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Research Article

Zeros, Poles, and Fixed Points of Meromorphic Solutions of Difference Painlevé Equations

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In this paper, we mainly study the properties of transcendental meromorphic solutions $f(z)$ of difference Painlevé equations $w(z+1)w(z-1)(w(z)-1)=\eta(z)w^2(z)-\lambda(z)w(z)$ and $w(z+1)w(z-1)(w(z)-1)=\eta(z)w(z)$ and obtain precise estimations of the exponents of convergence of zeros, poles of $\Delta f(z)$ and $\Delta f(z)/f(z)$, and of fixed points of $f(z+c)$ for any $c\in\mathbb{C}$.

1. Introduction and Main Results

At the beginning of last century, Painlevé, Gambier, and Fuchs classified a large number of second-order differential equations in terms of a characteristic which is now known as the Painlevé property [1–4]. Ablowitz et al. [5] considered discrete equations as delay equations in the complex plane which enabled them to utilize complex analytic methods. They looked at, for instance, difference equations of the type

$$y(z+1)+y(z-1)=R(z,y), \tag{1}$$

where R is rational in both of its arguments. It is shown that if (1) has at least one nonrational finite-order meromorphic solution, then $\deg_y R\le 2$.

In this paper, we use the basic notions of Nevanlinna's theory (see [6, 7]). In addition, we use the notations

$\sigma(w)$ to denote the order of growth of the meromorphic function $w(z)$;

$\lambda(w)$ and $\lambda(1/w)$, respectively, to denote the exponents of convergence of zeros and poles of $w(z)$;

$\tau(w)$ to denote the exponent of convergence of fixed points of $w(z)$.

The quantity $\delta(a,w)$ is called the deficiency of the value a to $w(z)$. Furthermore, we denote by $S(r,w)$ any quantity satisfying $S(r,w)=o(T(r,w))$ for all r outside of a set with finite logarithmic measure and by

$$\mathcal{S}(w)=\{\alpha \text{ meromorphic}: T(r,\alpha)=S(r,w)\} \tag{2}$$

the field of small functions with respect to w. A meromorphic solution w of a difference (or differential) equation is called *admissible* if all coefficients of the equation are in $\mathcal{S}(w)$.

Recently, Halburd and Korhonen [8] considered (1) and got the following theorem.

Theorem A. *Let $R(z,y)$ be rational in both of its arguments such that its denominator has at least two distinct roots. If the second-order difference equation* (1) *admits a nonrational meromorphic solution of finite order such that there is a finite real constant $c\ge 1$, such that for sufficiently large r,*

$$c^{-1}\overline{n}_I(r,w)\le \overline{n}_{II}(r,w)\le c\overline{n}_I(r,w) \tag{3}$$

holds, then (1) *is a difference Painlevé II equation*

$$y(z+1)+y(z-1)=\frac{(\lambda z+\mu)y+\upsilon}{1-y^2}, \tag{4}$$

where λ, μ, and υ are constants.

Remark 1. If w has a pole at $z=z_0$, we say the singularity at z_0 is of type I if $w(z_0\pm 1)=\pm\varepsilon(\varepsilon=\pm 1)$ and of type II if $w(z_0\pm 1)=\mp\varepsilon$. We denote by $\overline{n}_I(r,w)$ the number of type I poles (ignoring multiplicities) in the disc $\{z : |z| < r\}$. Similarly, the function $\overline{n}_{II}(r, w)$ counts poles of type II.

In 2010, Chen and Shon [9] researched the properties of finite-order meromorphic solutions of difference Painlevé I and II equations. They mainly discussed the existence and the forms of rational solutions and value distribution of transcendental meromorphic solutions.

For difference Painlevé III equations, we recall the following.

Theorem B (see [10]). *Assume that equation*

$$w(z+1)w(z-1) = R(z,w) \tag{5}$$

has an admissible meromorphic solution w of hyperorder less than one, where $R(z,w)$ is rational and irreducible in w and meromorphic in z, then either w satisfies a difference Riccati equation

$$w(z+1) = \frac{\alpha(z)w(z) + \beta(z)}{w(z) + \gamma(z)}, \tag{6}$$

where $\alpha(z)$, $\beta(z)$, $\gamma(z) \in \mathcal{S}(w)$ are algebroid functions or (5) *can be transformed to one of the following equations:*

$$w(z+1)w(z-1) = \frac{\eta(z)w^2(z) - \lambda(z)w(z) + \mu(z)}{(w(z)-1)(w(z)-\nu(z))}, \tag{7a}$$

$$w(z+1)w(z-1) = \frac{\eta(z)w^2(z) - \lambda(z)w(z)}{w(z)-1}, \tag{7b}$$

$$w(z+1)w(z-1) = \frac{\eta(z)(w(z) - \lambda(z))}{w(z)-1}, \tag{7c}$$

$$w(z+1)w(z-1) = h(z)w^m(z). \tag{7d}$$

In (7a), *the coefficients satisfy $\kappa^2(z)\mu(z+1)\mu(z-1) = \mu^2(z)$, $\lambda(z+1)\mu(z) = \kappa(z)\lambda(z-1)\mu(z+1)$, $\kappa(z)\lambda(z+2)\lambda(z-1) = \kappa(z-1)\lambda(z)\lambda(z+1)$, and one of the following:*

(1) *$\eta \equiv 1$, $\nu(z+1)\nu(z-1) = 1$, $\kappa(z) = \nu(z)$;*

(2) *$\eta(z+1) = \eta(z-1) = \nu(z)$, $\kappa \equiv 1$.*

In (7b), *$\eta(z)\eta(z+1) = 1$ and $\lambda(z+2)\lambda(z-1) = \lambda(z)\lambda(z+1)$. In* (7c), *the coefficients satisfy one of the following:*

(1) *$\eta \equiv 1$ and either $\lambda(z) = \lambda(z+1)\lambda(z-1)$ or $\lambda(z+3)\lambda(z-3) = \lambda(z+2)\lambda(z-2)$;*

(2) *$\lambda(z+1)\lambda(z-1) = \lambda(z+2)\lambda(z-2)$, $\eta(z+1)\lambda(z+1) = \lambda(z+2)\eta(z-1)$, and $\eta(z)\eta(z-1) = \eta(z+2)\eta(z-3)$;*

(3) *$\eta(z+2)\eta(z-2) = \eta(z)\eta(z-1)$, $\lambda(z) = \eta(z-1)$;*

(4) *$\lambda(z+3)\lambda(z-3) = \lambda(z+2)\lambda(z-2)\lambda(z)$, $\eta(z)\lambda(z) = \eta(z+2)\eta(z-2)$.*

In (7d), *$h(z) \in \mathcal{S}(w)$ and $m \in \mathbb{Z}$, $|m| \le 2$.*

Zhang and Yang [11] investigated the difference Painlevé III equations (7a)–(7d) with constant coefficients and obtained the following results.

Theorem C. *If w is a nonconstant meromorphic solution of difference equation* (7d), *where $m = -2, -1, 0, 1$ and h is a nonzero constant. Then*

(i) *w cannot be a rational function;*

(ii) *$\lambda(w) = \tau(w) = \sigma(w)$.*

Lan and Chen [12] studied some difference Painlevé III equations and proved the following.

Theorem D. *Suppose that $h(z)$ is a nonconstant rational function. Suppose $w(z)$ is a transcendental meromorphic solution with finite order of* (7d), *where $m = -2, -1, 0, 1$. Set $\Delta w(z) = w(z+1) - w(z)$. Then*

(i) *$w(z)$ has no Nevanlinna exceptional value;*

(ii) *$\lambda(\Delta w) = \lambda(1/\Delta w) = \sigma(w)$, $\lambda(\Delta w/w) = \lambda(1/(\Delta w/w)) = \sigma(w)$.*

In general, $\tau(w(z+c)) \ne \tau(w(z))$, where c is a nonzero constant. For example, $w(z) = e^z + z$, $w(z)$ has no fixed points, but $w(z+1) = ee^z + z + 1$ has infinitely many fixed points and $\tau(w(z+1)) = \sigma(w) = 1$. Combining Theorems C and D, we continue to study properties (including fixed points) of transcendental meromorphic solutions of difference Painlevé III equations (7b) and (7c), and obtain the following.

Theorem 2. *Suppose that $\eta(z)$ and $\lambda(z)$ are nonconstant polynomials. Suppose $w(z)$ is a transcendental meromorphic solution with finite order of difference Painlevé III equation*

$$w(z+1)w(z-1)(w(z)-1) = \eta(z)w^2(z) - \lambda(z)w(z). \tag{8}$$

Then

(i) *$w(z)$ has at most one Nevanlinna exceptional value;*

(ii) *for any $c \in \mathbb{C}$, $w(z+c)$ has infinitely many fixed points, and $\tau(w(z+c)) = \sigma(w)$;*

(iii) *$\lambda(\Delta w) = \lambda(1/\Delta w) = \sigma(w)$ and $\lambda(1/(\Delta w/w)) = \sigma(w)$;*

(iv) *if there exists some nonconstant rational function $b(z)$ such that $\lambda(z) = b(z)(\eta(z) - b(z) + 1)$, then $\lambda(\Delta w/w) = \sigma(w)$.*

Theorem 3. *Suppose that $\eta(z)$ is a nonconstant polynomial. Suppose $w(z)$ is a transcendental meromorphic solution with finite order of difference Painlevé III equation*

$$w(z+1)w(z-1)(w(z)-1) = \eta(z)w(z). \tag{9}$$

Then

(i) *$w(z)$ has no Nevanlinna exceptional value;*

(ii) *for any $c \in \mathbb{C}$, $w(z+c)$ has infinitely many fixed points, and $\tau(w(z+c)) = \sigma(w)$;*

(iii) *$\lambda(\Delta w) = \lambda(1/\Delta w) = \sigma(w)$ and $\lambda(1/(\Delta w/w)) = \sigma(w)$;*

(iv) *if there exists some rational function $b(z)$ such that $\eta(z) = b(z)(b(z) - 1)$, then $\lambda(\Delta w/w) = \sigma(w)$.*

2. Lemmas for the Proofs of Theorems

Lemma 4 (see [13]). *Let $f(z)$ be a meromorphic function of finite order σ and let η be a nonzero complex number. Then for each $\varepsilon > 0$, one has*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O\left(r^{\sigma-1+\varepsilon}\right). \quad (10)$$

Lemma 5 (see [13]). *Let $f(z)$ be a meromorphic function with order $\sigma = \sigma(f)$, $\sigma < \infty$, and let η be a fixed nonzero complex number, then for each $\varepsilon > 0$, one has*

$$T(r, f(z+\eta)) = T(r, f(z)) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r). \quad (11)$$

Lemmas 4 and 5 show the following.

Lemma 6. *Let c be a nonzero constant and $f(z)$ be a meromorphic function with finite order σ. Then for each $\varepsilon > 0$, one has*

$$N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f(z)}\right) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r). \quad (12)$$

Lemma 7 (see [14, 15]). *Let w be a transcendental meromorphic solution of finite order of difference equation*

$$P(z, w) = 0, \quad (13)$$

where $P(z,w)$ is a difference polynomial in $w(z)$. If $P(z,a) \not\equiv 0$ for a meromorphic function $a \in S(w)$, then

$$m\left(r, \frac{1}{w-a}\right) = S(r, w). \quad (14)$$

Lemma 8 (see [15]). *Let $f(z)$ be a transcendental meromorphic solution of finite order σ of difference equation of the form*

$$U(z, f) P(z, f) = Q(z, f), \quad (15)$$

where $U(z,f)$, $P(z,f)$, and $Q(z,f)$ are difference polynomials such that the total degree $\deg_f U(z,f) = n$ in $f(z)$ and its shifts, and $\deg_f Q(z,f) \le n$. If $U(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts, then for each $\varepsilon > 0$,

$$m(r, P(z, f)) = O\left(r^{\sigma-1+\varepsilon}\right) + S(r, f) \quad (16)$$

possibly outside of an exceptional set of finite logarithmic measure.

3. Proofs of Theorems

Proof of Theorem 2. (i) By (8), we have

$$\begin{aligned} & w(z+1)\, w(z-1)\, w(z) \\ & \quad = w(z+1)\, w(z-1) + \eta(z)\, w^2(z) - \lambda(z)\, w(z). \end{aligned} \quad (17)$$

Applying Lemma 8 to (17), we have

$$m(r, w) = S(r, w), \quad (18)$$

which yields $N(r,w) = T(r,w) + S(r,w)$, that is, $\delta(\infty, w) = 0$.

By (17), we have

$$\frac{\lambda(z)}{w(z)} = \eta(z) + \frac{w(z+1)}{w(z)} \frac{w(z-1)}{w(z)} (1 - w(z)). \quad (19)$$

By (18), (19), and Lemma 4, we obtain

$$\begin{aligned} m\left(r, \frac{1}{w(z)}\right) & \le m\left(r, \frac{w(z+1)}{w(z)}\right) + m\left(r, \frac{w(z-1)}{w(z)}\right) \\ & \quad + m(r, w(z)) + m(r, \eta(z)) + m\left(r, \frac{1}{\lambda(z)}\right) \\ & = S(r, w(z)). \end{aligned} \quad (20)$$

Thus,

$$N\left(r, \frac{1}{w(z)}\right) = T(r, w(z)) + S(r, w(z)), \quad (21)$$

which means that $\delta(0, w) = 0$.

Set

$$\begin{aligned} P(z, w) := {} & w(z+1)\, w(z-1)\, (w(z) - 1) \\ & - \eta(z)\, w^2(z) + \lambda(z)\, w(z) = 0. \end{aligned} \quad (22)$$

Assume that $w(z)$ has two Nevanlinna exceptional values $a, c (a \ne c)$. By $\delta(\infty, w) = 0$ and $\delta(0, w) = 0$, we see that $ac \ne 0, \infty$. By Lemma 7, we have $P(z,a) \equiv 0$ and $P(z,c) \equiv 0$. That is,

$$\begin{aligned} & a^2(a-1) - a^2\eta(z) + a\lambda(z) = 0, \\ & c^2(c-1) - c^2\eta(z) + c\lambda(z) = 0. \end{aligned} \quad (23)$$

Hence,

$$(a-c)\,\eta(z) = (a-c)(a+c-1). \quad (24)$$

Since $a \ne c$, then $\eta(z) = a + c - 1$ is a constant. This contradicts the fact that $\eta(z)$ is a nonconstant polynomial. So, $w(z)$ has at most one Nevanlinna exceptional value.

(ii) For any $c \in \mathbb{C}$, substituting $z + c$ for z in (8), we obtain

$$\begin{aligned} & w(z+c+1)\, w(z+c-1)\, (w(z+c) - 1) \\ & \quad = \eta(z+c)\, w^2(z+c) - \lambda(z+c)\, w(z+c). \end{aligned} \quad (25)$$

Set $g(z) = w(z+c)$. Thus, (25) can be written as

$$\begin{aligned} & g(z+1)\, g(z-1)\, (g(z) - 1) \\ & \quad = \eta(z+c)\, g(z)^2 - \lambda(z+c)\, g(z). \end{aligned} \quad (26)$$

Set

$$\begin{aligned} P_1(z, g) := {} & g(z+1)\, g(z-1)\, (g(z) - 1) \\ & - \eta(z+c)\, g(z)^2 + \lambda(z+c)\, g(z) = 0. \end{aligned} \quad (27)$$

Since $\eta(z)$ and $\lambda(z)$ are polynomials,

$$P_1(z,z) = z\left(\frac{(z+1)(z-1)^2}{z} - z\eta(z+c) + \lambda(z+c)\right) \not\equiv 0. \quad (28)$$

By $P_1(z,z) \not\equiv 0$ and Lemma 7, we have $m(r, 1/(g(z)-z)) = S(r,g)$, which follows $N(r,1/(g(z)-z)) = T(r,g)+S(r,g)$. By $g(z) = w(z+c)$ and Lemma 5, we have $N(r,1/(w(z+c)-z)) = T(r,w(z)) + S(r,w(z))$. Thus, $\tau(w(z+c)) = \sigma(w(z))$.

(iii) By (8), we have

$$\frac{w(z+1)}{w(z)}\frac{w(z-1)}{w(z)} = \frac{\eta(z)w(z) - \lambda(z)}{w(z)(w(z)-1)}. \quad (29)$$

Applying Valiron-Mohon'ko theorem and Lemma 5 to (29), we obtain

$$\begin{aligned} 2T(r,w(z)) &= T\left(r, \frac{\eta(z)w(z)-\lambda(z)}{w(z)(w(z)-1)}\right) + S(r,w(z)) \\ &= T\left(r, \frac{w(z+1)}{w(z)}\frac{w(z-1)}{w(z)}\right) + S(r,w(z)) \\ &\le T\left(r, \frac{w(z+1)}{w(z)}\right) + T\left(r, \frac{w(z)}{w(z-1)}\right) + S(r,w(z)) \\ &= 2T\left(r, \frac{w(z+1)}{w(z)}\right) + S\left(r, \frac{w(z+1)}{w(z)}\right) + S(r,w(z)) \\ &\le 2T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r,w(z)). \end{aligned} \quad (30)$$

Thus,

$$T(r,w(z)) \le T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r,w(z)). \quad (31)$$

By (31) and Lemma 4, we have

$$\begin{aligned} N\left(r, \frac{\Delta w}{w}\right) &= T\left(r, \frac{\Delta w}{w}\right) - m\left(r, \frac{\Delta w}{w}\right) \\ &= T\left(r, \frac{\Delta w}{w}\right) + S(r,w) \\ &\ge T(r,w) + S(r,w). \end{aligned} \quad (32)$$

Therefore, $\lambda(1/(\Delta w/w)) \ge \sigma(w)$, that is, $\lambda(1/(\Delta w/w)) = \sigma(w)$.

Substituting $w(z+1) = w(z) + \Delta w(z)$, $w(z-1) = w(z) - \Delta w(z-1)$ into (8), we see

$$(w(z) + \Delta w(z))(w(z) - \Delta w(z-1)) = \frac{\eta(z)w^2(z) - \lambda(z)w(z)}{w(z)-1}, \quad (33)$$

that is,

$$(\Delta w(z) - \Delta w(z-1))w(z) - \Delta w(z)\Delta w(z-1) = \frac{-w^3(z) + (\eta(z)+1)w^2(z) - \lambda(z)w(z)}{w(z)-1}. \quad (34)$$

Let z_0 be a zero of w, by (33), z_0 is a zero of $w(z) + \Delta w(z)$ or $w(z) - \Delta w(z-1)$. Since $w(z_0) = 0$, then z_0 must be a zero of $\Delta w(z)$ or $\Delta w(z-1)$. Thus, by (21) and Lemma 6, we obtain

$$\begin{aligned} T(r,w(z)) &= N\left(r, \frac{1}{w(z)}\right) + S(r,w(z)) \\ &\le N\left(r, \frac{1}{\Delta w(z)}\right) + N\left(r, \frac{1}{\Delta w(z-1)}\right) + S(r,w(z)) \\ &= 2N\left(r, \frac{1}{\Delta w(z)}\right) + S(r,\Delta w(z)) + S(r,w(z)) \\ &\le 2N\left(r, \frac{1}{\Delta w(z)}\right) + S(r,w(z)). \end{aligned} \quad (35)$$

Hence, $\sigma(w) \le \lambda(\Delta w)$, that is, $\lambda(\Delta w) = \sigma(w)$.

Applying Valiron-Mohon'ko Theorem and Lemma 5 to (34), we deduce

$$\begin{aligned} 3T(r,w(z)) &= T\left(r, \frac{-w^3(z) + (\eta(z)+1)w^2(z) - \lambda(z)w(z)}{w(z)-1}\right) \\ &\quad + S(r,w(z)) \\ &= T(r, (\Delta w(z) - \Delta w(z-1))w(z) - \Delta w(z)\Delta w(z-1)) \\ &\quad + S(r,w(z)) \\ &\le T(r,w(z)) + 2T(r,\Delta w(z)) + 2T(r,\Delta w(z-1)) \\ &\quad + S(r,w(z)) \\ &= T(r,w(z)) + 4T(r,\Delta w(z)) \\ &\quad + S(r,\Delta w(z)) + S(r,w(z)) \\ &\le T(r,w(z)) + 4T(r,\Delta w(z)) + S(r,w(z)). \end{aligned} \quad (36)$$

Thus,

$$\frac{1}{2}T(r,w(z)) \le T(r,\Delta w(z)) + S(r,w(z)). \quad (37)$$

By (18), (37), and Lemma 4, we have

$$\begin{aligned} N(r,\Delta w) &= T(r,\Delta w) - m(r,\Delta w) \\ &\ge T(r,\Delta w) - m\left(r, \frac{\Delta w}{w}\right) - m(r,w) \\ &= T(r,\Delta w) + S(r,w) \\ &\ge \frac{1}{2}T(r,w) + S(r,w). \end{aligned} \quad (38)$$

Hence, $\lambda(1/\Delta w) \ge \sigma(w)$, that is, $\lambda(1/\Delta w) = \sigma(w)$.

(iv) Suppose that $\lambda(z) = b(z)(\eta(z) - b(z) + 1)$, where $b(z)$ is some nonconstant rational function. Now we prove that $\lambda(\Delta w/w) = \sigma(w)$. Set

$$G(z) = (\Delta w(z) - \Delta w(z-1))\, w(z) - \Delta w(z)\, \Delta w(z-1). \tag{39}$$

By (34), (39), and $\lambda(z) = b(z)(\eta(z) - b(z) + 1)$, we have

$$\begin{aligned} G(z) &= \frac{-w^3(z) + (\eta(z)+1)\, w^2(z) - \lambda(z)\, w(z)}{w(z) - 1} \\ &= \frac{-w(z)(w(z) - b(z))(w(z) - \eta(z) + b(z) - 1)}{w(z) - 1}. \end{aligned} \tag{40}$$

Since $b(z)$ is a nonconstant rational function, then $b(z+1)/b(z) \not\equiv b(z)/b(z-1)$. By (22) and $\lambda(z) = b(z)(\eta(z) - b(z) + 1)$, we know

$$\begin{aligned} &P(z, b(z)) \\ &\quad = b(z+1)\, b(z-1)(b(z) - 1) \\ &\qquad - b^2(z)\,\eta(z) + b(z)\,\lambda(z) \\ &\quad = b(z+1)\, b(z-1)(b(z) - 1) - b^2(z)\,\eta(z) \\ &\qquad + b^2(z)(\eta(z) - b(z) + 1) \\ &\quad = b^2(z)(b(z) - 1)\left(\frac{b(z+1)}{b(z)}\frac{b(z-1)}{b(z)} - 1\right) \not\equiv 0. \end{aligned} \tag{41}$$

Similarly, we obtain $P(z, \eta(z) - b(z) + 1) \not\equiv 0$. By $P(z, b(z)) \not\equiv 0$, $P(z, \eta(z) - b(z) + 1) \not\equiv 0$ and Lemma 7, we have

$$\begin{aligned} m\left(r, \frac{1}{w - b(z)}\right) &= S(r, w), \\ m\left(r, \frac{1}{w - \eta(z) + b(z) - 1}\right) &= S(r, w). \end{aligned} \tag{42}$$

By (18), (20), (40) and (42), we obtain

$$\begin{aligned} m\left(r, \frac{1}{G}\right) &\le m\left(r, \frac{1}{w - b}\right) + m\left(r, \frac{1}{w - \eta + b - 1}\right) \\ &\quad + m(r, w) + m\left(r, \frac{1}{w}\right) \\ &= S(r, w). \end{aligned} \tag{43}$$

By (39), we obtain

$$\frac{G(z)}{\Delta w(z)\,\Delta w(z-1)} = \frac{1}{\Delta w(z)}\left(\frac{\Delta w(z)}{\Delta w(z-1)} - 1\right) w(z) - 1. \tag{44}$$

It sees from Lemma 4 that

$$\begin{aligned} &m\left(r, \frac{1}{\Delta w(z)\,\Delta w(z-1)}\right) \\ &\quad \le m\left(r, \frac{1}{(\Delta w(z))^2}\right) + m\left(r, \frac{\Delta w(z)}{\Delta w(z-1)}\right) \\ &\quad = 2m\left(r, \frac{1}{\Delta w(z)}\right) + S(r, \Delta w(z)), \\ &2m\left(r, \frac{1}{\Delta w(z)}\right) \\ &\quad = m\left(r, \frac{1}{(\Delta w(z))^2}\right) \\ &\quad \le m\left(r, \frac{1}{\Delta w(z)\,\Delta w(z-1)}\right) + m\left(r, \frac{\Delta w(z-1)}{\Delta w(z)}\right) \\ &\quad = m\left(r, \frac{1}{\Delta w(z)\,\Delta w(z-1)}\right) + S(r, \Delta w(z)). \end{aligned} \tag{45}$$

Hence,

$$\begin{aligned} m\left(r, \frac{1}{\Delta w(z)\,\Delta w(z-1)}\right) \\ = 2m\left(r, \frac{1}{\Delta w(z)}\right) + S(r, \Delta w(z)). \end{aligned} \tag{46}$$

From (18), (43), (44), (46) and Lemma 4, we deduce that

$$\begin{aligned} &2m\left(r, \frac{1}{\Delta w(z)}\right) \\ &\quad = m\left(r, \frac{1}{\Delta w(z)\,\Delta w(z-1)}\right) + S(r, \Delta w(z)) \\ &\quad \le m\left(r, \frac{1}{G(z)}\right) + m\left(r, \frac{1}{\Delta w(z)}\right) + m\left(r, \frac{\Delta w(z)}{\Delta w(z-1)}\right) \\ &\qquad + m(r, w(z)) + S(r, \Delta w(z)) \\ &\quad \le m\left(r, \frac{1}{\Delta w(z)}\right) + S(r, \Delta w(z)) + S(r, w(z)) \\ &\quad \le m\left(r, \frac{1}{\Delta w(z)}\right) + S(r, w(z)), \end{aligned} \tag{47}$$

which yields

$$m\left(r, \frac{1}{\Delta w(z)}\right) = S(r, w(z)). \tag{48}$$

By (18) and (48) we have

$$\begin{aligned} m\left(r, \frac{1}{\Delta w/w}\right) = m\left(r, \frac{w}{\Delta w}\right) \le m(r, w) \\ + m\left(r, \frac{1}{\Delta w}\right) = S(r, w), \end{aligned} \tag{49}$$

and by (31),

$$N\left(r,\frac{1}{\Delta w/w}\right)=T\left(r,\frac{\Delta w}{w}\right)+S(r,w)\geq T(r,w)+S(r,w). \tag{50}$$

Then $\lambda(\Delta w/w)\geq\sigma(w)$. So, $\lambda(\Delta w/w)=\sigma(w)$. □

Proof of Theorem 3. (i) By (9), we have

$$\begin{aligned}&w(z+1)w(z-1)w(z)\\&\quad=w(z+1)w(z-1)+\eta(z)w(z).\end{aligned} \tag{51}$$

Applying Lemma 8 to (51), we have

$$m(r,w)=S(r,w). \tag{52}$$

Thus, $N(r,w)=T(r,w)+S(r,w)$, which yields $\delta(\infty,w)=0$.

Again by (9), we have

$$\frac{\eta(z)}{w(z)}=\frac{w(z+1)}{w(z)}\frac{w(z-1)}{w(z)}(w(z)-1). \tag{53}$$

From (52), (53), and Lemma 4, we deduce that

$$\begin{aligned}m\left(r,\frac{1}{w}\right)&\leq m\left(r,\frac{w(z+1)}{w(z)}\right)+m\left(r,\frac{w(z-1)}{w(z)}\right)\\&\quad+m(r,w)+m\left(r,\frac{1}{\eta}\right)\\&=S(r,w),\end{aligned} \tag{54}$$

which follows

$$N\left(r,\frac{1}{w(z)}\right)=T(r,w(z))+S(r,w). \tag{55}$$

Thus, $\delta(0,w)=0$.

Set

$$P(z,w):=w(z+1)w(z-1)(w(z)-1)-\eta(z)w(z). \tag{56}$$

For any $a\in\mathbb{C}\setminus\{0\}$, since $\eta(z)$ is a nonconstant polynomial, then $P(z,a)=a^2(a-1)-a\eta(z)\not\equiv 0$. By $P(z,a)\not\equiv 0$ and Lemma 7, we know that $m(r,1/(w-a))=S(r,w)$, which means that $N(r,1/(w-a))=T(r,w)+S(r,w)$. Hence, $\delta(a,w)=0$. Combining $\delta(\infty,w)=0,\delta(0,w)=0$, we see that w has no Nevanlinna exceptional value.

(ii) For any $c\in\mathbb{C}$, substituting $z+c$ for z in (9), we see that

$$\begin{aligned}&w(z+c+1)w(z+c-1)(w(z+c)-1)\\&\quad=\eta(z+c)w(z+c).\end{aligned} \tag{57}$$

Set $g(z)=w(z+c)$. Thus, (57) can be written as

$$g(z+1)g(z-1)(g(z)-1)=\eta(z+c)g(z). \tag{58}$$

Set

$$\begin{aligned}P_1(z,g):=&\ g(z+1)g(z-1)(g(z)-1)\\&-\eta(z+c)g(z)=0.\end{aligned} \tag{59}$$

Since $\eta(z)$ is a polynomial, then

$$P_1(z,z)=z\left(\frac{(z+1)(z-1)^2}{z}-\eta(z+c)\right)\not\equiv 0. \tag{60}$$

By this and Lemma 7, we have $m(r,1/(g(z)-z))=S(r,g)$, which follows $N(r,1/(g(z)-z))=T(r,g(z))+S(r,g(z))$. By $g(z)=w(z+c)$ and Lemma 5, we see that $N(r,1/(w(z+c)-z))=T(r,w(z))+S(r,w(z))$. Thus, $\tau(w(z+c))=\sigma(w(z))$.

(iii) Substituting $w(z+1)=w(z)+\Delta w(z),w(z-1)=w(z)-\Delta w(z-1)$ into (9), we have

$$(w(z)+\Delta w(z))(w(z)-\Delta w(z-1))=\frac{\eta(z)w(z)}{w(z)-1}. \tag{61}$$

If z_0 is a zero of $w(z)$, by (61), z_0 must be a zero of $\Delta w(z)$ or $\Delta w(z-1)$. Thus, by (55) and Lemma 6, we have

$$\begin{aligned}&T(r,w(z))\\&\quad=N\left(r,\frac{1}{w(z)}\right)+S(r,w(z))\\&\quad\leq N\left(r,\frac{1}{\Delta w(z)}\right)+N\left(r,\frac{1}{\Delta w(z-1)}\right)+S(r,w(z))\\&\quad\leq 2N\left(r,\frac{1}{\Delta w(z)}\right)+S(r,w(z)).\end{aligned} \tag{62}$$

Hence, $\sigma(w)\leq\lambda(\Delta w)$, that is, $\lambda(\Delta w)=\sigma(w)$.

By (61), we have

$$\begin{aligned}&(\Delta w(z)-\Delta w(z-1))w(z)-\Delta w(z)\Delta w(z-1)\\&\quad=\frac{-w^3(z)+w^2(z)+\eta(z)w(z)}{w(z)-1}.\end{aligned} \tag{63}$$

Applying Valiron-Mohon'ko theorem and Lemma 5 to (63), we deduce

$$\begin{aligned}&3T(r,w(z))\\&\quad=T\left(r,\frac{-w^3(z)+w^2(z)+\eta(z)w(z)}{w(z)-1}\right)+S(r,w(z))\\&\quad=T(r,(\Delta w(z)-\Delta w(z-1))w(z)-\Delta w(z)\Delta w(z-1))\\&\qquad+S(r,w(z))\\&\quad\leq T(r,w(z))+2T(r,\Delta w(z))\\&\qquad+2T(r,\Delta w(z-1))+S(r,w(z))\\&\quad=T(r,w(z))+4T(r,\Delta w(z))\\&\qquad+S(r,\Delta w(z))+S(r,w(z))\\&\quad\leq T(r,w(z))+4T(r,\Delta w(z))+S(r,w(z)).\end{aligned} \tag{64}$$

Hence,

$$\frac{1}{2}T(r,w(z)) \le T(r,\Delta w(z)) + S(r,w(z)). \quad (65)$$

Combining (65) with (52) and Lemma 4, we have

$$\begin{aligned} N(r,\Delta w) &= T(r,\Delta w) - m(r,\Delta w) \\ &\ge T(r,\Delta w) - m\left(r,\frac{\Delta w}{w}\right) - m(r,w) \\ &= T(r,\Delta w) + S(r,w) \\ &\ge \frac{1}{2}T(r,w) + S(r,w), \end{aligned} \quad (66)$$

which yields $\lambda(1/\Delta w) \ge \sigma(w)$. So, $\lambda(1/\Delta w) = \sigma(w)$.

By (9), we have

$$\frac{w(z+1)}{w(z)}\frac{w(z-1)}{w(z)} = \frac{\eta(z)}{w(z)(w(z)-1)}. \quad (67)$$

Applying Valiron-Mohon'ko theorem and Lemma 5 to (67), we obtain

$$\begin{aligned} &2T(r,w(z)) \\ &= T\left(r,\frac{\eta(z)}{w(z)(w(z)-1)}\right) + S(r,w(z)) \\ &= T\left(r,\frac{w(z+1)}{w(z)}\frac{w(z-1)}{w(z)}\right) + S(r,w(z)) \\ &\le T\left(r,\frac{w(z+1)}{w(z)}\right) + T\left(r,\frac{w(z)}{w(z-1)}\right) + S(r,w(z)) \\ &= 2T\left(r,\frac{w(z+1)}{w(z)}\right) + S\left(r,\frac{w(z+1)}{w(z)}\right) + S(r,w(z)) \\ &\le 2T\left(r,\frac{\Delta w(z)}{w(z)}\right) + S(r,w(z)). \end{aligned} \quad (68)$$

Thus,

$$T(r,w(z)) \le T\left(r,\frac{\Delta w(z)}{w(z)}\right) + S(r,w(z)). \quad (69)$$

By (69) and Lemma 4, we see that

$$\begin{aligned} N\left(r,\frac{\Delta w}{w}\right) &= T\left(r,\frac{\Delta w}{w}\right) - m\left(r,\frac{\Delta w}{w}\right) \\ &= T\left(r,\frac{\Delta w}{w}\right) + S(r,w) \\ &\ge T(r,w) + S(r,w). \end{aligned} \quad (70)$$

Hence, $\lambda(1/(\Delta w/w)) \ge \sigma(w)$, that is, $\lambda(1/(\Delta w/w)) = \sigma(w)$.

(iv) Suppose that $\eta(z) = b(z)(b(z)-1)$, where $b(z)$ is some rational function. Now we prove that $\lambda(\Delta w/w) = \sigma(w)$. Set

$$G(z) = (\Delta w(z) - \Delta w(z-1))w(z) - \Delta w(z)\Delta w(z-1). \quad (71)$$

By (63), (71), and $\eta(z) = b(z)(b(z)-1)$, we have

$$\begin{aligned} G(z) &= \frac{-w(z)\left(w^2(z) - w(z) - \eta(z)\right)}{w(z)-1} \\ &= \frac{-w(z)(w(z)-b(z))(w(z)+b(z)-1)}{w(z)-1}. \end{aligned} \quad (72)$$

Since $\eta(z) = b(z)(b(z)-1)$ is a nonconstant polynomial, $b(z)$ is a nonconstant rational function. Then $b(z+1)/b(z) \not\equiv b(z)/b(z-1)$. By (56) and $\eta(z) = b(z)(b(z)-1)$, we know

$$\begin{aligned} &P(z,b(z)) \\ &= b^2(z)(b(z)-1)\left(\frac{b(z+1)}{b(z)}\frac{b(z-1)}{b(z)} - 1\right) \not\equiv 0. \end{aligned} \quad (73)$$

Similarly, we obtain $P(z,-b(z)+1) \not\equiv 0$. By $P(z,b(z)) \not\equiv 0$, $P(z,-b(z)+1) \not\equiv 0$, and Lemma 7, we have

$$\begin{aligned} m\left(r,\frac{1}{w(z)-b(z)}\right) &= S(r,w(z)), \\ m\left(r,\frac{1}{w(z)+b(z)-1}\right) &= S(r,w(z)). \end{aligned} \quad (74)$$

'By (52), (54), (74), and (72), we obtain

$$\begin{aligned} m\left(r,\frac{1}{G}\right) &\le m(r,w) + m\left(r,\frac{1}{w}\right) + m\left(r,\frac{1}{w-b}\right) \\ &\quad + m\left(r,\frac{1}{w+b-1}\right) = S(r,w). \end{aligned} \quad (75)$$

Using the same method as in the proof of (iv) in Theorem 2, we may obtain $m(r,1/\Delta w) = S(r,w)$. By this and (52), we have

$$\begin{aligned} m\left(r,\frac{1}{\Delta w/w}\right) &= m\left(r,\frac{w}{\Delta w}\right) \le m(r,w) + m\left(r,\frac{1}{\Delta w}\right) \\ &= S(r,w), \end{aligned} \quad (76)$$

and by (69),

$$N\left(r,\frac{1}{\Delta w/w}\right) = T\left(r,\frac{\Delta w}{w}\right) + S(r,w) \ge T(r,w) + S(r,w). \quad (77)$$

Then $\lambda(\Delta w/w) \ge \sigma(w)$. So, $\lambda(\Delta w/w) = \sigma(w)$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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