

(iv) the map $(n,m) \mapsto \mathbb{U}(n,m)$: $\{(n,m) : n,m \in \mathbb{Z}_+, n \geq m\} \to X$ is continuous, for all $n \geq m$.

It is well known that any p-periodic discrete evolution family $\mathscr{U}$ is exponentially bounded; that is, there exist $\omega \in \mathbb{R}$ and an $M_\omega \geq 0$ such that

$$\|\mathbb{U}(n,m)\| \leq M_\omega e^{\omega(n-m)}, \quad \forall n \geq m \in \mathbb{Z}_+. \tag{2}$$

When a family $\mathscr{U}$ is exponentially bounded, its growth bound $\omega_0(\mathscr{U})$ is the infimum of all $\omega \in \mathbb{R}$ for which there exists an $M_\omega \geq 1$ such that inequality (2) is fulfilled. It is known that

$$\begin{aligned}\omega_0(\mathscr{U}) &= \lim_{n\to\infty} \frac{\ln\|\mathbb{U}(n,0)\|}{n} \\ &= \frac{1}{p}\ln\left(r\left(\mathbb{U}(p,0)\right)\right).\end{aligned} \tag{3}$$

As a matter of fact,

$$\begin{aligned}\omega_0(\mathscr{U}) &= \lim_{n\to\infty} \frac{\ln\|\mathbb{U}(n,0)\|}{n} \\ &= \lim_{n\to\infty} \frac{\ln\|\mathbb{U}(np,0)\|}{np} \\ &= \frac{1}{p}\lim_{n\to\infty} \ln\|\mathbb{U}^n(p,0)\|^{1/n} \\ &= \frac{1}{p}\ln\lim_{n\to\infty}\|\mathbb{U}^n(p,0)\|^{1/n} \\ &= \frac{1}{p}\ln\left(r\left(\mathbb{U}(p,0)\right)\right).\end{aligned} \tag{4}$$

A family $\mathscr{U}$ is termed uniformly exponentially stable if $\omega_0(\mathscr{U})$ is negative or, equivalently, there exist an $M > 0$ and $\omega > 0$ such that $\|\mathbb{U}(n,m)\| \leq Me^{-\omega(n-m)}$, for all $n \geq m \in \mathbb{Z}_+$. Therefore, we have the following lemma.

Lemma 2. *The discrete evolution family $\mathscr{U}$ is uniformly exponentially stable if and only if $r(\mathbb{U}(p,0)) < 1$.*

The map $\mathbb{U}(p,0)$ is also called the Poincare map of the evolution family $\mathscr{U}$.

Consider the following discrete Cauchy problem:

$$\begin{aligned}\zeta_{n+1} &= \mathscr{A}_n\zeta_n + e^{i\theta(n+1)}z(n+1), \quad n \in \mathbb{Z}_+, \\ \zeta_0 &= 0,\end{aligned} \tag{$\mathscr{A}_n, \theta, 0$}$$

where the sequence $(\mathscr{A}_n)$ is p-periodic; that is, $\mathscr{A}(n+p) = \mathscr{A}_n$ for all $n \in \mathbb{Z}_+$ and a fixed $p \in \{2,3,\ldots\}$.

Let

$$\mathbb{U}(n,k) := \begin{cases}\mathscr{A}_{n-1}\mathscr{A}_{n-2}\cdots\mathscr{A}_k, & \text{if } k \leq n-1, \\ I, & \text{if } k = n.\end{cases} \tag{5}$$

Then, the family $\{\mathbb{U}(n,k)\}_{n\geq k\geq 0}$ is a discrete p-periodic evolution family. Over finite dimensional spaces, the uniform exponential stability of the Cauchy problem $(\mathscr{A}_n, \theta, 0)$ in discrete and continuous autonomous cases has been studied in [9, 10].

Let us divide n by p, that is, $n = lp + r$ for some $l \in \mathbb{Z}_+$, where $r \in \{0,1,\ldots,p-1\}$. We consider the following sets which will be useful along this work:

$$\mathscr{A}_j := \{1+jp, 2+jp, \ldots, (j+1)p-1\}, \quad \forall j \in \mathbb{Z}_+. \tag{6}$$

If $r \in \{1,2,\ldots,p-1\}$, then define

$$\begin{aligned}B_l &:= \{lp+1, lp+2, \ldots, lp+r\}, \\ C &:= \{0, p, 2p, \ldots, lp\}.\end{aligned} \tag{7}$$

It is clear that

$$\bigcup_{j=0}^{l-1} A_j \bigcup B_l \bigcup C = \{0,1,2,\ldots,n\}. \tag{8}$$

With the help of partition (8), we construct the space $\mathscr{W}$ which consists of all the sequences of the form

$$z(k) := \begin{cases}(k-jp)\left[(1+j)p-k\right]\mathbb{U}(k-jp,0), & \text{if } k \in \mathscr{A}_j, \\ k(p-k)\mathbb{U}(k,0), & \text{if } k \in B_l, \\ 0, & \text{if } k \in C.\end{cases} \tag{9}$$

That is,

$$\mathscr{W} := \{z(n) : z(n) \text{ has the property (9)}\}. \tag{10}$$

Obviously, $\mathscr{W}$ is the subspace of $\mathbb{P}_0^p(\mathbb{Z}_+, X)$.

Our result is stated as follows.

Theorem 3. *Let $\mathscr{U} := \{\mathbb{U}(n,m) : n,m \in \mathbb{Z}_+, n \geq m\}$ be a discrete evolution family on X. If the sequence*

$$\zeta_n = \sum_{k=0}^{n} e^{i\theta k}\mathbb{U}(n,k)z(k) \tag{11}$$

is bounded for each real number θ and each p-periodic sequence $z(n) \in \mathscr{W}$, then $\mathscr{U}$ is uniformly exponentially stable.

Proof. Let $z(n) \in \mathscr{W}$. By virtue of (9),

$$\begin{aligned}\zeta_n &= \sum_{k=1}^{n} e^{i\theta k}\mathbb{U}(n,k)\, z(k)\\ &= \sum_{k=1}^{lp+r} e^{i\theta k}\mathbb{U}(lp+r,k)\, z(k)\\ &= \sum_{k\in\cup_{j=0}^{l-1}A_j\cup B_l\cup C} e^{i\theta k}\mathbb{U}(lp+r,k)\, z(k)\\ &= \sum_{k\in\cup_{j=0}^{l-1}A_j} e^{i\theta k}\mathbb{U}(lp+r,k)\, z(k)\\ &\quad + \sum_{k\in B_l} e^{i\theta k}\mathbb{U}(lp+r,k)\, z(k)\\ &\quad + \sum_{k\in C} e^{i\theta k}\mathbb{U}(lp+r,k)\, z(k)\\ &= \sum_{j=0}^{l-1}\sum_{k=1+jp}^{p-1+jp} e^{i\theta k}\mathbb{U}(lp+r,k)\, z(k)\\ &\quad + \sum_{k=lp+1}^{lp+r} e^{i\theta k}\mathbb{U}(lp+r,k)\, z(k)\\ &\quad + \sum_{k\in C} e^{i\theta k}\mathbb{U}(lp+r,k)\, z(k).\end{aligned} \tag{12}$$

By virtue of the definition of $z(k)$, we have

$$\begin{aligned}\zeta_n &= \sum_{k=1}^{n} e^{i\theta k}\mathbb{U}(n,k)\, z(k)\\ &= \sum_{j=0}^{l-1}\sum_{k=1+jp}^{p-1+jp} e^{i\theta k}\mathbb{U}(lp+r,k)(k-jp)\\ &\qquad \times [(1+j)p-k]\,\mathbb{U}(k-jp,0)\\ &\quad + \sum_{k=lp+1}^{lp+r} e^{i\theta k}\mathbb{U}(lp+r,k)\, k(p-k)\,\mathbb{U}(k,0)\\ &= \sum_{j=0}^{l-1}\sum_{k=1+jp}^{p-1+jp} e^{i\theta k}\mathbb{U}(lp+r,k)(k-jp)\\ &\qquad \times [(1+j)p-k]\,\mathbb{U}(k-jp,0)\\ &\quad + \sum_{k=lp+1}^{lp+r} e^{i\theta k}\mathbb{U}(lp+r,k)\, k(p-k)\,\mathbb{U}(k,0)\\ &= I_1 + I_2,\end{aligned} \tag{13}$$

where

$$\begin{aligned}I_1 &:= \sum_{j=0}^{l-1}\sum_{k=1+jp}^{p-1+jp} e^{i\theta k}\mathbb{U}(lp+r,k)(k-jp)\\ &\qquad \times [(1+j)p-k]\,\mathbb{U}(k-jp,0),\\ I_2 &:= \sum_{k=lp+1}^{lp+r} e^{i\theta k}\mathbb{U}(lp+r,k)\, k(p-k)\,\mathbb{U}(k,0).\end{aligned} \tag{14}$$

Write I_1 in the form

$$\begin{aligned}I_1 &= \sum_{j=0}^{l-1}\sum_{k=1+jp}^{p-1+jp} e^{i\theta k}\mathbb{U}(lp+r,k)(k-jp)\\ &\qquad \times [(1+j)p-k]\,\mathbb{U}(k-jp,0)\\ &= \sum_{j=0}^{l-1}\sum_{k=1+jp}^{p-1+jp} e^{i\theta k}\mathbb{U}(lp+r,k)(k-jp)\\ &\qquad \times [(1+j)p-k]\,\mathbb{U}(k,jp)\\ &= \sum_{j=0}^{l-1}\sum_{k=1+jp}^{p-1+jp} e^{i\theta k}\mathbb{U}(lp+r,jp)(k-jp)\\ &\qquad \times [(1+j)p-k]\\ &= \sum_{j=0}^{l-1}\mathbb{U}(lp+r,jp)\\ &\qquad \times \sum_{k=1+jp}^{p-1+jp} e^{i\theta k}(k-jp)[(1+j)p-k]\\ &= \sum_{j=0}^{l-1}\mathbb{U}(lp+r,jp)\\ &\qquad \times \sum_{k=1+jp}^{p-1+jp} e^{i\theta k}(k-jp)[p-(k-jp)]\\ &= \sum_{j=0}^{l-1}\mathbb{U}(r,0)\,\mathbb{U}^{l-j}(p,0)\, e^{i\theta jp}\sum_{\nu=1}^{p-1} e^{i\theta\nu}\nu(p-\nu)\\ &= \mathbb{U}(r,0)\sum_{\nu=1}^{p-1} e^{i\theta\nu}\nu(p-\nu)\sum_{j=0}^{l-1} e^{i\theta jp}\mathbb{U}^{l-j}(p,0)\\ &= \mathbb{U}(r,0)\sum_{\nu=1}^{p-1} e^{i\theta\nu}\nu(p-\nu)\sum_{\alpha=1}^{l} e^{i\theta p(l-\alpha)}\mathbb{U}^{\alpha}(p,0)\\ &= \mathbb{U}(r,0)\sum_{\nu=1}^{p-1} e^{i\theta\nu}\nu(p-\nu)\, e^{i\theta pl}\end{aligned}$$

$$\times \sum_{\alpha=1}^{l} e^{-i\theta p\alpha} \mathbb{U}^{\alpha}(p,0)$$

$$= G(\theta,p) \sum_{\alpha=1}^{l} e^{-i\theta p\alpha} \mathbb{U}^{\alpha}(p,0), \tag{15}$$

where $G(\theta,p) := \mathbb{U}(r,0)\sum_{\nu=1}^{p-1} e^{i\theta\nu} v(p-\nu)e^{i\theta pl} \neq 0$. Furthermore,

$$\begin{aligned} I_2 &= \sum_{k=lp+1}^{lp+r} e^{i\theta k} \mathbb{U}(lp+r,k)\, k(p-k)\, \mathbb{U}(k,0) \\ &= \sum_{k=lp+1}^{lp+r} e^{i\theta k} \mathbb{U}(lp+r,0)\, k(p-k) \\ &= \mathbb{U}(lp+r,0) \sum_{k=lp+1}^{lp+r} e^{i\theta k} k(p-k). \end{aligned} \tag{16}$$

Therefore,

$$\begin{aligned} &\sum_{k=0}^{n} e^{i\theta k} \mathbb{U}(n,k)\, z(k) \\ &\quad = G(\theta,p) \sum_{\alpha=1}^{l} e^{-i\theta p\alpha} \mathbb{U}^{\alpha}(p,0) \\ &\qquad + \mathbb{U}(lp+r,0) \sum_{k=lp+1}^{lp+r} e^{i\theta k} k(p-k). \end{aligned} \tag{17}$$

Since ζ_n is bounded, we conclude that I_1 is bounded; that is,

$$\sup_{l\geq 0} \left\| \sum_{\alpha=0}^{l} e^{-i\theta p\alpha} \mathbb{U}^{\alpha}(p,0) \right\| < \infty. \tag{18}$$

Using Lemma 1, we deduce that $r(\mathbb{U}(p,0)) < 1$. Hence, by Lemma 2, $\mathcal{U}$ is uniformly exponentially stable. This completes the proof. □

Conflict of Interests

The authors declare that they have no competing interests.

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