Research Article

On Exact Series Solution for Strongly Coupled Mixed Parabolic Boundary Value Problems

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This paper continues with the construction of the exact solution for parabolic coupled systems of the type $u_t = Au_{xx}$, $A_1 u(0,t) + B_1 u_x(0,t) = 0$, $A_2 u(l,t) + B_2 u_x(l,t) = 0$, $0 < x < 1$, $t > 0$, and $u(x,0) = f(x)$, where A_1, A_2, B_1, and B_2 are arbitrary matrices for which the block matrix $\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}$ is nonsingular, and A is a positive stable matrix. Although this problem has been solved in the literature (Soler et al., 2013), in this work we are using completely new conditions.

1. Introduction

Coupled partial differential systems with coupled boundary-value conditions are frequent in different areas of science and technology, as in chemical physics [1–3], in scattering problems in quantum mechanics [4–6], thermoelastoplastic modelling [7], coupled diffusion problems [8–10], and so forth.

In [11], eigenfunctions of problems of the type

$$u_t(x,t) = Au_{xx}(x,t), \quad 0 < x < 1, \ t > 0,$$
$$A_1 u(0,t) + B_1 u_x(0,t) = 0 \in \mathbb{C}^m, \quad t > 0, \quad (1)$$
$$A_2 u(1,t) + B_2 u_x(1,t) = 0 \in \mathbb{C}^m, \quad t > 0,$$

where the unknown $u = (u_1, u_2, \ldots, u_m)^T$ is a m-dimensional vector, are constructed under the following hypotheses.

(i) The matrix coefficient A is a matrix which satisfies the following condition:

$$\mathrm{Re}(z) > 0 \text{ for all eigenvalues } z \text{ of } A, \quad (2)$$

and thus A is a positive stable matrix (where $\mathrm{Re}(z)$ denotes the real part of $z \in \mathbb{C}$).

(ii) Matrices A_i, B_i, $i = 1, 2$, are $m \times m$ complex matrices, and we assume that the block matrix

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \text{ is regular}, \quad (3)$$

and also the matrix pencil

$$A_1 + \rho B_1 \text{ is regular}. \quad (4)$$

Observe that condition (4) involves the existence of some $\rho_0 \in \mathbb{C}$, matrix $A_1 + \rho_0 B_1$ being invertible [12].

In order to construct the eigenfunctions in [11], the following matrices $\widetilde{A}_1$ and $\widetilde{B}_1$ were defined by

$$\widetilde{A}_1 = (A_1 + \rho_0 B_1)^{-1} A_1, \qquad \widetilde{B}_1 = (A_1 + \rho_0 B_1)^{-1} B_1, \quad (5)$$

thus satisfying the condition

$$\widetilde{A}_1 + \rho_0 \widetilde{B}_1 = I, \quad (6)$$

where matrix I denotes, as usual, the identity matrix. Under hypothesis (3), matrix $B_2 - (A_2 + \rho_0 B_2)\widetilde{B}_1$ is regular; see [11, page 431], and $\widetilde{A}_2$ and $\widetilde{B}_2$ are the matrices defined by

$$\widetilde{A}_2 = \left[B_2 - (A_2 + \rho_0 B_2)\widetilde{B}_1 \right]^{-1} A_2,$$
$$\widetilde{B}_2 = \left[B_2 - (A_2 + \rho_0 B_2)\widetilde{B}_1 \right]^{-1} B_2, \quad (7)$$

which satisfy the following conditions:

$$\widetilde{B}_2 - \left(\widetilde{A}_2 + \rho_0 \widetilde{B}_2\right) \widetilde{B}_1 = I, \qquad \widetilde{B}_2 \widetilde{A}_1 - \widetilde{A}_2 \widetilde{B}_1 = I. \quad (8)$$

In [11] authors also assumed the following essential hypothesis:

$$\text{exist } b_1 \in \sigma\left(\widetilde{B}_1\right) - \{0\}, \qquad b_2 \in \sigma\left(\widetilde{B}_2\right), \qquad v \in \mathbb{C}^m - \{0\},$$

$$\text{such that } \left(\widetilde{B}_1 - b_1 I\right) v = \left(\widetilde{B}_2 - b_2 I\right) v = 0, \quad (9)$$

where $\sigma(C)$ denotes the set of all the eigenvalues of a matrix C in $\mathbb{C}^{m \times m}$. These eigenfunctions introduced in [11] were also used in [13] to construct a series solution of the initial-value problem:

$$u_t(x,t) = A u_{xx}(x,t), \quad 0 < x < 1, \; t > 0, \quad (10)$$

$$A_1 u(0,t) + B_1 u_x(0,t) = 0, \quad t > 0, \quad (11)$$

$$A_2 u(1,t) + B_2 u_x(1,t) = 0, \quad t > 0, \quad (12)$$

$$u(x,0) = f(x), \quad 0 \le x \le 1, \quad (13)$$

where $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))^T$ is an m-dimensional vector, under the additional hypothesis:

$$\text{Ker}\left(\widetilde{B}_1 - b_1 I\right) \bigcap \text{Ker}\left(\widetilde{B}_2 - b_2 I\right) \quad (14)$$

is an invariant subspace with respect to matrix A,

where a subspace E of $\mathbb{C}^m$ is invariant by the matrix $A \in \mathbb{C}^{m \times m}$, if $A(E) \subset E$.

It is not difficult to show examples where this assumption (9) is held but (14) is not held. Let us consider the following example:

Example 1. We will consider the homogeneous parabolic problem with homogeneous conditions (10)–(13), where matrix $A \in \mathbb{C}^{4 \times 4}$ is chosen:

$$A = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 2 & 0 & -2 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (15)$$

and the 4×4 matrices $A_i, B_i, i \in \{1, 2\}$, are

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (16)$$

Due to (5)–(7) we obtain

$$\widetilde{A_1} = (A_1 + \rho_0 B_1)^{-1} A_1 = A_1,$$
$$\widetilde{B_1} = (A_1 + \rho_0 B_1)^{-1} B_1 = B_1, \quad (17)$$

$$\widetilde{A_2} = \left(B_2 - (A_2 + \rho_0 B_2) \widetilde{B_1}\right)^{-1} A_2$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (18)$$

$$\widetilde{B_2} = \left(B_2 - (A_2 + \rho_0 B_2) \widetilde{B_1}\right)^{-1} B_2$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to verify that $\sigma(\widetilde{B}_1) = \{0, 1\}$ and $\sigma(\widetilde{B}_2) = \{1, 0, -1\}$. If we take $b_1 = 1$, one gets

$$\text{Ker}\left(\widetilde{B}_1 - I\right) = \left\langle \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \right\rangle. \quad (19)$$

Taking into account that $\sigma(\widetilde{B}_2)$, we will have three possible values for b_2.

(i) For $b_2 = 1$ there is

$$\text{Ker}\left(\widetilde{B}_2 - I\right) = \left\langle \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \right\rangle, \quad (20)$$

and then condition (9) is not fulfilled.

(ii) For $b_2 = 0$, one gets

$$\text{Ker}\left(\widetilde{B}_2\right) = \left\langle \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \right\rangle. \quad (21)$$

Thus, condition (9) is satisfied with

$$v = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \in \text{Ker}\left(\widetilde{B}_1 - I\right) \bigcap \text{Ker}\left(\widetilde{B}_2\right) = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}. \quad (22)$$

Next, let us verify if subspace $\text{Ker}(\widetilde{B}_1 - I) \bigcap \text{Ker}(\widetilde{B}_2)$ is an invariant subspace for A. Let $x \in \text{Ker}(\widetilde{B}_1 - I) \bigcap \text{Ker}(\widetilde{B}_2)$; then x takes the form $x = \begin{pmatrix} 0 \\ \lambda \\ 0 \\ 0 \end{pmatrix}, \lambda \in \mathbb{C}$. In this case we obtain

$$Ax = \begin{pmatrix} \lambda \\ 2\lambda \\ 0 \\ 0 \end{pmatrix} \notin \text{Ker}\left(\widetilde{B}_1 - I\right) \bigcap \text{Ker}\left(\widetilde{B}_2\right). \quad (23)$$

Thus, condition (14) is not fulfilled.

(iii) For $b_2 = -1$ see

$$\operatorname{Ker}\left(\widetilde{B}_2 + I\right) = \left\langle \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \right\rangle. \tag{24}$$

Thus, condition (9) is fulfilled with $v = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \in \operatorname{Ker}(\widetilde{B}_1 -$

$I) \bigcap \operatorname{Ker}(\widetilde{B}_2 + I)$.

Now, we verify if subspace $\operatorname{Ker}(\widetilde{B}_1 - I) \bigcap \operatorname{Ker}(\widetilde{B}_2 + I)$ is an invariant subspace for A. Let $x \in \operatorname{Ker}(\widetilde{B}_1 - I) \bigcap \operatorname{Ker}(\widetilde{B}_2 + I)$; then x takes the form $x = \begin{pmatrix} \lambda \\ \lambda \\ 0 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{C}$. In this case we have

$$Ax = \begin{pmatrix} 3\lambda \\ 3\lambda \\ -\lambda \\ 0 \end{pmatrix} \notin \operatorname{Ker}\left(\widetilde{B}_1 - I\right) \bigcap \operatorname{Ker}\left(\widetilde{B}_2 + I\right). \tag{25}$$

Thus, condition (14) is not fulfilled.

Observe that, in this example, hypothesis (9) is satisfied but in (14) it is not satisfied. Thus, the method proposed in [13] cannot be used to solve this problem.

This paper deals with the construction of eigenfunctions of problem (10)–(12) by assuming hypotheses (2), (3), and (4) but not hypothesis (9). This set of eigenfunctions allows us to construct a series solution of the problem (10)–(13). We provide conditions for the function $f(x)$ and the matrix coefficients, in order to ensure the existence of a series solution of the problem.

The paper is organized as follows: in Section 2 a set of eigenfunctions will be constructed under a new condition, different from condition (9); in Section 3 a series solution for the problem is presented. In Section 4 we will introduce an algorithm and give an illustrative example.

Throughout this paper we will assume the results and nomenclature given in [11]. If B is a matrix in $\mathbb{C}^{n \times m}$, we denote by $B^\dagger$ its Moore-Penrose pseudoinverse [12]. A collection of examples, properties, and applications of this concept may be found in [14], and $B^\dagger$ can be efficiently computed with the *MATLAB* and *Mathematica* computer algebra systems.

2. The New Conditions

In Section 2 of [11] the eigenfunctions of problem (10)–(12) were constructed by using a matrix variable separation technique. We can repeat the calculations in this section to reach condition (44):

$$\lambda \cot(\lambda) \in \sigma\left(-\widetilde{A}_2 \widetilde{A}_1 - \lambda^2 \widetilde{B}_2 \widetilde{B}_1\right). \tag{26}$$

Instead of (9), we will assume the following new condition:

$$0 \in \sigma\left(\widetilde{B}_1\right), \qquad a_2 \in \sigma\left(\widetilde{A}_2\right),$$

and we have $w \in \mathbb{C}^m - \{0\}$, $\tag{27}$

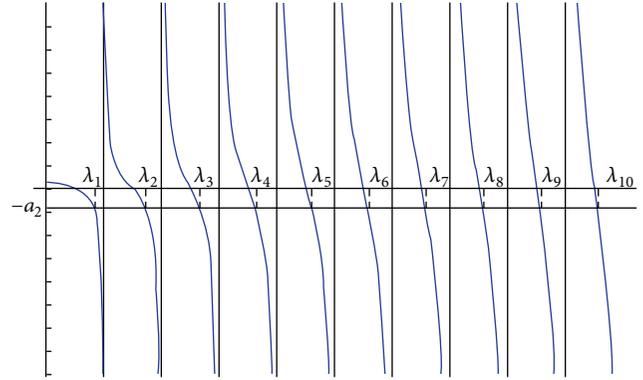being $\widetilde{B}_1 w = \left(\widetilde{A}_2 - a_2 I\right) w = 0$.

FIGURE 1: Graphical representation of $y = \lambda \cot(\lambda)$ and determination of the eigenvalues λ_n for $-a_2 < 0$.

From relation (6) $\widetilde{A}_1 w = w$ is obtained (because, obviously, $\widetilde{B}_1 w = 0$). Considering (8) $\widetilde{B}_2 w = w$ is obtained. Thus $(-\widetilde{A}_2 \widetilde{A}_1 - \lambda^2 \widetilde{B}_2 \widetilde{B}_1) w = -a_2 w$; that is, $-a_2$ is the eigenvalue which will be equal to $\lambda \cot(\lambda)$ in (26):

$$\lambda \cot(\lambda) = -a_2. \tag{28}$$

Let us assume that a_2 given in (28) satisfies

$$a_2 \in \mathbb{R}. \tag{29}$$

We will observe that, under hypothesis (29), we have guaranteed the existence of the solutions for the equation

$$\lambda \cot(\lambda) = -a_2. \tag{30}$$

Equation (30) has a unique solution λ_k in each interval $(k\pi, (k+1)\pi)$ for $k \geq 1$, as seen in Figure 1. Also, the following lemma is easily demonstrated.

Lemma 2. *Under hypothesis (29), the roots λ_k of (30) fulfil* $\lim_{n \to \infty} \lambda_n = +\infty$. *Also,*

$$\lim_{n \to \infty} \left| \sin\left(\lambda_n\right) \right| = 1, \qquad \lim_{n \to \infty} \cos\left(\lambda_n\right) = 0,$$
$$\lim_{n \to \infty} \left(\lambda_{n+1} - \lambda_n\right) = \pi. \tag{31}$$

Proof. Function $f(\lambda) = \lambda \cot(\lambda)$ has got vertical asymptotes at the points $\lambda = k\pi$, $k \in \mathbb{N}$ and $f(\lambda)$ has got zeros at the points $\lambda = \pi/2 + k\pi$, $k \in \mathbb{N}$. Therefore, as we have stated, the real coefficient function $-a_2$ intersects the graph of the function $f(\lambda)$ in each interval $(k\pi, (k+1)\pi)$, where $\lambda_k \in (k\pi, (k+1)\pi)$ is the point of intersection. Therefore, the sequence $\{\lambda_k\}_{k \geq 1}$ is monotone increasing with $\lim_{k \to \infty} \lambda_k = \infty$. First, we have to consider two possibilities.

(i) If $-a_2 > 0$, as can be seen in Figure 1, for k being large enough, $\lambda_k \in (k\pi, \pi/2 + k\pi)$.

(ii) If $-a_2 < 0$, as can be seen in Figure 1, for k being large enough, $\lambda_k \in (\pi/2 + k\pi, (k+1)\pi)$.

Therefore, observe that $\pi/2 < \lambda_{k+1} - \lambda_k < 3\pi/2$ for k being large enough. For λ_k, substituted in (30), we get

$$\lambda_k \cot\left(\lambda_k\right) = -a_2, \tag{32}$$

by dividing by λ_k. By taking limits where $k \rightarrow \infty$, $\lim_{k \rightarrow \infty} \cot(\lambda_k) = 0$ is obtained, and in this way the sequence $\{\sin(\lambda_k)\}_{k \geq 1}$ is bounded, $\lim_{k \rightarrow \infty} \cos(\lambda_k) = 0$, and $\lim_{k \rightarrow \infty} |\sin(\lambda_k)| = 1$. Moreover,

$$\cot(\lambda_{k+1} - \lambda_k) = \frac{\cot(\lambda_{k+1})\cot(\lambda_k) + 1}{\cot(\lambda_k) - \cot(\lambda_{k+1})}, \quad (33)$$

so, considering limits where $k \rightarrow \infty$, one gets $\lim_{k \rightarrow \infty} \cot(\lambda_{k+1} - \lambda_k) = \infty$, and the sequence $\{\cos(\lambda_{k+1} - \lambda_k)\}_{k \geq 1}$ is also bounded. Moreover $\lim_{k \rightarrow \infty} \sin(\lambda_{k+1} - \lambda_k) = 0$, and with $\pi/2 < \lambda_{k+1} - \lambda_k < 3\pi/2$, one gets that $\lim_{k \rightarrow \infty}(\lambda_{k+1} - \lambda_k) = \pi$.

Finally, if $a_2 = 0$, (30) is reduced to $\lambda \cot(\lambda) = 0$, whose roots are $\lambda_k = \pi/2 + k\pi$, $k \in \mathbb{N}$, and trivially $\lambda_{k+1} - \lambda_k = \pi$, then $\lim_{k \rightarrow \infty}(\lambda_{k+1} - \lambda_k) = \pi$. $\qquad \square$

Observe that under hypothesis $-a_2 < 1$ there is a root $\lambda_0 \in (0, \pi)$, and we can define the set of eigenvalues of the problem (10)–(12) as

$$\mathscr{F} = \{\lambda_k \in (k\pi, (k+1)\pi) ; \lambda_k \cot(\lambda_k) = -a_2, \ k \geq 1\} \cup \mathscr{F}_0, \quad (34)$$

where

$$\mathscr{F}_0 = \begin{cases} \emptyset, & \text{if } -a_2 > 1 \\ \{0\}, & \text{if } -a_2 = 1 \\ \lambda_0 \in (0, \pi), & \text{if } -a_2 < 1. \end{cases} \quad (35)$$

Thus, by [11, page 433] a set of solutions of problem (10)–(12) is given by

$$u(x, t, \lambda_k) = e^{-\lambda_k A t}\left\{\sin(\lambda_k x)\widetilde{A}_1 - \lambda_k \cos(\lambda_k x)\widetilde{B}_1\right\}C(\lambda_k),$$
$$\lambda_k \in \mathscr{F}, \quad (36)$$

where $C(\lambda_k)$ satisfies

$$\left(\widetilde{B}_1 A^j - A^j \widetilde{B}_1\right)C(\lambda_k) = 0, \quad 0 < j < p,$$
$$\left(-a_2 I + \widetilde{A}_2 \widetilde{A}_1 + \lambda^2 \widetilde{B}_2 \widetilde{B}_1\right)A^j C(\lambda_k) = 0, \quad 0 \leq j < p, \quad (37)$$

and where p is the degree of minimal polynomial of A. Formulas (37) are equivalent to the matrix equation:

$$G_{\lambda_k}(\rho_0) C(\lambda_k) = 0, \quad (38)$$

where

$$G_{\lambda_k}(\rho_0) = \begin{pmatrix} \widetilde{B}_1 A - A\widetilde{B}_1 \\ \vdots \\ \widetilde{B}_1 A^{p-1} - A^{p-1}\widetilde{B}_1 \\ \widetilde{A}_2 \widetilde{A}_1 + \lambda_k^2 \widetilde{B}_2 \widetilde{B}_1 - a_2 I \\ \left(\widetilde{A}_2 \widetilde{A}_1 + \lambda_k^2 \widetilde{B}_2 \widetilde{B}_1 - a_2 I\right)A \\ \vdots \\ \left(\widetilde{A}_2 \widetilde{A}_1 + \lambda_k^2 \widetilde{B}_2 \widetilde{B}_1 - a_2 I\right)A^{p-1} \end{pmatrix}. \quad (39)$$

In order to ensure that $C(\lambda_k) \neq 0$ fulfils (37) we have

$$\text{rank } G_{\lambda_k}(\rho_0) < m, \quad (40)$$

and under condition (40), the solution of (37) is given by

$$C(\lambda_k) = \left(I - G_{\lambda_k}(\rho_0)^{\dagger} G_{\lambda_k}(\rho_0)\right)S, \quad S \in \mathbb{C}^m. \quad (41)$$

The eigenfunctions associated to the problem (10)–(12) are then given by

$$u(x, t, \lambda_k) = e^{-\lambda_k A t}\left\{\sin(\lambda_k x)\widetilde{A}_1 - \lambda_k \cos(\lambda_k x)\widetilde{B}_1\right\}C(\lambda_k),$$
$$\lambda_k \in \mathscr{F}. \quad (42)$$

Working in a similar form to that in [11, page 433], we can show that also $\lambda = 0$ is an eigenvalue of problem (10), if

$$1 \in \sigma\left(-\widetilde{A}_2 \widetilde{A}_1\right). \quad (43)$$

Under hypothesis (43), if we denote

$$C(0) = \left(I - G_0(\rho_0)^{\dagger} G_0(\rho_0)\right)S, \quad S \in \mathbb{C}^m, \quad (44)$$

one gets that function

$$u(x, 0) = \left(x\widetilde{A}_1 - \widetilde{B}_1\right)C(0) \quad (45)$$

is an eigenfunction of problem (10) associated to eigenvalue $\lambda = 0$.

As a conclusion, the following theorem has been demonstrated.

Theorem 3. *Consider problem* (10)–(12) *which fulfils conditions* (2) *and* (3). *Let p be the degree of the minimal polynomial of matrix A and let ρ_0 be a real number satisfying* (4). *Let $\widetilde{A}_1$, $\widetilde{B}_1$ be the matrices defined by* (5) *and $\widetilde{A}_2$, $\widetilde{B}_2$ by* (7), *respectively.*

(1) *Let us assume conditions* (27) *and* (29). *Then* (28) *admits a set of real positive solutions denoted by $\mathscr{F}$ and defined by* (34). *Let $G_{\lambda_k}(\rho_0)$ be the matrix defined by* (39) *where $\lambda_k \in \mathscr{F}$ and suppose that condition* (40) *is fulfilled. Then, problem* (10)–(12) *admits eigenfunctions $u(x, t, \lambda_k)$ associated to positive eigenvalues $\lambda_k \in \mathscr{F}$ and defined by* (42), *where $C(\lambda_k)$ is given by* (41) *where S is a vector in $\mathbb{C}^m$.*

(2) $\lambda = 0$ *is an eigenvalue of problem* (10)–(12) *if condition* (43) *is fulfilled. Under hypothesis* (43), *if $G_0(\rho_0) = \widetilde{A}_2 \widetilde{A}_1 + I$, then expression* (45) *provides eigenfunction of problem* (10)–(12) *associated to eigenvalue $\lambda = 0$.*

3. A Series Solution

By assuming superposition principle, a possible series solution of problem (10)–(13) is given by

$$u(x, t) = \begin{cases} u(x, 0) + \displaystyle\sum_{\lambda_n \in \mathscr{F}} u(x, t, \lambda_k), & 0 \in \mathscr{F}, \\ \displaystyle\sum_{\lambda_n \in \mathscr{F}} u(x, t, \lambda_k), & 0 \notin \mathscr{F}, \end{cases} \quad (46)$$

where $u(x, t, \lambda_k)$ and $u(x, 0)$ are defined by (42) and (45), respectively, for suitable vectors $C(\lambda_n)$ and $C(0)$.

Assuming that series (46) and the corresponding derivatives $u_x(x, t)$, $u_{xx}(x, t)$, and $u_t(x, t)$ are convergent (we will demonstrate this later), (46) will be a solution of (10)–(12). Now, we need to determine vectors $C(\lambda)$ and $C(0)$ so that (46) satisfies (13).

Note that, by taking w to fulfil (27), from (8) we have

$$\widetilde{A}_2 w = a_2 w, \qquad \widetilde{A}_1 w = w. \tag{47}$$

Under condition (47), we will consider the scalar Sturm-Liouville problem:

$$X''(x) + \lambda^2 X(x) = 0,$$
$$X(0) = 0, \tag{48}$$
$$a_2 X(1) + X'(1) = 0,$$

which provides a family of eigenvalues $\mathscr{F}$ given in (34). Then, the associated eigenfunctions are

$$X_{\lambda_n}(x) = \sin(\lambda_n x), \quad \lambda_n > 0,$$
$$X_0(x) = x, \quad \text{if } \lambda_0 = 0. \tag{49}$$

According to the theorem of convergence of Sturm-Liouville for functional series [15, chapter 11], with the initial condition $f(x) = (f_1(x), \dots, f_m(x))^t$ given in (13) which fulfils the following properties:

$$f \in \mathscr{C}^2([0, 1]),$$
$$f(0) = 0, \tag{50}$$
$$a_2 f(1) + f'(1) = 0,$$

each component f_i of f, for $1 \le i \le m$, has got a series expansion which converges absolutely and uniformly to the interval $[0, 1]$; namely,

$$f_i(x) = \alpha x e_{0i} + \sum_{\lambda_n \in \mathscr{F}} \sin(\lambda_n x) e_{\lambda_n i}, \tag{51}$$

where

$$\alpha = \begin{cases} 1, & \text{si } \lambda_0 = 0, \\ 0, & \text{si } \lambda_0 \ne 0, \end{cases}$$

$$e_{0i} = \frac{\int_0^1 x f_i(x)\, dx}{\int_0^1 x^2\, dx} \quad \text{if } \lambda_0 = 0, \tag{52}$$

$$e_{\lambda_n i} = \frac{\int_0^1 \sin(\lambda_n x) f_i(x)\, dx}{\int_0^1 (\sin(\lambda_n x))^2\, dx} \quad \text{if } \lambda_n > 0.$$

Thus,

$$f(x) = \alpha x E(0) + \sum_{\lambda_n \in \mathscr{F}} \sin(\lambda_n x) E(\lambda_n), \tag{53}$$

where $E(0) = \begin{pmatrix} e_{01} \\ \vdots \\ e_{0m} \end{pmatrix}$ and $E(\lambda_n) = \begin{pmatrix} e_{\lambda_n 1} \\ \vdots \\ e_{\lambda_n m} \end{pmatrix}$. On the other hand, from (46) and taking into account (42) and (45), we obtain

$$u(x, 0) = \alpha \left(x\widetilde{A}_1 - \widetilde{B}_1 \right) C(0) + \sum_{\lambda_n \in \mathscr{F}} \left(\sin(\lambda_n x) \widetilde{A}_1 - \lambda_n \cos(\lambda_n x) \widetilde{B}_1 \right) C(\lambda_n). \tag{54}$$

We can equate the two expressions (53) and (54) if $C(0)$ and $C(\lambda_n)$, apart from conditions (41) and (44), satisfy $\{C(0), C(\lambda)\} \subset \mathrm{Ker}(\widetilde{B}_1)$. Then, we have

$$C(\lambda_n) = E(\lambda_n) = \frac{\int_0^1 \sin(\lambda_n x) f(x)\, dx}{\int_0^1 (\sin(\lambda_n x))^2\, dx}, \quad \text{if } \lambda_n > 0,$$
$$\tag{55}$$
$$C(0) = E(0) = \frac{\int_0^1 x f(x)\, dx}{\int_0^1 x^2\, dx} \quad \text{if } \lambda_0 = 0.$$

Note $C(0)$ and $C(\lambda) \in \mathrm{Ker}(\widetilde{B}_1)$, if

$$f(x) \in \mathrm{Ker}\left(\widetilde{B}_1\right). \tag{56}$$

Then $u(x, t)$ is defined by

$$u(x, t) = \alpha x C(0) + \sum_{\lambda_n \in \mathscr{F}} e^{-\lambda_n^2 A t} \sin(\lambda_n x) C(\lambda_n), \tag{57}$$

where α and $C(\lambda_n)$ are defined by (52) and (55), respectively, and $u(x, y)$ fulfils the initial condition (13). Note that conditions (37)–(40) are held if

$$A^j f(x) \in \mathrm{Ker}\left(\widetilde{B}_1\right) \cap \mathrm{Ker}\left(\widetilde{A}_2 - a_2 I\right), \quad 0 \le j < p - 1. \tag{58}$$

Condition (58) is equivalent to

$$f(x) \in \mathrm{Ker}\left(\widetilde{B}_1\right) \cap \mathrm{Ker}\left(\widetilde{A}_2 - a_2 I\right), \quad 0 \le x \le 1,$$
$$\mathrm{Ker}\left(\widetilde{B}_1\right) \cap \mathrm{Ker}\left(\widetilde{A}_2 - a_2 I\right) \tag{59}$$

is an invariant subspace with respect to matrix A.

The study of the convergence of the series solution (57) with α defined by (52) and $C(\lambda_n)$ by (55), by using Lemma 2, can be reduced to that made in [13] for the case $b_1 = 0$. Similarly, independence of the series solution (57) with respect to the chosen $\rho_0 \in \mathbb{R}$ can be demonstrated with the same technique as given in [16].

We can summarize the results in the following theorem.

Theorem 4. *Consider the homogeneous problem with homogeneous conditions (10)–(13) under hypotheses given in Theorem 3. Assume that function f of (13) fulfils conditions (50) and (59). Then, the series defined in (46) is a solution of problem (10)–(13).*

Input data: $A, A_1, A_2, B_1, B_2 \in \mathbb{C}^{m \times m}, f(x) \in \mathbb{C}^m$.
Result: $u(x, t)$.

(1) Verify that matrix A satisfies (2).
(2) Verify that matrices $A_i, B_i \in \mathbb{C}^{m \times m}, i \in \{1, 2\}$ are singular, and check that the block matrix $\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}$ is regular.
(3) Determine a number $\rho_0 \in \mathbb{R}$ so that the matrix pencil $A_1 + \rho_0 B_1$ is regular.
(4) Determine matrices $\widetilde{A_1}$ and $\widetilde{B_1}$ defined in (5).
(5) Determine matrices $\widetilde{A_2}$ and $\widetilde{B_2}$ defined in (7).
(6) Consider the following cases:
    (i) **Case 1:** Condition (27) is fulfilled, that is, matrices $\widetilde{B_1}$ and $\widetilde{A_2}$ have a common eigenvector $w \neq 0$ associated with
        eigenvalues $0 \in \sigma\left(\widetilde{B_1}\right)$ and $a_2 \in \sigma\left(\widetilde{A_2}\right)$. In this case we will continue with step (7).
    (ii) **Case 2:** Condition (27) is not fulfilled. In this case the algorithm is not completed because it is not possible to find the
        solution of (10)–(13) for the given data.
(7) Take $0 \in \sigma\left(\widetilde{B_1}\right)$ and determine $a_2 \in \sigma\left(\widetilde{A_2}\right) \bigcap \mathbb{R}$, and vector $w \neq 0$ when $w \in \mathrm{Ker}\left(\widetilde{B_1}\right) \bigcap \mathrm{Ker}\left(\widetilde{A_2}\right)$ in a way that:
    (i) Conditions (59) fulfilled, that is:
    (1.1) $\mathrm{Ker}\left(\widetilde{B_1}\right) \bigcap \mathrm{Ker}\left(\widetilde{A_2} - a_2 I\right)$ is an invariant subspace respect to matrix A.
    (1.2) $f(x) \in \mathrm{Ker}\left(\widetilde{B_1}\right) \bigcap \mathrm{Ker}\left(\widetilde{A_2} - a_2 I\right), \forall x \in [0, 1]$.
    (ii) The vectorial function $f(x)$ fulfils (50), that is:
    (1.3) $f \in \mathscr{C}^2\left([0, 1]\right)$.
    (1.4) $f(0) = 0$.
    (1.5) $a_2 f(1) + f'(1) = 0$.
If these conditions are not satisfied, go back over step (6) of Algorithm (1) and discard the value taken for a_2.
(8) Determine the positive solutions of (30) and determine $\mathscr{F}$ defined in (34).
(9) Determine degree p of minimal polynomial of matrix A.
(10) Build block matrix $G_{\lambda_k}(\rho_0)$ defined in (39).
(11) Determine $\lambda_k \in \mathscr{F}$ so that rank $G_{\lambda_k}(\rho_0) < m$.
(12) Include the eigenvalue $\lambda = 0$ if $1 \in \sigma\left(-\widetilde{A_2}\widetilde{A_1}\right)$.
(13) Determine α given in (52).
(14) Determine vectors $C(\lambda_n)$ defined in (55).
(15) Determine the series solution $u(x, t)$ of problem (10)–(13) defined in (57).

ALGORITHM 1: Solution of the homogeneous problem with homogeneous conditions (10)–(13).

4. Algorithm and Example

We can summarize the process to calculate the solution of the homogeneous problem with homogeneous conditions (10)–(13) in Algorithm 1.

Example 5. We will consider the homogeneous parabolic problem with homogeneous conditions (10)–(13), given in Example 1, that is, where the matrix $A \in \mathbb{C}^{4 \times 4}$ is given in (15) and matrices $A_i, B_i, i \in \{1, 2\}$ are given in (16). We consider the vector-valued function $f(x)$ to be defined as

$$f(x) = \begin{pmatrix} 0 \\ 0 \\ x^2 - 2x \\ 0 \end{pmatrix}. \tag{60}$$

Observe that, as demonstrated in Example 1, hypothesis (9) is fulfilled but (14) is not fulfilled. Thus, the method proposed in [13] cannot be used to solve this problem.

Algorithm 1 (step by step). Consider

(1) Matrix A satisfies the condition (2), because $\sigma(A) = \{1, 2, 3\}$; that is, A is positive stable.

(2) Each of the matrices $A_i, B_i, i \in \{1, 2\}$ is singular, and the block matrix,

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right), \tag{61}$$

is regular.

(3) Note that although A_1 is singular, if we take $\rho_0 = 1 \in \mathbb{R}$, the matrix pencil,

$$A_1 + \rho_0 B_1 = I_{4 \times 4}, \tag{62}$$

is regular. Therefore, $\rho_0 = 1$.

(4) In (17) we have

$$\widetilde{A_1} = (A_1 + \rho_0 B_1)^{-1} A_1 = A_1,$$

$$\widetilde{B_1} = (A_1 + \rho_0 B_1)^{-1} B_1 = B_1. \tag{63}$$

(5) In (18) we have

$$\widetilde{A_2} = \left(B_2 - (A_2 + \rho_0 B_2)\, \widetilde{B_1}\right)^{-1} A_2$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{B_2} = \left(B_2 - (A_2 + \rho_0 B_2)\, \widetilde{B_1}\right)^{-1} B_2 \tag{64}$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(6) Also $\sigma(\widetilde{B_1}) = \{0,1\}$ and $\sigma(\widetilde{A_2}) = \{0,-1\}$. Note that in this case the condition (27) is fulfilled because with $0 \in \sigma(\widetilde{B_1})$ and $a_2 = 0 \in \sigma(\widetilde{A_2})$ there is a common eigenvector $w \in \mathbb{C}^4$, $w = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, and thus $\mathrm{Ker}(\widetilde{B_1}) \cap \mathrm{Ker}(\widetilde{A_2}) \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Therefore, we are in case 1 of **Algorithm 1**.

(7) We take the values $b_1 = 0$, $a_2 = 0$, and we will check the conditions given in step 7 of the algorithm:

(1.1)

$$\mathrm{Ker}\left(\widetilde{B_1}\right) \bigcap \mathrm{Ker}\left(\widetilde{A_2}\right) = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle. \tag{65}$$

Let $x \in \mathrm{Ker}(\widetilde{B_1}) \cap \mathrm{Ker}(\widetilde{A_2})$; then $x = \begin{pmatrix} 0 \\ 0 \\ \lambda \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{C}$. In this case we have

$$Ax = \begin{pmatrix} 0 \\ 0 \\ 2\lambda \\ 0 \end{pmatrix} \in \mathrm{Ker}\left(\widetilde{B_1}\right) \bigcap \mathrm{Ker}\left(\widetilde{A_2}\right), \tag{66}$$

and then the subspace $\mathrm{Ker}(\widetilde{B_1}) \cap \mathrm{Ker}(\widetilde{A_2})$ is invariant by matrix A.

It is trivial to verify the following:

(1.2)

$$f(x) \in \mathrm{Ker}\left(\widetilde{B_1}\right) \bigcap \mathrm{Ker}\left(\widetilde{A_2}\right), \quad \forall x \in [0,1]. \tag{67}$$

(1.3) $f(x) \in \mathscr{C}^2([0,1])$,

(1.4) $f(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$,

(1.5) $a_2 f(1) + f'(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$,

(8) Equation (30) is as follows:

$$\lambda \cot(\lambda) = 0. \tag{68}$$

We can solve (68) exactly, $\lambda_k = \pi/2 + k\pi$, with an additional solution $\lambda_0 \in]0, \pi[$, because

$$-a_2 = 0 < 1, \tag{69}$$

and then $\lambda_0 = \pi/2$. Thus, we have a numerable family of solutions for (68) which we denote $\mathscr{F}$, given by

$$\mathscr{F} = \left\{\lambda_k = \frac{\pi}{2} + k\pi; \lambda_k \in (k\pi, (k+1)\pi), k \geq 1\right\} \cup \mathscr{F}_0,$$

$$\mathscr{F}_0 = \left\{\lambda_0 = \frac{\pi}{2}\right\}. \tag{70}$$

(9) The minimal polynomial of matrix A is given by $p(x) = (x-1)^2(x-3)(x-2)$, and then $p = 4$.

(10) If λ_k is a positive solution of (68), the matrix $G_{\lambda_k}(\rho_0)$ given by (39) takes the form

$$G_{\lambda_k}(\rho_0) = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -2 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -5 \\
0 & 0 & 0 & -7 \\
4 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -18 \\
0 & 0 & 0 & -21 \\
13 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
-\lambda_k^2 & 0 & 0 & 0 \\
-\lambda_k^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\hline
-2\lambda_k^2 & -\lambda_k^2 & 0 & \lambda_k^2 \\
-2\lambda_k^2 & -\lambda_k^2 & 0 & \lambda_k^2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\hline
-5\lambda_k^2 & -4\lambda_k^2 & 0 & 5\lambda_k^2 \\
-5\lambda_k^2 & -4\lambda_k^2 & 0 & 5\lambda_k^2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\hline
-14\lambda_k^2 & -13\lambda_k^2 & 0 & 18\lambda_k^2 \\
-14\lambda_k^2 & -13\lambda_k^2 & 0 & 18\lambda_k^2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{71}$$

(11) Since the third column $G_{\lambda_k}(\rho_0)$ is zero, rank $G_{\lambda_k}(\rho_0) < 4$. Thus, each of the positive solutions given in (70) is an eigenvalue.

(12) It is trivial to verify that $1 \notin \sigma(-\widetilde{A}_2 \widetilde{A}_1)$, because

$$-\widetilde{A}_2 \widetilde{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \sigma\left(-\widetilde{A}_2 \widetilde{A}_1\right) = \{0\}, \quad (72)$$

and thus we will not include 0 as an eigenvalue.

(13) Taking into account that $\lambda_0 \neq 0$, $\alpha = 0$.

$$e^{Au} = \begin{pmatrix} e^{2u}\cosh(u) & e^{2u}\sinh(u) & 0 & \dfrac{1}{4}e^{u}\left(3 - 3e^{2u} + 2u\right) \\ e^{2u}\sinh(u) & e^{2u}\cosh(u) & 0 & -\dfrac{1}{4}e^{u}\left(-3 + 3e^{2u} + 2u\right) \\ -e^{2u}\sinh(u) & -\dfrac{1}{2}e^{u}(-1 + e^{u})^2 & e^{2u} & \dfrac{1}{4}e^{u}\left(1 + e^{u}\left(-4 + 3e^{u}\right) + 2u\right) \\ 0 & 0 & 0 & e^{u} \end{pmatrix}. \quad (74)$$

Next, by considering (74) with $u = -(\pi/2 + n\pi)^2 t$ and simplifying, we obtain the value of $e^{-(\pi/2 + n\pi)^2 At}$.

(16) Values of $C(\lambda_n)$ given in (73) are replaced in (57), and we take into account the value of the matrix $e^{-(\pi/2 + n\pi)^2 At}$. After simplification, we finally obtain the solution of problem (10)–(13) given by

$$u(x, t)$$

$$= \left(\sum_{n \geq 0} -\frac{32 e^{-(1/2)(\pi + 2n\pi)^2 t} \sin\left((1/2)(\pi + 2n\pi)x\right)}{\pi^3 (2n + 1)^3} \right)$$

$$\times \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad (x, t) \in [0, 1] \times [0, +\infty). \quad (75)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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(14) Vectors $C(\lambda_n)$ defined in (55) take the following values:

$$C(\lambda_n) = -\frac{32}{\pi^3 (2k + 1)^3} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \quad (73)$$

(15) When the minimal theorem [17, page 571] is used,

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