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Research Article

Strong Inequalities for Hermite-Fejér Interpolations and Characterization of K-Functionals

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The works of Smale and Zhou (2003, 2007), Cucker and Smale (2002), and Cucker and Zhou (2007) indicate that approximation operators serve as cores of many machine learning algorithms. In this paper we study the Hermite-Fejér interpolation operator which has this potential of applications. The interpolation is defined by zeros of the Jacobi polynomials with parameters $-1 < \alpha$, $\beta < 0$. Approximation rate is obtained for continuous functions. Asymptotic expression of the K-functional associated with the interpolation operators is given.

1. Introduction

Zhou and Jetter [1] used Bernstein-Durrmeyer operators for studying support vector machine classification algorithms. This work initiates the direction of applying more linear operators from approximation theory to learning theory. We will follow this direction and study Hermite-Fejér interpolation operator. It would be interesting to derive explicit learning rates by means of these operators for some specific learning algorithms.

Let $P_n^{(\alpha,\beta)}$ denote the Jacobi polynomial of order n. Let $\{x_k\}_{k=1}^n$ be the zeros of $P_n^{(\alpha,\beta)}$. We assume that $\alpha, \beta > -1$. For any continuous function f on $[-1,1]$, the Hermite-Fejér interpolation $H_n^{(\alpha,\beta)}(f,\cdot)$ is a polynomial of order $2n-1$ that satisfies

$$H_n^{(\alpha,\beta)}(f,x_k) = f(x_k), \qquad H_n^{(\alpha,\beta)'}(f,x_k) = 0, \tag{1}$$

for any $k = 1, 2, \ldots, n$. Let $\|\cdot\|$ be the norm on $C[-1,1]$ ($\forall f \in C[-1,1]$, $\|f\| := \max\{|f(x)|, -1 \le x \le 1\}$). Without introducing ambiguity we also use $\|\cdot\|$ to denote the norm on $C_{2\pi}$ (which is the totality of the continuous functions on $\mathbb{R}$ with period 2π, and in this case $\|f\| := \max\{|f(x)|, x \in \mathbb{R}\}$).

One has (see e.g., [2])

$$\lim_{n\to\infty} \left\| H_n^{(\alpha,\beta)}(f) - f \right\| = 0, \quad \forall f \in C[-1,1], \tag{2}$$

if and only if $\alpha, \beta < 0$.

Define

$$w^{(\alpha,\beta)}(t) = \left|\sin\frac{t}{2}\right|^{2\alpha+1}\left|\cos\frac{t}{2}\right|^{2\beta+1}. \tag{3}$$

Denote by $\widehat{F}$ the conjugate function of F, and write $g(t) = f(\cos t)$. When $-1 < \alpha, \beta < 0$, one has (see [3, 4])

$$\begin{gathered} \left\| H_n^{(\alpha,\beta)}(f) - f \right\| = \mathcal{O}\left(\frac{1}{n}\right) \Longleftrightarrow g \in \text{Lip}1, \\ \frac{\widehat{g' w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}} = \mathcal{O}(1), \\ \left\| H_n^{(\alpha,\beta)}(f) - f \right\| = o\left(\frac{1}{n}\right) \Longleftrightarrow f \text{ is a constant.} \end{gathered} \tag{4}$$

So when $-1 < \alpha, \beta < 0$, $H_n^{(\alpha,\beta)}$ is saturation with $\{1/n\}$ and the saturation class is

$$\begin{aligned} S(\alpha,\beta) = \Bigg\{ & f \in C[-1,1] \mid g(t) = f(\cos t), \\ & g \in \text{Lip}1, \frac{\widehat{g' w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}} = \mathcal{O}(1) \Bigg\}. \end{aligned} \tag{5}$$

Note that for all $\alpha, \beta \in (-1, -1/2)$, the associated classes $S(\alpha,\beta)$ are identical ([4, Theorem 6]). See [2, 5–7] for related works.

Denote that $\Delta_t h(x) = h(x+t/2) - h(x-t/2)$ and recursively $\Delta_t^k h(x) = \Delta_t \Delta_t^{k-1} h(x)$ with $\Delta_t^1 = \Delta_t$. Write $\varphi(x) = \sqrt{1-x^2}$. Define

$$\omega_\varphi(f,\delta) = \max_{0<t\le\delta} \left\|\Delta_{\varphi t} f\right\|. \tag{6}$$

For any $f \in C[-1,1]$, one has [8]

$$\inf_{h\in C^2[-1,1]} \left(\|f-h\| + \delta\left\|\varphi h'\right\|\right) \asymp \omega_\varphi(f,\delta). \tag{7}$$

Let $f \in C[-1,1]$ and $h(\theta) = H(\cos\theta)$. We use the following definition of K-functional from [9]:

$$K_{\alpha,\beta}(f,\delta) = \inf_{H\in C^2[-1,1]} \left(\|f-H\| + \delta\left\|h'\right\| + \delta\left\|\frac{\widehat{h' w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|\right). \tag{8}$$

We cite the following three Theorems from [9].

Theorem 1. *Let $-1 < \alpha, \beta < -1/2$ be fixed. Then there is a constant $1 < A < \infty$ such that for all $f \in C[-1,1]$ and all $n \ge 1$, one has*

$$\begin{aligned} \max_{k\ge n} \left\|H_k^{(\alpha,\beta)}(f) - f\right\| &\asymp \max_{n\le k\le An} \left\|H_k^{(\alpha,\beta)}(f) - f\right\| \\ &\asymp \frac{1}{n}\left\|\int_{1/n}^{1} \frac{\Delta_{t\varphi}^2 f}{t^2} dt\right\| + \omega_\varphi\left(f,\frac{1}{n}\right), \end{aligned} \tag{9}$$

in which the symbol $\asymp$ does not rely on n and f.

Theorem 2. *Let $-1 < \alpha, \beta < -1/2$ be fixed. Then the following relation holds:*

$$\begin{aligned} K_{\alpha,\beta}(f,\delta) \asymp \delta\left\|\int_{\delta}^{1} \frac{\Delta_{\varphi u}^2 f}{u^2} du\right\| + \omega_\varphi(f,\delta), \\ \forall f \in C[-1,1], \quad \delta > 0. \end{aligned} \tag{10}$$

Theorem 3. *Let $-1 < \alpha, \beta < 0$ be fixed. Then there is a constant $1 < A < \infty$ such that, for all $f \in C[-1,1]$ and all $n \ge 1$, one has*

$$\begin{aligned} \max_{k\ge n} \left\|H_k^{(\alpha,\beta)}(f) - f\right\| &\asymp \max_{n\le k\le An} \left\|H_k^{(\alpha,\beta)}(f) - f\right\| \\ &\asymp K_{\alpha,\beta}\left(f,\frac{1}{n}\right). \end{aligned} \tag{11}$$

Here the symbol $\asymp$ does not rely on n and f.

The K-functional $K_{\alpha,\beta}(f,\delta)$ for all $-1 < \alpha, \beta < 0$ is characterized by the following

Theorem 4. *Let $-1 < \alpha, \beta < 0$ and $-1 < \alpha_0, \beta_0 < -1/2$ be fixed. Then, for all $\delta > 0$ and all $f \in C[-1,1]$, the following holds:*

$$K_{\alpha,\beta}(f,\delta) \asymp K_{\alpha_0,\beta_0}(f,\delta) + J_{\alpha,\beta,1}(f,\delta) + J_{\alpha,\beta,2}(f,\delta), \tag{12}$$

in which

$$\begin{aligned} J_{\alpha,\beta,1}&(f,\delta) \\ &= \max_{0\le x\le 1-\delta^2} \frac{\delta}{(1-x)^{\alpha+1/2}} \\ &\quad\times\left|\int_{-1}^{x} \frac{f(t)-f(1)}{t-1}\right. \\ &\quad\left.\times\left(1-\alpha+\beta-(1+\alpha+\beta)t\right)(1-t)^\alpha(1+t)^\beta dt\right|, \end{aligned} \tag{13}$$

$$\begin{aligned} J_{\alpha,\beta,2}&(f,\delta) \\ &= \max_{-1+\delta^2\le x\le 0} \frac{\delta}{(1+x)^{\beta+1/2}} \\ &\quad\times\left|\int_{x}^{1} \frac{f(t)-f(-1)}{t+1}\left(1+\alpha-\beta+(1+\alpha+\beta)t\right)\right. \\ &\quad\left.\times(1-t)^\alpha(1+t)^\beta dt\right|. \end{aligned} \tag{14}$$

Theorems 3 and 4 give

$$\begin{aligned} K_{-1/2,-1/2}&(f,\delta) \\ &\asymp \delta\left\|\int_{\delta}^{1} \frac{\Delta_{t\varphi} f}{t^2} dt\right\| \\ &\quad+ \delta\max_{0\le x\le 1-\delta^2}\left|\int_{-1}^{x} \frac{f(t)-f(1)}{t-1}\left(1-t^2\right)^{-1/2} dt\right| \\ &\quad+ \delta\max_{-1+\delta^2\le x\le 0}\left|\int_{x}^{1} \frac{f(t)-f(-1)}{t+1}\left(1-t^2\right)^{-1/2} dt\right| \\ &\quad+ \omega_\varphi(f,\delta). \end{aligned} \tag{15}$$

Moreover, if $-1 < \alpha, \beta < -1/2$, then, due to $|f(1) - f(t)| \le C\omega_\varphi(f, \sqrt{1-t})$ (see (2.6) of [9]), we have

$$J_{\alpha,\beta,1}(f,\delta) + J_{\alpha,\beta,2}(f,\delta) \le C\omega_\varphi(f,\delta). \tag{16}$$

Theorem 4 will be proved in Section 3. In Section 2 we will discuss some properties of Jacobi polynomials and make some remarks concerning the conjugate function $\widehat{g' w^{(\alpha,\beta)}}$.

2. Estimates for Jacobi Polynomials and Conjugate Functions

Chapter 8 of [10] gave the following.

Lemma 5. *Let $C > 0$ be fixed. For all $\theta \in [C/n, \pi - C/n]$ and $n \ge C/\pi + 1$, one has*

$$\begin{aligned} P_n^{(\alpha,\beta)}&(\cos\theta) \\ &= \left(\pi n w^{(\alpha,\beta)}(\theta)\right)^{-1/2}\left\{\cos(N\theta + r) + \mathcal{O}(n\sin\theta)^{-1}\right\}, \end{aligned} \tag{17}$$

in which $N = n + (\alpha+\beta+1)/2$ *and* $r = -(\alpha+1/2)\pi/2$. *If* $0 \le \theta \le \pi$, *then there exists a* θ' *making* $P_n^{(\alpha,\beta)}(\cos\theta') = 0$ *and* $|\theta-\theta'| \asymp 1/n$. *Moreover, let* $M = \{\theta \mid \theta = \arccos x, P_n^{(\alpha,\beta)'}(x) = 0\}$. *Then*

$$\left\{P_n^{(\alpha,\beta)}(\cos\theta)\right\}^2 w^{(\alpha,\beta)}(\theta) \asymp \frac{1}{n}, \quad \forall\theta \in M, \tag{18}$$

and, for $0 \le \theta \le \pi$, *one has* $\theta' \in M$ *with* $|\theta-\theta'| \asymp 1/n$.

Define

$$\begin{aligned} G(\theta,t) &= P_n^{(\alpha,\beta)}(\cos(\theta+t))\, w^{(\alpha,\beta)}(\theta+t) \\ &\quad - P_n^{(\alpha,\beta)}(\cos(\theta-t))\, w^{(\alpha,\beta)}(\theta-t), \end{aligned} \tag{19}$$

and we have

$$G_n^{(\alpha,\beta)}(\theta) = \frac{1}{2\pi}\int_0^\pi \frac{G(\theta,t)}{\tan(t/2)}dt. \tag{20}$$

For $G_n^{(\alpha,\beta)}$ we have a conclusion similar to (17) (see [11]) as follows.

Lemma 6. *For all fixed* $\alpha,\beta > -1$ *and* $a > 0$, *there is* $C > 0$ *such that*

$$\max_{a/n\le\theta\le\pi-a/n}\left|\left(w^{(\alpha,\beta)}(\theta)\right)^{-1/2} G_n^{(\alpha,\beta)}(\theta)\right| \le \frac{C}{\sqrt{n}}, \quad \forall n \ge \frac{a}{\pi}+1. \tag{21}$$

Denote by Π_n the set of n-order algebraic polynomials and by π_n the set of n-order trigonometric polynomials. Denote further that

$$\Delta_n^{(\alpha,\beta)}(f,x) = H_n^{(\alpha,\beta)}(f,x) - f(x). \tag{22}$$

The following identity can be found in [9, 12] (see Lemma 5 of [12] and its proof).

Lemma 7. *Let* $-1 < \alpha, \beta < 0$ *be fixed. Then for all* $T \in \Pi_{2n-1}$

$$\begin{aligned} &\Delta_n^{(\alpha,\beta)}(T,x) \\ &\quad = 4\pi C_n\left\{P_n^{(\alpha,\beta)2}(x)\,\widehat{t' w^{(\alpha,\beta)}}(\theta) - t'(\theta)\,P_n^{(\alpha,\beta)}(x)\,G_n^{(\alpha,\beta)}(\theta)\right\}. \end{aligned} \tag{23}$$

For our purpose, we need the following result, which improves the estimate of [4] (see Lemma 4 of [4]).

Lemma 8. *Let* $-1 < \alpha, \beta < 0$ *be fixed. If* $g \in C_{2\pi}^2$ *and* $\|\widehat{g' w^{(\alpha,\beta)}}/w^{(\alpha,\beta)}\| < \infty$, *then for all* $0 < \delta \le \pi/20$, *the following holds:*

$$\begin{aligned} &\left\|\frac{1}{w^{(\alpha,\beta)}}\int_\delta^\pi \frac{\Delta_{2t}\left(g' w^{(\alpha,\beta)}\right)}{\tan t/2}dt\right\|_{[\delta,\pi-\delta]} \\ &\quad \le C\left(\left\|g'\right\| + \left\|\frac{\widehat{g' w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|\right). \end{aligned} \tag{24}$$

Proof. Write $F'(\theta) = g'(\theta)w^{(\alpha,\beta)}(\theta)$. So $|\widehat{F}'/w^{(\alpha,\beta)}| \le C$ and for $\delta \le \theta \le \pi/20$

$$\begin{aligned} &\frac{1}{w^{(\alpha,\beta)}(\theta)}\left|\frac{\widehat{F}(\theta+\delta) - \widehat{F}(\theta)}{\delta}\right| \\ &\quad = \frac{1}{w^{(\alpha,\beta)}}\left|\frac{1}{\delta}\int_0^\delta \widehat{F}'(\theta+t)\,dt\right| \le C\left\|\frac{\widehat{g' w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|. \end{aligned} \tag{25}$$

On the other hand,

$$\begin{aligned} &\frac{\widehat{F}(\theta+\delta) - \widehat{F}(\theta)}{\delta} \\ &\quad = -\frac{1}{2\pi\delta}\left\{\int_0^\pi \frac{\Delta_{2t}F(\theta+\delta)}{\tan t/2}dt - \int_0^\pi \frac{\Delta_{2t}F(\theta)}{\tan t/2}dt\right\} \\ &\quad = -\frac{1}{2\pi\delta}\int_0^\pi dt\int_0^\delta \frac{\Delta_{2t}F'(\theta+u)}{\tan t/2}du \\ &\quad = -\frac{1}{2\pi\delta}\int_0^\delta du\int_\delta^\pi \frac{\Delta_{2t}F'(\theta+u)}{\tan t/2}dt \\ &\qquad + \mathcal{O}\left(\left\|g'\right\| w^{(\alpha,\beta)}(\theta)\right). \end{aligned} \tag{26}$$

Through integration by parts,

$$\begin{aligned} &-\frac{1}{2\pi\delta}\int_0^\delta du\int_\delta^\pi \frac{\Delta_{2t}F'(\theta+u)}{\tan t/2}dt \\ &\quad = -\frac{1}{2\pi\delta}\int_0^\delta du\int_\delta^\pi \frac{\left[\Delta_t^2 F(\theta+u)\right]'}{\tan t/2}dt \\ &\quad = -\frac{1}{4\pi\delta}\int_0^\delta du\int_\delta^\pi \frac{\Delta_t^2 F(\theta+u)}{\sin^2 t/2}dt + \mathcal{O}\left(\left\|g'\right\| w^{(\alpha,\beta)}(\theta)\right). \end{aligned} \tag{27}$$

Moreover, we have

$$\begin{aligned} &-\frac{1}{4\pi\delta}\int_0^\delta du\int_\delta^\pi \frac{\Delta_t^2 F(\theta+u)}{\sin^2 t/2}dt \\ &\quad = -\frac{1}{4\pi}\int_\delta^\pi \frac{\Delta_t^2 F(\theta)}{\sin^2 t/2}dt \\ &\qquad - \frac{1}{4\pi\delta}\int_0^\delta du\int_\delta^\pi \frac{\Delta_t^2\left(F(\theta+u) - F(\theta)\right)}{\sin^2 t/2}dt, \end{aligned} \tag{28}$$

but

$$\begin{aligned} &-\frac{1}{4\pi\delta}\int_0^\delta du\int_\delta^\pi \frac{\Delta_t^2\left(F(\theta+u) - F(\theta)\right)}{\sin^2 t/2}dt \\ &\quad = -\frac{1}{4\pi\delta}\int_0^\delta du\int_\delta^{\pi/2} \frac{\Delta_t^2\left(F(\theta+u) - F(\theta)\right)}{\sin^2 t/2}dt \\ &\qquad - \frac{1}{4\pi\delta}\int_0^\delta du\int_{\pi/2}^\pi \frac{\Delta_t^2\left(F(\theta+u) - F(\theta)\right)}{\sin^2 t/2}dt \\ &\quad = \mathcal{O}\left(\left\|g'\right\| w^{(\alpha,\beta)}(\theta)\right), \end{aligned}$$

$$-\frac{1}{4\pi}\int_{\delta}^{\pi}\frac{\Delta_t^2 F(\theta)}{\sin^2 t/2}dt = -\frac{1}{2\pi}\left\{\int_{\delta}^{\pi}\frac{\Delta_{2t}F'(\theta)}{\tan t/2}dt - \frac{\Delta_\delta^2 F(\theta)}{\tan\delta/2}\right\} = -\frac{1}{2\pi}\int_{\delta}^{\pi}\frac{\Delta_{2t}F'(\theta)}{\tan t/2}dt + \mathcal{O}\left(\left\|g'\right\| w^{(\alpha,\beta)}(\theta)\right). \tag{29}$$

So we have

$$\frac{\widehat{F}(\theta+\delta)-\widehat{F}(\theta)}{\delta} = -\frac{1}{2\pi}\int_{\delta}^{\pi}\frac{\Delta_{2t}F'(\theta)}{\tan t/2}dt + \mathcal{O}\left(\left\|g'\right\| w^{(\alpha,\beta)}(\theta)\right). \tag{30}$$

Thus, if $\delta \le \theta \le \pi/20$, then

$$\left|\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\delta}^{\pi}\frac{\Delta_{2t}\left(g'(\theta)w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt\right| \le \frac{2\pi}{w^{(\alpha,\beta)}(\theta)}\left|\frac{\widehat{F}(\theta+\delta)-\widehat{F}(\theta)}{\delta}\right| + C\left\|g'\right\| \le C\left(\left\|g'\right\| + \left\|\frac{\widehat{g'w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|\right). \tag{31}$$

If $\pi/20 \le \theta \le \pi/2$, then $w^{(\alpha,\beta)}(\theta) \asymp 1$ and

$$\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\theta/2}^{\pi}\frac{\Delta_{2t}\left(g'(\theta)w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt = \mathcal{O}\left(\left\|g'\right\|\right). \tag{32}$$

In this case, for $0 \le t \le \theta/2$, we have

$$\frac{w^{(\alpha,\beta)}(\theta\pm t) - w^{(\alpha,\beta)}(\theta)}{t} = \mathcal{O}(1), \tag{33}$$

$$\begin{aligned}\Delta_{2t}&\left(g'(\theta)w^{(\alpha,\beta)}(\theta)\right)\\ &= g'(\theta+t)w^{(\alpha,\beta)}(\theta+t) - g'(\theta-t)w^{(\alpha,\beta)}(\theta-t)\\ &= \Delta_{2t}\left(g'(\theta)\right)w^{(\alpha,\beta)}(\theta) + g'(\theta+t)\\ &\quad\times\left[w^{(\alpha,\beta)}(\theta+t) - w^{(\alpha,\beta)}(\theta)\right]\\ &\quad - g'(\theta-t)\left[w^{(\alpha,\beta)}(\theta-t) - w^{(\alpha,\beta)}(\theta)\right];\end{aligned} \tag{34}$$

therefore,

$$\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\delta}^{\theta/2}\frac{\Delta_{2t}\left(g'(\theta)w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt = \int_{\delta}^{\theta/2}\frac{\Delta_{2t}g'(\theta)}{\tan t/2}dt + \mathcal{O}\left(\left\|g'\right\|\right). \tag{35}$$

Let $\delta \to 0$ in (35) and, with (32), we obtain

$$\int_{0}^{\theta/2}\frac{\Delta_{2t}g'(\theta)}{\tan t/2}dt = \mathcal{O}\left(\left\|g'\right\| + \left\|\frac{\widehat{g'w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|\right). \tag{36}$$

For $\pi/20 \le \theta \le \pi/2$, it is clear that if

$$\int_{\delta}^{\theta/2}\frac{\Delta_{2t}g'(\theta)}{\tan t/2}dt = \mathcal{O}\left(\left\|g'\right\| + \left\|\frac{\widehat{g'w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|\right), \quad \frac{\pi}{20} \le \theta \le \frac{\pi}{2}, \tag{37}$$

then (31) and (37) imply that

$$\left|\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\delta}^{\pi}\frac{\Delta_{2t}\left(g'(\theta)w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt\right| \le C\left(\left\|g'\right\| + \left\|\frac{\widehat{g'w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|\right), \quad \delta \le \theta \le \frac{\pi}{2}. \tag{38}$$

On the other hand, it is easy to see that the above estimate also holds for $\pi/2 \le \theta \le \pi - \delta$. Thus, the desired inequality is obtained.

Next, we are going to prove (37). This time we define

$$F(\theta) = \int_{0}^{\theta/2}\frac{\Delta_{2t}g(\theta)}{\tan t/2}dt. \tag{39}$$

Simple calculation shows that

$$\begin{aligned}&\frac{F(\theta+\delta)-F(\theta)}{\delta}\\ &\quad= \frac{1}{\delta}\left(\int_{0}^{(\theta+\delta)/2}\frac{\Delta_{2t}g(\theta+\delta)}{\tan t/2}dt - \int_{0}^{\theta/2}\frac{\Delta_{2t}g(\theta)}{\tan t/2}dt\right)\\ &\quad= \frac{1}{\delta}\int_{0}^{\theta/2}\int_{0}^{\delta}\frac{\Delta_{2t}g'(\theta+u)}{\tan t/2}du\,dt + \mathcal{O}\left(\left\|g'\right\|\right).\end{aligned} \tag{40}$$

With (36), we obtain

$$\frac{F(\theta+\delta)-F(\theta)}{\delta} = \mathcal{O}\left(\left\|g'\right\| + \left\|\frac{\widehat{g'w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|\right), \quad \frac{\pi}{20} \le \theta \le \frac{\pi}{2}. \tag{41}$$

Hence,

$$\frac{1}{\delta}\int_{0}^{\theta/2}\int_{0}^{\delta}\frac{\Delta_{2t}g'(\theta+u)}{\tan t/2}du\,dt = \mathcal{O}\left(\left\|g'\right\| + \left\|\frac{\widehat{g'w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|\right), \quad \frac{\pi}{20} \le \theta \le \frac{\pi}{2}. \tag{42}$$

Similar to the case of $\delta \le \theta \le \pi/20$, we have

$$\begin{aligned}&\frac{1}{\delta}\int_{0}^{\theta/2}\int_{0}^{\delta}\frac{\Delta_{2t}g'(\theta+u)}{\tan t/2}du\,dt\\ &\quad= \frac{1}{2\delta}\int_{0}^{\delta}\int_{\delta}^{\theta/2}\frac{\Delta_{2t}^2 g(\theta+u)}{\sin^2 t/2}dt\,du + \mathcal{O}\left(\left\|g'\right\|\right)\\ &\quad= \frac{1}{2\delta}\int_{0}^{\delta}\int_{\delta}^{\theta/2}\frac{\Delta_{2t}^2 g(\theta)}{\sin^2 t/2}dt\,du\end{aligned}$$

$$+\frac{1}{2\delta}\int_0^{\delta}\int_{\delta}^{\theta/2}\frac{\Delta_{2t}^2\left(g(\theta+u)-g(\theta)\right)}{\sin^2 t/2}dt\,du+\mathcal{O}\left(\left\|g'\right\|\right)$$
$$=\frac{1}{2}\int_{\delta}^{\theta/2}\frac{\Delta_{2t}^2 g(\theta)}{\sin^2 t/2}dt+\mathcal{O}\left(\left\|g'\right\|\right), \tag{43}$$

which obviously implies (37). □

In what follows, we will give an estimate of the conjugate function defined in the saturation class $S(\alpha,\beta)$.

Lemma 9. *Let $-1<\alpha,\ \beta<0$ be fixed. Then, for all $0<\delta\le\theta\le\pi/2$ and even $g\in C_{2\pi}^2$, one has*

$$\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\delta}^{\pi}\frac{\Delta_{2t}\left(g'(\theta)\,w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt$$
$$=2\int_{\delta}^{\sin\theta}\frac{\Delta_t^2 g(\theta)}{t^2}dt+\frac{2}{w^{(\alpha,\beta)}(\theta)}\times\int_{\theta}^{\pi}g'(t)\frac{w^{(\alpha,\beta)}(t)\sin t}{1-\cos t}dt+\mathcal{O}\left(\left\|g'\right\|\right)$$
$$=2\int_{\delta}^{\sin\theta}\frac{\Delta_t^2 g(\theta)}{t^2}dt-\frac{2}{w^{(\alpha,\beta)}(\theta)}\times\int_{\theta}^{\pi}\left(g(t)-g(0)\right)\left(\frac{w^{(\alpha,\beta)}(t)\sin t}{1-\cos t}\right)'dt+\mathcal{O}\left(\left\|g'\right\|\right), \tag{44}$$

in which $\mathcal{O}$ does not depend on $\|g'\|$ and δ. Moreover, let $t_n\in\pi_n$ be the best approximation of g. Then, for all $1/n\le\theta\le\pi-1/n$, one gets

$$\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{1/n}^{\pi}\frac{\Delta_{2t}\left(t_n'(\theta)\,w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt$$
$$=\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{1/n}^{\pi}\frac{\Delta_{2t}\left(g'(\theta)\,w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt+\mathcal{O}\left(\left\|g'\right\|\right). \tag{45}$$

Proof. Since

$$\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\delta}^{\pi}\frac{\Delta_{2t}\left(g'(\theta)\,w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt$$
$$=\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\delta}^{\sin\theta}\frac{\Delta_{2t}\left(g'(\theta)\,w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt+\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\sin\theta}^{\pi}\frac{\Delta_{2t}\left(g'(\theta)\,w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt, \tag{46}$$

$$\int_{\delta}^{\sin\theta}\frac{\left|w^{(\alpha,\beta)}(\theta\pm t)-w^{(\alpha,\beta)}(\theta)\right|}{\tan t/2}dt=\mathcal{O}\left(w^{(\alpha,\beta)}(\theta)\right),$$

we have

$$\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\delta}^{\sin\theta}\frac{\Delta_{2t}\left(g'(\theta)\,w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt$$
$$=\int_{\delta}^{\sin\theta}\frac{\Delta_{2t}g'(\theta)}{\tan t/2}dt+\mathcal{O}\left(\left\|g'\right\|\right)$$
$$=\int_{\delta}^{\sin\theta}\frac{\left[\Delta_t^2 g(\theta)\right]'}{\tan t/2}dt+\mathcal{O}\left(\left\|g'\right\|\right). \tag{47}$$

Integrating by parts, we obtain

$$\int_{\delta}^{\sin\theta}\frac{\Delta_{2t}g'(\theta)}{\tan t/2}dt=2\int_{\delta}^{\sin\theta}\frac{\Delta_t^2 g(\theta)}{t^2}dt+\mathcal{O}\left(\left\|g'\right\|\right). \tag{48}$$

Therefore,

$$\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\delta}^{\pi}\frac{\Delta_{2t}\left(g'(\theta)\,w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt$$
$$=2\int_{\delta}^{\sin\theta}\frac{\Delta_t^2 g(\theta)}{t^2}dt+\frac{1}{w^{(\alpha,\beta)}(\theta)}\times\int_{\sin\theta}^{\pi}\frac{\Delta_{2t}\left(g'(\theta)\,w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt+\mathcal{O}\left(\left\|g'\right\|\right). \tag{49}$$

To deal with the second term of the above estimate, we note that, if $\pi/20\le\theta\le\pi/2$, then $w^{(\alpha,\beta)}(\theta)\asymp 1$, and

$$\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\sin\theta}^{\pi}\frac{\Delta_{2t}\left(g'(\theta)\,w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt=\mathcal{O}\left(\left\|g'\right\|\right),$$
$$\frac{2}{w^{(\alpha,\beta)}(\theta)}\int_{\theta}^{\pi}g'(t)\frac{w^{(\alpha,\beta)}(t)\sin t}{1-\cos t}dt=\mathcal{O}\left(\left\|g'\right\|\right). \tag{50}$$

Thus,

$$\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\sin\theta}^{\pi}\frac{\Delta_{2t}\left(g'(\theta)\,w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt$$
$$=\frac{2}{w^{(\alpha,\beta)}(\theta)}\int_{\theta}^{\pi}g'(t)\frac{w^{(\alpha,\beta)}(t)\sin t}{1-\cos t}dt+\mathcal{O}\left(\left\|g'\right\|\right). \tag{51}$$

If $\delta\le\theta\le\pi/20$, rewrite the previous term as

$$\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\sin\theta}^{\pi}\frac{\Delta_{2t}\left(g'(\theta)\,w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt$$
$$=\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\sin\theta}^{2\theta}\frac{\Delta_{2t}\left(g'(\theta)\,w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt+\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{2\theta}^{\pi}\frac{\Delta_{2t}\left(g'(\theta)\,w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt. \tag{52}$$

Obviously, we get

$$\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\sin\theta}^{2\theta}\frac{\Delta_{2t}\left(g'(\theta)\,w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2}dt=\mathcal{O}\left(\left\|g'\right\|\right). \tag{53}$$

On the other hand, we have

$$\begin{aligned}
&\int_{2\theta}^{\pi} \frac{\Delta_{2t}\left(g'(\theta)\, w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2} dt \\
&= \int_{3\theta}^{\pi+\theta} \frac{g'(t)\, w^{(\alpha,\beta)}(t)}{\tan (t-\theta)/2} dt + \int_{\theta}^{\pi-\theta} \frac{g'(t)\, w^{(\alpha,\beta)}(t)}{\tan (t+\theta)/2} dt \\
&= 2\int_{3\theta}^{\pi} \frac{g'(t)\, w^{(\alpha,\beta)}(t) \sin t}{\cos\theta - \cos t} dt + \int_{\theta}^{3\theta} \frac{g'(t)\, w^{(\alpha,\beta)}(t)}{\tan (t+\theta)/2} dt \\
&\quad - \int_{\pi-\theta}^{\pi} \frac{g'(t)\, w^{(\alpha,\beta)}(t)}{\tan (t+\theta)/2} dt + \int_{\pi}^{\pi+\theta} \frac{g'(t)\, w^{(\alpha,\beta)}(t)}{\tan (t-\theta)/2} dt.
\end{aligned} \tag{54}$$

Since

$$\sin^2\frac{\theta}{2}\int_{3\theta}^{\pi} \frac{w^{(\alpha,\beta)}(t)\sin t}{|\cos\theta - \cos t|(1-\cos t)} dt = \mathcal{O}\left(w^{(\alpha,\beta)}(\theta)\right), \tag{55}$$

we obtain

$$\begin{aligned}
&2\int_{3\theta}^{\pi} \frac{g'(t)\, w^{(\alpha,\beta)}(t)\sin t}{\cos\theta - \cos t} dt \\
&= 2\int_{\theta}^{\pi} \frac{g'(t)\, w^{(\alpha,\beta)}(t)\sin t}{1-\cos t} dt + \mathcal{O}\left(\left\|g'\right\| w^{(\alpha,\beta)}(\theta)\right).
\end{aligned} \tag{56}$$

Moreover, the following estimates hold:

$$\begin{aligned}
&\int_{\theta}^{3\theta} \frac{g'(t)\, w^{(\alpha,\beta)}(t)}{\tan (t+\theta)/2} dt = \mathcal{O}\left(\left\|g'\right\| w^{(\alpha,\beta)}(\theta)\right), \\
&\int_{\pi-\theta}^{\pi} \frac{g'(t)\, w^{(\alpha,\beta)}(t)}{\tan (t+\theta)/2} dt = \mathcal{O}\left(\left\|g'\right\| w^{(\alpha,\beta)}(\theta)\right), \\
&\int_{\pi}^{\pi+\theta} \frac{g'(t)\, w^{(\alpha,\beta)}(t)}{\tan (t-\theta)/2} dt = \mathcal{O}\left(\left\|g'\right\| w^{(\alpha,\beta)}(\theta)\right).
\end{aligned} \tag{57}$$

Consequently, for all $0 < \delta \le \theta \le \pi/2$, we get

$$\begin{aligned}
&\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\delta}^{\pi} \frac{\Delta_{2t}\left(g'(\theta)\, w^{(\alpha,\beta)}(\theta)\right)}{\tan t/2} dt \\
&= 2\int_{\delta}^{\sin\theta} \frac{\Delta_t^2 g(\theta)}{t^2} dt + \frac{2}{w^{(\alpha,\beta)}(\theta)} \\
&\quad \times \int_{\theta}^{\pi} g'(t)\, \frac{w^{(\alpha,\beta)}(t)\sin t}{1-\cos t} dt + \mathcal{O}\left(\left\|g'\right\|\right),
\end{aligned} \tag{58}$$

which proves the first assertion of the lemma.

The second estimate can be obtained from Lemma 8, the first estimate, and integration by parts. □

Lemma 10. *There exists an absolute constant $C > 0$ such that, for all even $g \in C_{2\pi}^2$,*

$$\begin{aligned}
&\left|\int_{\theta}^{\pi} \frac{\Delta_t^2 g(0)}{t^2} dt\right| \le C\left(\left\|g'\right\| + \max_{\delta\le\eta\le\pi/2}\left|\int_{\delta}^{\pi} \frac{\Delta_t^2 g(\eta)}{t^2} dt\right|\right), \\
&\qquad \forall 0 \le \delta \le \theta \le \frac{\pi}{2}.
\end{aligned} \tag{59}$$

Proof. We may assume that $\delta \le \theta < 1$ and $1/\theta, 1/\delta \in \mathbb{N}$. Thus, by [13, 14], for Fejér mean $\sigma_{1/\theta} g$ of g, we have

$$\begin{aligned}
&\theta\left|\int_{\theta}^{\pi} \frac{\Delta_t^2 g(0)}{\sin^2(t/2)} dt\right| \\
&\le \theta\left\|\int_{\theta}^{\pi} \frac{\Delta_t^2 g}{\sin^2(t/2)} dt\right\| \le C\left\|\sigma_{1/\theta} g - g\right\| \le C\theta\left\|\widehat{g'}\right\|;
\end{aligned} \tag{60}$$

therefore,

$$\left|\int_{\theta}^{\pi} \frac{\Delta_t^2 g(0)}{t^2} dt\right| \le C\left(\left\|\int_{0}^{\pi} \frac{\Delta_t^2 g}{t^2} dt\right\| + \left\|g'\right\|\right). \tag{61}$$

For $T \in \pi_{1/\delta}$, we have

$$\begin{aligned}
\left|\int_{\theta}^{\pi} \frac{\Delta_t^2 T(0)}{t^2} dt\right| &\le C\left(\left\|\int_{\delta}^{\pi} \frac{\Delta_t^2 T}{t^2} dt\right\| + \left\|T'\right\| + \delta\left\|T''\right\|\right) \\
&\le C\left(\left\|\int_{\delta}^{\pi} \frac{\Delta_t^2 T}{t^2} dt\right\| + \left\|T'\right\|\right).
\end{aligned} \tag{62}$$

Consequently, for $T \in \pi_{1/\delta}$ with $\|T - g\| \le C\delta\|g'\|$ and $\|T'\| \le C\|g'\|$, we obtain

$$\begin{aligned}
&\left|\int_{\theta}^{\pi} \frac{\Delta_t^2 g(0)}{t^2} dt\right| \\
&\le C\left(\frac{1}{\theta}\|g - T\| + \left\|\int_{\delta}^{\pi} \frac{\Delta_t^2 T}{t^2} dt\right\| + \left\|g'\right\|\right) \\
&\le C\left(\frac{1}{\theta}\|g - T\| + \frac{1}{\delta}\|T - g\| + \left\|\int_{\delta}^{\pi} \frac{\Delta_t^2 g}{t^2} dt\right\| + \left\|g'\right\|\right) \\
&\le C\left(\left\|\int_{\delta}^{\pi} \frac{\Delta_t^2 g}{t^2} dt\right\| + \left\|g'\right\|\right).
\end{aligned} \tag{63}$$

We may assume that

$$\left\|\int_{\delta}^{\pi} \frac{\Delta_t^2 g}{t^2} dt\right\| = \max_{\delta\le\eta\le\pi/2}\left|\int_{\delta}^{\pi} \frac{\Delta_t^2 g(\eta)}{t^2} dt\right|. \tag{64}$$

Otherwise, choose $h(\eta) = g(\eta)$ $(|\eta| \le \pi/20)$ and $h(\eta) = g(2\pi)(\pi/10 \le |\eta| \le 2\pi)$ to make $h \in C_{2\pi}^2$ even and $\|h'\| \le C\|g'\|$. Then

$$\begin{aligned}
\left\|\int_{\delta}^{\pi} \frac{\Delta_t^2 h}{t^2} dt\right\| &\le C\left(\max_{\delta\le\eta\le\pi/2}\left|\int_{\delta}^{\pi} \frac{\Delta_t^2 h(\eta)}{t^2} dt\right| + \left\|h'\right\|\right) \\
&\le C\left(\max_{\delta\le\eta\le\pi/2}\left|\int_{\delta}^{\pi} \frac{\Delta_t^2 g(\eta)}{t^2} dt\right| + \left\|g'\right\|\right).
\end{aligned} \tag{65}$$

We conclude that

$$\left|\int_\theta^\pi \frac{\Delta_t^2 g(0)}{t^2}dt\right| \le C\left(\left|\int_\theta^\pi \frac{\Delta_t^2 h(0)}{t^2}dt\right| + \|g'\|\right)$$
$$\le C\left(\left\|\int_\delta^\pi \frac{\Delta_t^2 h}{t^2}dt\right\| + \|g'\|\right) \quad (66)$$
$$\le C\left(\max_{\delta\le\eta\le\pi/2}\left|\int_\delta^\pi \frac{\Delta_t^2 g(\eta)}{t^2}dt\right| + \|g'\|\right),$$

which gives the desired inequality. □

3. Proof of Theorem 4

We need to prove the following Lemma before Theorem 4.

Lemma 11. *Given $-1 < \alpha, \beta < 0$, there is $C > 0$ such that, for all $f \in C[-1,1]$ and all $n \ge 1$, one has*

$$\frac{1}{n}\left\|\varphi H_n^{(\alpha,\beta)\prime}(f)\right\| \le CK_{\alpha,\beta}\left(f,\frac{1}{n}\right),$$
$$\left\|f - H_n^{(\alpha,\beta)}(f)\right\| + \frac{1}{n}\left\|\varphi H_n^{(\alpha,\beta)\prime}(f)\right\| \asymp K_{\alpha,\beta}\left(f,\frac{1}{n}\right). \quad (67)$$

Proof. Denote that $T = H_n^{(\alpha,\beta)}(f)$. Then from Theorem 3, we have

$$\|f - T\| \le CK_{\alpha,\beta}\left(f,\frac{1}{n}\right). \quad (68)$$

Next, let us estimate $(1/n)\|\varphi T'\|$.

We know that $T'(x)\varphi(x) = -t_n'(\theta)$ with $\theta = \arccos x$ and $t_n(\theta) = T(\cos\theta)$. But (1) tells that $T'(x_k) = 0$. Hence, $t_n'(\theta_k) = 0$ for $\theta_k = \arccos x_k$. Lemma 5 tells that, for each $0 \le \theta \le \pi$, there is θ_k satisfying $|\theta - \theta_k| \le C/n$. Assume that $|t_n'(\theta)| = \|t_n'\|$. Then $\|t_n'\| = |t_n'(\theta) - t_n'(\theta_k)|$ and further

$$\|t_n'\| \le \frac{C}{n}\|t_n''\|. \quad (69)$$

We may assume that $n = 2^m$. Thus, the Bernstein inequality for trigonometric polynomials yields

$$\|t_n''\| \le \sum_{j=1}^m \left\|t_{2^j}'' - t_{2^{j-1}}''\right\|$$
$$\le C\sum_{j=0}^m 2^{2j}\left\|f - H_{2^j}^{(\alpha,\beta)}(f)\right\| \quad (70)$$
$$\le Cn\max_{1\le m\le n} m\left\|f - H_m^{(\alpha,\beta)}(f)\right\|.$$

But

$$\frac{1}{n}\max_{1\le m\le n} m\left\|f - H_m^{(\alpha,\beta)}(f)\right\|$$
$$\le \frac{C}{n}\max_{1\le m\le n}\left(mK_{\alpha,\beta}\left(f,\frac{1}{m}\right)\right) \le CK_{\alpha,\beta}\left(f,\frac{1}{n}\right). \quad (71)$$

So, we have

$$\frac{1}{n}\|\varphi T'\| \le CK_{\alpha,\beta}\left(f,\frac{1}{n}\right). \quad (72)$$

Combining this inequality with (68), we get

$$\|f - T\| + \frac{1}{n}\|\varphi T'\| \le CK_{\alpha,\beta}\left(f,\frac{1}{n}\right). \quad (73)$$

Now we need only to prove

$$K_{\alpha,\beta}\left(f,\frac{1}{n}\right) \le C\left(\|f - T\| + \frac{1}{n}\|\varphi T'\|\right). \quad (74)$$

Obviously,

$$K_{\alpha,\beta}\left(f,\frac{1}{n}\right) \le \|f - T\| + \frac{1}{n}\|\varphi T'\| + \frac{1}{n}\left\|\frac{\widehat{t'w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|. \quad (75)$$

Lemma 5 tells the following. Let $a > 0$ be fixed, then for $\theta \in M \cap \{\theta \mid a/n \le \theta \le \pi - a/n\}$,

$$\frac{1}{n} \asymp \left\{P_n^{(\alpha,\beta)}(\cos\theta)\right\}^2 w^{(\alpha,\beta)}(\theta). \quad (76)$$

Thus, for those θ,

$$\frac{1}{n}\left|\frac{\widehat{t'w^{(\alpha,\beta)}}(\theta)}{w^{(\alpha,\beta)}(\theta)}\right| \asymp \left\{P_n^{(\alpha,\beta)}(\cos\theta)\right\}^2\left|\widehat{t'w^{(\alpha,\beta)}}(\theta)\right|. \quad (77)$$

Since $\Delta_n^{(\alpha,\beta)}(T) = 0$, following (23), (18), and Lemma 6, we have, for those θ,

$$\frac{1}{n}\left|\frac{\widehat{t'w^{(\alpha,\beta)}}(\theta)}{w^{(\alpha,\beta)}(\theta)}\right| \le \frac{1}{n}\|\varphi T'\|. \quad (78)$$

If $a/n \le \theta \le \pi - a/n$ and $\theta \notin M$, then there is a $\theta' \in M \cap \{\theta \mid a/n \le \theta \le \pi - a/n\}$ satisfying $|\theta - \theta'| \le C/n$ (see Lemma 5). Hence, from Bernstein inequality $\|t''\| \le Cn\|t'\|$, we have

$$\frac{1}{n}\int_{1/n}^{\sin\theta}\frac{\Delta_u^2 t(\theta)}{\sin^2(u/2)}du$$
$$= \frac{1}{n}\int_{1/n}^{\sin\theta'}\frac{\Delta_u^2 t(\theta')}{\sin^2(u/2)}du + \frac{1}{n}\int_{\sin\theta'}^{\sin\theta}\frac{\Delta_u^2 t(\theta)}{\sin^2(u/2)}du$$
$$+ \frac{1}{n}\int_{1/n}^{\sin\theta'}\frac{\Delta_u^2\left(t(\theta) - t(\theta')\right)}{\sin^2(u/2)}du \quad (79)$$
$$= \frac{1}{n}\int_{1/n}^{\sin\theta'}\frac{\Delta_u^2 t(\theta')}{\sin^2(u/2)}d + \mathcal{O}\left(\frac{1}{n}\|\varphi T'\|\right).$$

Moreover,

$$\frac{1}{nw^{(\alpha,\beta)}(\theta)}\int_\theta^\pi t'(u)\frac{w^{(\alpha,\beta)}(u)\sin u}{1-\cos u}du$$
$$= \frac{1}{nw^{(\alpha,\beta)}(\theta')}\int_{\theta'}^\pi t'(u)\frac{w^{(\alpha,\beta)}(u)\sin u}{1-\cos u}du \quad (80)$$
$$+ \mathcal{O}\left(\frac{1}{n}\|\varphi T'\|\right).$$

Consequently, (78) obtained from Lemma 9 holds for all $\theta \in [1/n, \pi - 1/n]$. Finally, from (78) and (75) we obtain (74). □

Proof of Theorem 4. Firstly, we prove that

$$K_{\alpha,\beta}(f,\delta) \le C\left(K_{\alpha_0,\beta_0}(f,\delta) + J_{\alpha,\beta,1}(f,\delta) + J_{\alpha,\beta,2}(f,\delta)\right). \tag{81}$$

Let $T = H_n^{(\alpha_0,\beta_0)}(f)$, $t(\theta) = T(\cos\theta)$, and $g(\theta) = f(\cos\theta)$. From Theorems 1 and 2 we conclude that, for some $n \le n' \le An$,

$$\begin{aligned} K_{\alpha,\beta}\left(f,\frac{1}{n}\right) &\le C\left\|f - H_{n'}^{(\alpha,\beta)}(f)\right\| \\ &\le C\left(\left\|f - H_n^{(\alpha_0,\beta_0)}(f)\right\| \right. \\ &\quad \left. + \left\|H_n^{(\alpha_0,\beta_0)}(f) - H_{n'}^{(\alpha,\beta)}\left(H_n^{(\alpha_0,\beta_0)}(f)\right)\right\|\right). \end{aligned} \tag{82}$$

We know that $H_n^{(\alpha_0,\beta_0)}(f) - H_{n'}^{(\alpha,\beta)}(H_n^{(\alpha_0,\beta_0)}(f)) \in \Pi_{2n'-1}$, so (see [14, page 43])

$$\begin{aligned} &\left\|H_n^{(\alpha_0,\beta_0)}(f) - H_{n'}^{(\alpha,\beta)}\left(H_n^{(\alpha_0,\beta_0)}(f)\right)\right\| \\ &\quad \le C\left\|H_n^{(\alpha_0,\beta_0)}(f) - H_{n'}^{(\alpha,\beta)}\left(H_n^{(\alpha_0,\beta_0)}(f)\right)\right\|_{[-1+1/n^2,1-1/n^2]}. \end{aligned} \tag{83}$$

Thus, (18) and Lemmas 6 and 7 imply that

$$\begin{aligned} &\left\|H_n^{(\alpha_0,\beta_0)}(f) - H_{n'}^{(\alpha,\beta)}\left(H_n^{(\alpha_0,\beta_0)}(f)\right)\right\| \\ &\quad \le \frac{C}{n}\left(\left\|\frac{\widehat{t'w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|_{[1/n,\pi-1/n]} + \left\|t'\right\|\right). \end{aligned} \tag{84}$$

Consequently, by Lemma 11, we get

$$K_{\alpha,\beta}\left(f,\frac{1}{n}\right) \le C\left(K_{\alpha_0,\beta_0}\left(f,\frac{1}{n}\right) + \frac{1}{n}\left\|\frac{\widehat{t'w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|_{[1/n,\pi-1/n]}\right). \tag{85}$$

Obviously,

$$\begin{aligned} &\left\|\frac{\widehat{t'w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|_{[1/n,\pi-1/n]} \\ &\quad \le C\left\|\frac{1}{w^{(\alpha,\beta)}}\int_{1/n}^{\pi}\frac{\Delta_{2u}\left(t'w^{(\alpha,\beta)}\right)}{\tan(u/2)}du\right\|_{[1/n,\pi-1/n]} + C\left\|t'\right\|. \end{aligned} \tag{86}$$

If $1/n \le \theta \le \pi/2$, then Lemma 9 implies that

$$\begin{aligned} &\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{1/n}^{\pi}\frac{\Delta_{2u}\left(t'(\theta)w^{(\alpha,\beta)}(\theta)\right)}{\tan(u/2)}du \\ &\quad = 2\int_{1/n}^{\sin\theta}\frac{\Delta_u^2 t(\theta)}{u^2}du - \frac{2}{w^{(\alpha,\beta)}(\theta)} \\ &\qquad \times \int_{\theta}^{\pi}(t(u) - t(0))\left(\frac{w^{(\alpha,\beta)}(u)\sin u}{1-\cos u}\right)'du + \mathcal{O}\left(\left\|t'\right\|\right). \end{aligned} \tag{87}$$

Since

$$\begin{aligned} &\frac{1}{n}\left|\int_{1/n}^{\sin\theta}\frac{\Delta_u^2 t(\theta)}{u^2}du\right| \le CK_{\alpha_0,\beta_0}\left(f,\frac{1}{n}\right), \\ &\frac{1}{n}\left|\frac{2}{w^{(\alpha,\beta)}(\theta)}\int_{\theta}^{\pi}(t(u) - t(0))\left(\frac{w^{(\alpha,\beta)}(u)\sin u}{1-\cos u}\right)'du\right| \\ &\quad \le CJ_{\alpha,\beta,1}\left(f,\frac{1}{n}\right), \end{aligned} \tag{88}$$

therefore, for $1/n \le \theta \le \pi/2$, we have

$$\begin{aligned} &\frac{1}{n}\left|\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{1/n}^{\pi}\frac{\Delta_{2u}\left(t'(\theta)w^{(\alpha,\beta)}(\theta)\right)}{\tan(u/2)}du\right| \\ &\quad \le C\left(K_{\alpha_0,\beta_0}\left(f,\frac{1}{n}\right) + J_{\alpha,\beta,1}\left(f,\frac{1}{n}\right)\right). \end{aligned} \tag{89}$$

In the same way, if $\pi/2 \le \theta \le \pi - 1/n$, we obtain

$$\begin{aligned} &\frac{1}{n}\left|\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{1/n}^{\pi}\frac{\Delta_{2u}\left(t'(\theta)w^{(\alpha,\beta)}(\theta)\right)}{\tan(u/2)}du\right| \\ &\quad \le C\left(K_{\alpha_0,\beta_0}\left(f,\frac{1}{n}\right) + J_{\alpha,\beta,2}\left(f,\frac{1}{n}\right)\right). \end{aligned} \tag{90}$$

Consequently,

$$\begin{aligned} &\frac{1}{n}\left\|\frac{\widehat{t'w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|_{[1/n,\pi-1/n]} \\ &\quad \le C\left(K_{\alpha_0,\beta_0}\left(f,\frac{1}{n}\right) + J_{\alpha,\beta,1}\left(f,\frac{1}{n}\right) + J_{\alpha,\beta,2}\left(f,\frac{1}{n}\right)\right). \end{aligned} \tag{91}$$

Therefore,

$$\begin{aligned} &K_{\alpha,\beta}\left(f,\frac{1}{n}\right) \\ &\quad \le C\left(K_{\alpha_0,\beta_0}\left(f,\frac{1}{n}\right) + J_{\alpha,\beta,1}\left(f,\frac{1}{n}\right) + J_{\alpha,\beta,2}\left(f,\frac{1}{n}\right)\right). \end{aligned} \tag{92}$$

Next, we prove that

$$K_{\alpha_0,\beta_0}(f,\delta) + J_{\alpha,\beta,1}(f,\delta) + J_{\alpha,\beta,2}(f,\delta) \le CK_{\alpha,\beta}(f,\delta). \tag{93}$$

Firstly, we assume that $-1/2 < \alpha < 0$ and $0 < 1/n \le \theta \le \pi/2$. Thus, for $T \in \Pi_n$ and $t(\theta) = T(\cos\theta)$, we have

$$\begin{aligned} &\max_{1/n\le\theta\le\pi/2}\left|\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{\theta}^{\pi}t'(u)\frac{w^{(\alpha,\beta)}(u)\sin u}{1-\cos u}du\right| \\ &\quad \le \max_{1/n\le\theta\le\pi/2}\left|\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_{0}^{\pi}t'(u)\frac{w^{(\alpha,\beta)}(u)\sin u}{1-\cos u}du\right| + C\left\|t'\right\| \end{aligned}$$

$$\leq C\left|\frac{1}{w^{(\alpha,\beta)}(1/n)}\int_0^\pi t'(u)\frac{w^{(\alpha,\beta)}(u)\sin u}{1-\cos u}du\right|+C\left\|t'\right\|$$
$$\leq C\max_{1/n\leq\theta\leq\pi/2}\left|\int_{1/n}^{\sin\theta}\frac{\Delta_u^2 t(\theta)}{u^2}du+\frac{1}{w^{(\alpha,\beta)}(\theta)}\times\int_\theta^\pi t'(u)\frac{w^{(\alpha,\beta)}(u)\sin u}{1-\cos u}du\right|+C\left\|t'\right\|. \tag{94}$$

For $H\in C^2[-1,1]$, denote that $h(\theta)=H(\cos\theta)$ and let T be the best approximation of H. Following from (94), Lemmas 8 and 9, we have

$$\frac{1}{n}\max_{1/n\leq\theta\leq\pi/2}\left|\frac{1}{w^{(\alpha,\beta)}(\theta)}\times\int_\theta^\pi(h(u)-h(0))\left(\frac{w^{(\alpha,\beta)}(u)\sin u}{1-\cos u}\right)'du\right|\leq\frac{C}{n}\left(\left\|h'\right\|+\left\|\frac{\widehat{h'w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|\right). \tag{95}$$

Thus,

$$J_{\alpha,\beta,1}\left(f,\frac{1}{n}\right)\leq C\left(\|f-H\|+\frac{1}{n}\left(\left\|h'\right\|+\left\|\frac{\widehat{h'w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|\right)\right). \tag{96}$$

Therefore,

$$J_{\alpha,\beta,1}\left(f,\frac{1}{n}\right)\leq CK_{\alpha,\beta}\left(f,\frac{1}{n}\right). \tag{97}$$

Next, suppose that $\alpha=-1/2$. Clearly, we have

$$\left(\frac{w^{(\alpha,\beta)}(t)\sin t}{1-\cos t}\right)'=-(\beta+1)\cos^{2\beta+1}\frac{t}{2}-\frac{1}{2}\cos^{2\beta+3}\frac{t}{2}\sin^{-2}\frac{t}{2}, \tag{98}$$

and $w^{(\alpha,\beta)}(\theta)\asymp 1$ for $\delta\leq\theta\leq\pi/2$. Hence, let $H=H_n^{(\alpha,\beta)}(f)$, $h(\theta)=H(\cos\theta)$, and $n=1/\delta$. From Lemma 9 we obtain

$$\left|\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_\theta^\pi(h(t)-h(0))\left(\frac{w^{(\alpha,\beta)}(t)\sin t}{1-\cos t}\right)'dt\right|\leq C\left(\left|\int_\theta^\pi\frac{\Delta_t^2 h(0)}{t^2}dt\right|+\left\|h'\right\|\right), \tag{99}$$

for $\delta\leq\theta\leq\pi/2$.

On the other hand, since $\alpha=-1/2$, we have

$$\max_{\delta\leq\eta\leq\pi/2}\left|\int_\delta^\pi\frac{\Delta_t^2 h(\eta)}{t^2}dt\right|+\left\|h'\right\|\asymp\max_{\delta\leq\eta\leq\pi/2}\left|\frac{1}{w^{(\alpha,\beta)}(\eta)}\int_\delta^\pi\frac{\Delta_{2t}\left(h'(\eta)w^{(\alpha,\beta)}(\eta)\right)}{\tan t/2}dt\right|+\left\|h'\right\|. \tag{100}$$

We know that, for $g(\theta)=f(\cos\theta)$,

$$J_{\alpha,\beta,1}(f,\delta)\asymp\max_{\delta\leq\theta\leq\pi/2}\left|\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_\theta^\pi(g(t)-g(0))\left(\frac{w^{(\alpha,\beta)}(t)\sin t}{1-\cos t}\right)'dt\right|. \tag{101}$$

Lemma 10 shows that the last three estimates imply that

$$J_{\alpha,\beta,1}(f,\delta)\leq C\left(\|f-H\|+\delta\max_{\delta\leq\theta\leq\pi/2}\left|\frac{1}{w^{(\alpha,\beta)}(\theta)}\int_\theta^\pi(h(t)-h(0))\times\left(\frac{w^{(\alpha,\beta)}(t)\sin t}{1-\cos t}\right)'dt\right|\right)+C\delta\left\|h'\right\|$$
$$\leq C\left(\|f-H\|+\delta\max_{\delta\leq\eta\leq\pi/2}\left|\frac{1}{w^{(\alpha,\beta)}(\eta)}\int_\delta^\pi\frac{\Delta_{2t}\left(h'(\eta)w^{(\alpha,\beta)}(\eta)\right)}{\tan t/2}dt\right|+\delta\left\|h'\right\|\right)$$
$$\leq C\left(\|f-H\|+\delta\left\|h'\right\|+\delta\left\|\frac{\widehat{h'w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}}\right\|_{[\delta,\pi-\delta]}\right). \tag{102}$$

So by Lemma 7 (see (78)) and Lemma 11 we obtain again

$$J_{\alpha,\beta,1}(f,\delta)\leq CK_{\alpha,\beta}(f,\delta). \tag{103}$$

If $-1<\alpha<-1/2$, then, due to $|f(1)-f(t)|\leq C\omega_\varphi(f,\sqrt{1-t})$ (see (2.6) of [9]), we have

$$J_{\alpha,\beta,1}(f,\delta)\leq C\omega_\varphi(f,\delta)\leq C\inf_{H\in C^2[-1,1]}\left(\|f-H\|+\delta\left\|h'\right\|\right)\leq CK_{\alpha,\beta}(f,\delta). \tag{104}$$

Thus, (103) holds for all $-1<\alpha<0$.

In the same way one can verify that

$$J_{\alpha,\beta,2}(f,\delta)\leq CK_{\alpha,\beta}(f,\delta). \tag{105}$$

To complete the proof we have to verify that

$$K_{\alpha_0,\beta_0}(f,\delta)\leq CK_{\alpha,\beta}(f,\delta), \tag{106}$$

where $-1<\alpha_0,\beta_0<-1/2$ as given by this theorem. Write $g(\theta)=f(\cos\theta)$. Theorem 2 shows that

$$K_{\alpha_0,\beta_0}(f,\delta)\asymp\max_{\delta\leq\theta\leq\pi-\delta}\left|\int_\delta^{\sin\theta}\frac{\Delta_u^2 g(\theta)}{u^2}du\right|+\omega(g,\delta). \tag{107}$$

Clearly, we have

$$\omega(g,\delta) \le C \inf_{H\in C^2} \left(\|f-H\| + \delta \|h'\|\right) \le CK_{\alpha,\beta}(f,\delta), \tag{108}$$

where $h(\theta) = H(\cos\theta)$. We need only to prove that

$$\max_{\delta\le\theta\le\pi-\delta} \left| \int_{\delta}^{\sin\theta} \frac{\Delta_u^2 g(\theta)}{u^2} du \right| \le CK_{\alpha,\beta}(f,\delta). \tag{109}$$

Finally, let $t \in \pi_n$ be even and $n = 1/\delta$. Notice that, for $\alpha = -1/2$,

$$\begin{aligned} &\left| \frac{1}{w^{(\alpha,\beta)}(\theta)} \int_{\theta}^{\pi} t'(u) \frac{w^{(\alpha,\beta)}(u)\sin u}{1-\cos u} du \right| \\ &\quad \le C \left| \int_{\theta}^{\pi} \frac{\Delta_u^2 t(0)}{u^2} du \right| + C \|t'\|. \end{aligned} \tag{110}$$

It follows from Lemmas 10 and 9 that

$$\begin{aligned} &\max_{1/n\le\theta\le\pi/2} \left| \int_{1/n}^{\sin\theta} \frac{\Delta_u^2 t(\theta)}{u^2} du \right| \\ &\quad \le C \max_{1/n\le\theta\le\pi/2} \left| \int_{1/n}^{\sin\theta} \frac{\Delta_u^2 t(\theta)}{u^2} du + \frac{1}{w^{(\alpha,\beta)}(\theta)} \right. \\ &\qquad \left. \times \int_{\theta}^{\pi} t'(u) \frac{w^{(\alpha,\beta)}(u)\sin u}{1-\cos u} du \right| + C\|t'\|. \end{aligned} \tag{111}$$

If $-1/2 < \alpha < 0$, then (94) implies (111). If $-1 < \alpha < -1/2$, then

$$\max_{1/n\le\theta\le\pi/2} \left| \frac{1}{w^{(\alpha,\beta)}(\theta)} \int_{\theta}^{\pi} t'(u) \frac{w^{(\alpha,\beta)}(u)\sin u}{1-\cos u} du \right| \le C\|t'\|. \tag{112}$$

Thus, (111) is valid for all $-1 < \alpha < 0$. Consequently, now let $t \in \pi_n$ be the best approximation of $h(\theta) = H(\cos\theta)$ and $H \in C^2[-1,1]$. From (111) and Lemmas 8 and 9, we have

$$\frac{1}{n} \max_{1/n\le\theta\le\pi/2} \left| \int_{1/n}^{\sin\theta} \frac{\Delta_u^2 t(\theta)}{u^2} du \right| \le \frac{C}{n} \left(\|h'\| + \left\| \frac{\widehat{h' w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}} \right\| \right). \tag{113}$$

When we use Lemma 9 for $\pi - \theta$, the above is true for $\pi/2 \le \theta \le \pi - 1/n$ instead of $1/n \le \theta \le \pi/2$. Thus,

$$\frac{1}{n} \max_{1/n\le\theta\le\pi-1/n} \left| \int_{1/n}^{\sin\theta} \frac{\Delta_u^2 h(\theta)}{u^2} du \right| \le \frac{C}{n} \left(\|h'\| + \left\| \frac{\widehat{h' w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}} \right\| \right). \tag{114}$$

Therefore,

$$\begin{aligned} &\frac{1}{n} \max_{1/n\le\theta\le\pi-1/n} \left| \int_{1/n}^{\sin\theta} \frac{\Delta_u^2 g(\theta)}{u^2} du \right| \\ &\quad \le C\|f-H\| + \frac{C}{n} \left(\|h'\| + \left\| \frac{\widehat{h' w^{(\alpha,\beta)}}}{w^{(\alpha,\beta)}} \right\| \right), \end{aligned} \tag{115}$$

which gives (109). □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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