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Research Article

Algebraic Type Approximation to the Blasius Velocity Profile

Beong In Yun

Department of Statistics and Computer Science, Kunsan National University, Gunsan 573-701, Republic of Korea

Correspondence should be addressed to Beong In Yun; paulllyun@gmail.com

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For the Blasius velocity profile we propose a simple algebraic type approximate function which is uniformly accurate over the whole region. Moreover, for further improvement a correction method based on a weight function is introduced. The availability of the proposed method is shown by the result of numerical experiments.

1. Introduction

We consider the well-known Blasius problem

$$Nf(x) := 2f'''(x) + f(x)f''(x) = 0, \quad 0 \le x < \infty, \tag{1}$$

subject to the boundary conditions

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1. \tag{2}$$

The so-called Blasius function $f(x)$ describes the stream on the boundary layer over a flat plate. There are lots of analytical approximation methods to the Blasius function $f(x)$ such as the variational iteration method [1–6], the Adomian decomposition method [7–9], and the homotopy analysis method [10–12]. Recently, spectral methods based on orthogonal functions have been applied in approximation of solutions for the nonlinear boundary value problems like the Blasius problem [13–17]. In addition, numerical solutions of the nonlinear differential equations for the boundary layer problems such as Falkner-Skan equations, including the Blasius equation as a special case, have been studied by many researchers [18–25].

Concerning the streamwise velocity profile $f'(x)$, we note an approximate analytical solution proposed in the literature [26] of the form

$$\begin{aligned} f'_Y(x) = {} & \tanh(bx) \\ & + 2cr\,\mathrm{sech}^2(rx)\left\{\tanh^3(rx) + 2\tanh^7(rx)\right\}, \end{aligned} \tag{3}$$

where b and c are determined by the known properties of the Blasius function $f(x)$ at the wall $x = 0$ and far from the wall, respectively. The parameter r is chosen by minimizing the residual function

$$Nf_Y(x) = 2f_Y'''(x) + f_Y(x)f_Y''(x). \tag{4}$$

Recently, Savaş [27] introduced another approximate analytical solution for the streamwise velocity profile as

$$f'_{S,n}(x) = \left(\tanh\left[(\alpha x)^n\right]\right)^{1/n}, \tag{5}$$

for the constants $(\alpha, n) = (0.33206, 3/2)$ or $(0.33245, 5/3)$.

In the next section, motivated by the analytical solutions (3) and (5), we propose another algebraic type approximate analytical solution for the velocity profile as given by (6) and explore its properties with a method to determine the parameters therein. In Section 3, by using an appropriate weight function, we introduce a correction method to improve the accuracy of the presented approximation. Moreover, for further improvement we employ an auxiliary term which appropriately reflects the error of the presented approximation. Some numerical experiments are performed to demonstrate the efficiency of the presented method.

2. Approximation to the Velocity Profile

To approximate the velocity profile $f'(x)$ directly we suggest an algebraic type analytical function as

$$f'_{A,m}(x) = \left[\frac{(ax)^m}{(ax)^m + 1}\right]^{1/m} \tag{6}$$

for a constant $a > 0$ and an exponent $m > 1$. We note that $f'_{A,m}$ satisfies the boundary conditions $f'_{A,m}(0) = 0$ and $f'_{A,m}(\infty) = 1$ given in (2), and its derivative is

$$f''_{A,m}(x) = \frac{f'_{A,m}(x)}{x\{(ax)^m + 1\}} = \frac{a}{\{(ax)^m + 1\}^{1+(1/m)}}. \quad (7)$$

Since $f''_{A,m}(0) = a$, we may set

$$a = f''(0) = 0.332057\ldots \quad (8)$$

which is a well-known Blasius constant [28]. In addition, the velocity profile $y = f'_{A,m}(x)$ has an inversion of a simple form as

$$x = \frac{1}{a}\left[\frac{y^m}{1-y^m}\right]^{1/m}, \quad 0 \le y < 1. \quad (9)$$

The related approximation $f_{A,m}$ to the Blasius stream function $f(x)$ can be obtained by the formula

$$f_{A,m}(x) = \int_0^x f'_{A,m}(t)\,dt = \int_0^x \left[\frac{(at)^m}{(at)^m + 1}\right]^{1/m} dt. \quad (10)$$

In fact, using the symbolic computational software Mathematica (version 9), one can find the analytical form of $f_{A,m}$ as

$$f_{A,m}(x) = \frac{a}{2}x^2 F\left(\frac{1}{m}, \frac{2}{m}, \frac{m+2}{m}; -(ax)^m\right), \quad (11)$$

where $F(p, q, r; z)$ is the hypergeometric function [29] whose series expansion is

$$F(p, q, r; z) = \sum_{k=0}^{\infty} \frac{(p)_k (q)_k}{(r)_k k!} z^k \quad (12)$$

and $(s)_k$ is the shifted factorial defined by

$$(s)_k = s(s+1)(s+2)\cdots(s+k-1), \quad k \ge 1 \quad (13)$$

with $(s)_0 = 1$.

For an appropriate parameter m in $f'_{A,m}(x)$ we may choose a value $m = m^*$ at which the L_2-norm of the residual function $Nf_{A,m}$,

$$\|Nf_{A,m}\|_2^2 = \int_0^{\infty} \{Nf_{A,m}(x)\}^2\,dx \quad (14)$$

is minimized. To find m^* one can use a package, Mathematica, for example, and we will obtain the local minimum in $\|Nf_{A,m}\|_2$ at the value $m^* \approx 4.216$.

Figure 1 shows the errors of the presented approximate velocity profile, $f'_{A,m}(x)$, with integers $m = 4$ and $m = 5$ near the value $m^* \approx 4.216$. The error means difference between $f'_{A,m}(x)$ and the numerical solution for the velocity profile $f'(x)$ which is regarded as an exact solution. By numerical experiments for various values of m, we can see that the accuracy of $f'_{A,m}(x)$ becomes better far from the wall $x = 0$ as m goes large while it becomes better near $x = 0$ as m goes small.

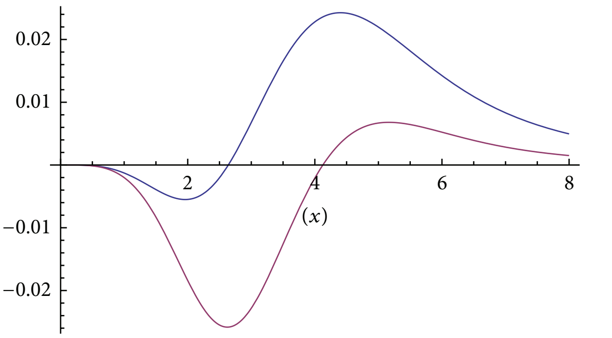


FIGURE 1: Errors of the presented approximate velocity profiles, $f'_{A,4}(x)$ (upper line) and $f'_{A,5}(x)$ (lower line).

3. Improvement by a Weighted Average

In order to improve the accuracy of the proposed approximate velocity profile over the whole region, we introduce a weighted average

$$\tilde{f}'_{A,m,n}(x) = (1-\theta(x)) f'_{A,m}(x) + \theta(x) f'_{A,n}(x) \quad (15)$$

for $1 < m < n$, where $\theta(x)$ is a weight function defined as

$$\theta(x) = \theta(b, k; x) = \frac{(x/b)^k}{(x/b)^k + 1}, \quad 0 \le x < \infty \quad (16)$$

for $b > 0$ and $k > 0$. It follows that $0 \le \theta(x) < 1$ for $0 \le x < \infty$ with $\theta(b) = 1/2$. Moreover, it should be noticed that for a sufficiently large k

$$\theta(x) \sim \begin{cases} 0, & \text{for } x < b, \\ 1, & \text{for } x > b \end{cases} \quad (17)$$

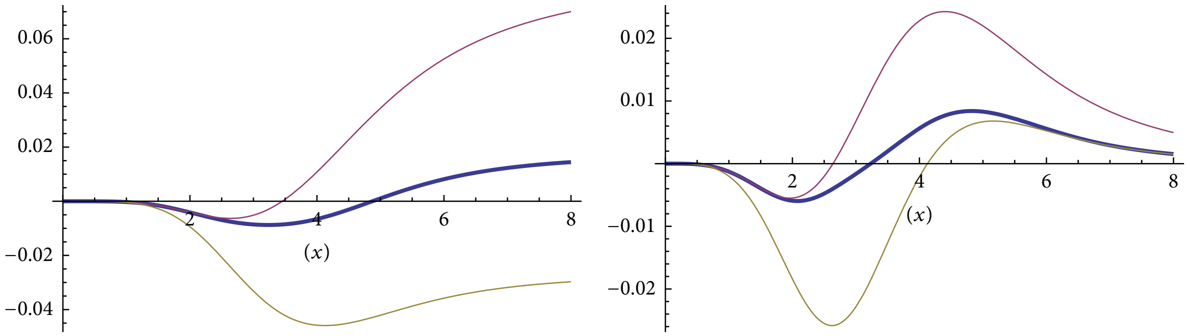
and thus

$$\tilde{f}'_{A,m,n}(x) \sim \begin{cases} f'_{A,m}(x), & \text{for } x < b, \\ f'_{A,n}(x), & \text{for } x > b. \end{cases} \quad (18)$$

This implies that the point $x = b$ plays the role of a threshold between two approximate velocity profiles $f'_{A,m}$ and $f'_{A,n}$. On the other hand, the related approximate stream function $\tilde{f}_{A,m,n}$ can be obtained by numerical integration in the equation

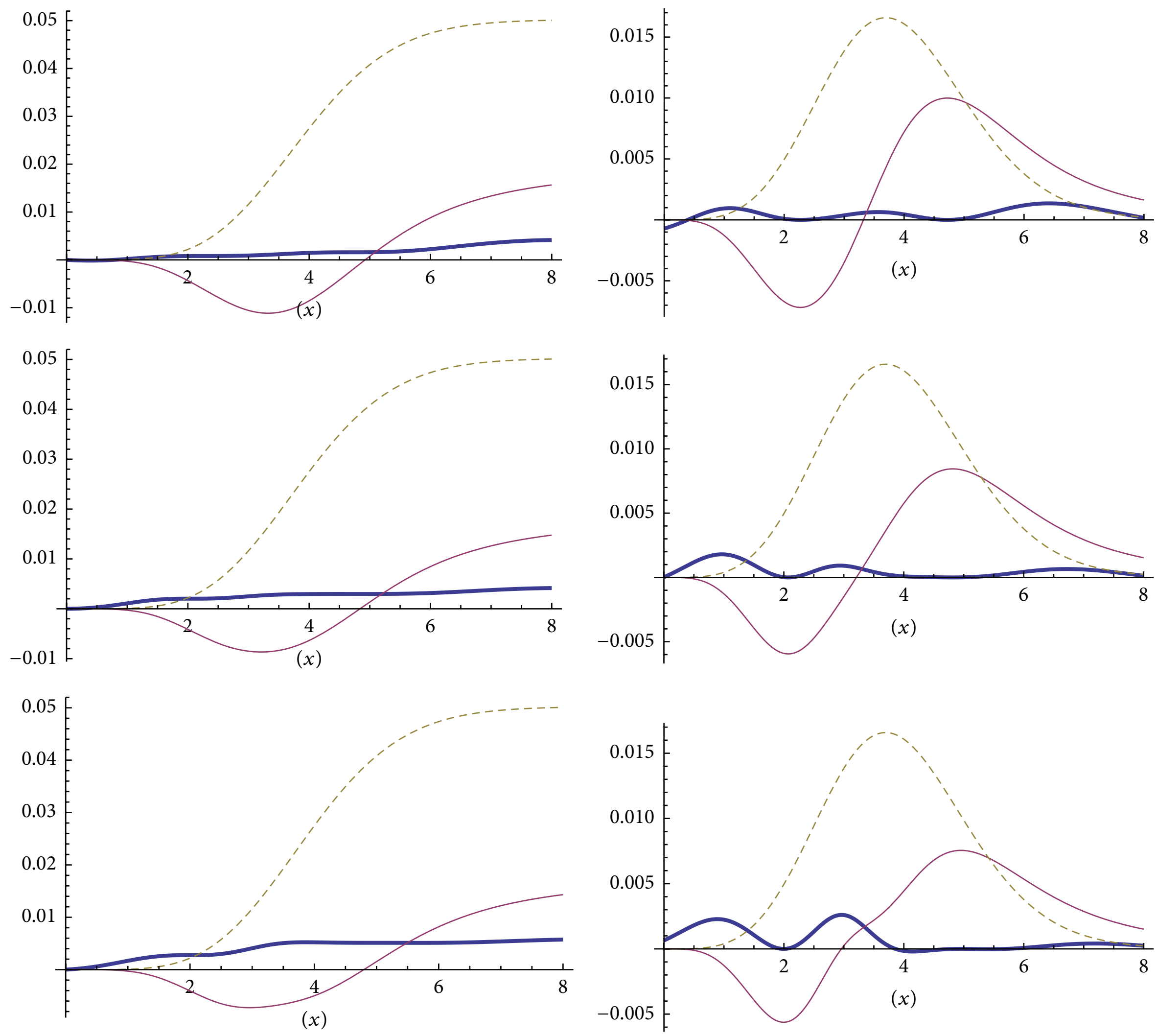
$$\tilde{f}_{A,m,n}(x) = \int_0^x \tilde{f}'_{A,m,n}(t)\,dt. \quad (19)$$

Referring to Figure 1 for the cases of $m = 4$ and $n = 5$, we may take $b = 3.5134$ in (16) which is a center of the points $x = 4.4033$ and 2.6234 at which $f_{A,4}(x)$ and $f_{A,5}(x)$, respectively, have the maximum absolute errors. Thick lines in Figure 2 indicate errors (i.e., differences from the numerical solution) of the corrected approximate stream function $\tilde{f}_{A,4,5}(x)$ and the velocity profile $\tilde{f}'_{A,4,5}(x)$ with $b = 3.5134$ and $k = 6$ in the weight function $\theta(x) = \theta(b, k; x)$. We can see that the maximum error is about 0.01 in the velocity profile and



(a) (b)

Figure 2: Errors of the corrected approximate stream function $\widetilde{f}_{A,4,5}(x)$ (in (a)) and the velocity profile $\widetilde{f}'_{A,4,5}(x)$ (in (b)), indicated by thick lines. In addition, errors of $f_{A,4}(x)$ and $f_{A,5}(x)$ and those of $f'_{A,4}(x)$ and $f'_{A,5}(x)$ are, respectively, included in (a) and (b), indicated by thin lines.



(a) (b)

Figure 3: Errors of the corrected approximate stream function $\widehat{f}_{A,4,5}(x)$ (in (a)) and the velocity profile $\widehat{f}'_{A,4,5}(x)$ (in (b)), indicated by thick lines, for the parameters $k = 4, 6, 8$ from the top row. Thin lines indicate errors of $\widetilde{f}_{A,4,5}(x)$ and $\widetilde{f}'_{A,4,5}(x)$ and dashed lines indicate Savas's approximations $f_{S,1.5}(x)$ and $f'_{S,1.5}(x)$.

Table 1: Numerical values of the constants c_1, c_2, M_1, and M_2 in (20) for the parameters $3 \le k \le 8$.

k	c_1	c_2	M_1	M_2
3	2.3510	4.7054	−0.008444	0.011006
4	2.2706	4.7209	−0.007185	0.009994
5	2.1595	4.7584	−0.006378	0.009134
6	2.0739	4.8167	−0.005948	0.008443
7	2.0241	4.8859	−0.005734	0.007923
8	1.9964	4.9526	−0.005624	0.007553

about 0.02 in the stream function. Comparing these with the errors of the approximate stream functions $f_{A,4}(x)$ and $f_{A,5}(x)$ in Figure 2(a) and the velocity profiles $f'_{A,4}(x)$ and $f'_{A,5}(x)$ in Figure 2(b), one can find distinct improvement of the corrected velocity profile $\tilde{f}'_{A,4,5}(x)$ defined in (15).

In practice, by numerical experiments, we can find better case of parameters like $(m, n) = (3.8, 5.8)$, for example, which results in more accurate approximation with the maximum errors about 0.003 and 0.005 in the velocity profile and the stream function, respectively. However, this choice of the parameters looks rather ambiguous. Thus, for development of plausible further improvement, we refer to the correction method proposed in the literature [30] which uses an auxiliary term reflecting the error of the presented approximation. First, observing the behavior of the error $E(x) = f'(x) - \tilde{f}'_{A,m,n}(x)$ given in Figure 2, for example, we can have the numerical values of the critical points $(c_1, M_1) = (c_1, E(c_1))$ and $(c_2, M_2) = (c_2, E(c_2))$ of $E(x)$. Then, to approximate $E(x)$ appropriately, we suggest a function $g_{m,n}(x) \approx E(x)$ of the form

$$g_{m,n}(x) = M(x - c)\, e^{-r(x-c)^2} + \overline{M}, \qquad (20)$$

where

$$M = \frac{M_2 - M_1}{2}, \qquad \overline{M} = \frac{M_1 + M_2}{2}, \qquad c = \frac{c_1 + c_2}{2}. \qquad (21)$$

The value of r in (20) can be determined by the condition $g'_{m,n}(c_1) = g'_{m,n}(c_2) = 0$ which implies that

$$r = \frac{2}{(c_2 - c_1)^2}. \qquad (22)$$

We consider a corrected approximation

$$\hat{f}'_{A,m,n}(x) = \tilde{f}'_{A,m,n}(x) + g_{m,n}(x). \qquad (23)$$

One may expect that the accuracy of $\hat{f}'_{A,m,n}(x)$ goes higher as $g_{m,n}(x)$ becomes closer to the error $E(x)$.

For example, for the case of $(m, n, k) = (4, 5, k)$, $3 \le k \le 8$, we can evaluate numerical values of the constants c_1, c_2, M_1, and M_2 as given in Table 1. Figure 3 shows errors of $\hat{f}_{A,4,5}(x)$ and the velocity profile $\hat{f}'_{A,4,5}(x)$ indicated by thick lines, compared with those of $\tilde{f}_{A,4,5}(x)$ and $\tilde{f}'_{A,4,5}(x)$ indicated by thin lines, for the parameters $k = 4, 6, 8$. Additionally, dashed lines indicate Savas's approximations $f_{S,1.5}(x)$ and $f'_{S,1.5}(x)$. We can find that $\hat{f}'_{A,4,5}(x)$ and $\hat{f}_{A,4,5}(x)$ with $k = 4, 6$ have the maximum errors about 0.002 and 0.005, respectively. This implies that the correction method (23) can highly improve the proposed method (15) as a result.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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