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Research Article

Existence of Wave Front Solutions of an Integral Differential Equation in Nonlinear Nonlocal Neuronal Network

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An integral-differential model equation arising from neuronal networks with very general kernel functions is considered in this paper. The kernel functions we study here include pure excitations, lateral inhibition, lateral excitations, and more general synaptic couplings (e.g., oscillating kernel functions). The main goal of this paper is to prove the existence and uniqueness of the traveling wave front solutions. The main idea we apply here is to reduce the nonlinear integral-differential equation into a solvable differential equation and test whether the solution we get is really a wave front solution of the model equation.

1. Introduction

1.1. The Model Equation and Its Biological Background. To describe and study the propagation of nerve impulses in synaptically coupled neuronal networks, some integral differential mathematical model equations have been proposed and their traveling wave solutions have been studied in the recent thirty years since traveling waves share the same properties as nerve impulses. In 1977, to study the dynamics of pattern formation in lateral-inhibition type homogeneous neural fields with general connections, Amari [1] derived the following nonlocal equation by statistical considerations:

$$u_t + u = \int_R K(x-y) H(u(y,t) - \theta)\, dy. \tag{1}$$

This model equation can be used to describe the simple fields which are one dimensional and homogeneous, have negligible time lag, and consist of only one layer. In this model equation, $u(x,t)$ represents the average membrane potential of the neurons located at position x and at time t, the nonlinear output function $H(u)$ is always chosen to be the Heaviside gain function: $H(u-\theta) = 0$ for all $u < \theta$, $H(0) = 1/2$, and $H(u-\theta) = 1$ for all $u > \theta$. Here $H(u-\theta)$ denotes the output firing rate of a neuron, which means that a neuron fires at its maximum rate when the potential exceeds a threshold and does not fire otherwise. $K(x)$ represents synaptic coupling between neurons in the tissue. The weight function $K(x)$, also named as kernel function now, considered in [1] is Mexican hat function because the field considered in his paper is of lateral inhibition type. See Atay and Hutt [2], Coombes et al. [3] Ermentrout [4] for more details of the biological background. Due to the finite propagation velocity, the following nonlocal nonlinear scalar integral model equation incorporating spatial temporal delay was proposed to describe the dynamics of an effective postsynaptic potential $u(x,t)$ at position x and time t:

$$u_t + u = \alpha \int_R K(x-y) H\left(u\left(y, t - \frac{1}{c}|x-y|\right) - \theta\right) dy + I(x,t), \tag{2}$$

where $0 < c \le \infty$, $\alpha > 0$, and $\theta > 0$ are constants. The parameter α denotes the synaptic rate constant in a neuronal network, θ represents the threshold for excitation of the neuronal network, c represents the finite propagation speed of action potentials along axons, and $|x-y|/c$ denotes the spatial temporal delay. There are also some related nonlinear singularly perturbed systems of integral-differential model equations proposed for the study of synaptically coupled neuronal networks. Pinto and Ermentrout [5, 6] proposed and studied the system:

$$u_t + u + w = \int_R K(x-y) H(u(y,t)-\theta)\, dy, \qquad w_t = \epsilon(u-\gamma w). \tag{3}$$

In [7], Hutt proposed and considered the model equation:

$$u_t + u = \alpha \int_R K(x-y) H\left(u\left(y, t-\frac{1}{c}|x-y|\right)-\theta\right) dy + \beta \int_R J(x-y) H(u(y,t-\tau)-\theta)\, dy, \tag{4}$$

which is a modification and generalization of some known models. Here c and τ are the propagation velocity of action potential and feedback delay, respectively. Hutt considered both the nonlocal axonal connections and nonlocal feedback connections with a time delay and examined briefly the dependence of the speed of the front on various parameters in this model equation. He also assumed the same firing threshold for all neurons function and thus the transfer function was chosen to be the Heaviside step function. Some more general model equations have been introduced and attracted a lot of research interest recently. See, for example, [2, 3, 5, 6, 8, 9].

Clearly, all these model equations (1)–(4) we mentioned above are related to the equation:

$$u_t + u = \alpha \int_R K(x-y) H\left(u\left(y, t-\frac{1}{c}|x-y|\right)-\theta\right) dy, \tag{5}$$

which can be derived by (2) when $I(x,t)$ is constant or by (4) when $\beta = 0$. As c approaches to infinity and (5) reduces to (1). Equation (5) is an important model equation because the wave fronts of (4) can be obtained by its solutions though it is a special case of (4) [10].

The studies on these model equations mostly focused on the following three typical classes of kernel functions in previous literature.

(A) The first class consists of nonnegative kernel functions (pure excitation).

(B) The second class consists of Mexican hat kernel functions (lateral inhibition); that is, $K \geq 0$ on $(-M, N)$ and $K \leq 0$ on $(-\infty,-M)\cup(N,\infty)$, for some positive constants M and N.

(C) The third class consists of upside down Mexican hat kernel functions (lateral excitation); that is, $K \leq 0$ on $(-M, N)$ and $K \geq 0$ on $(-\infty,-M)\cup(N,\infty)$, for two positive constants M and N.

The kernel functions are always supposed to be continuous at $x = 0$, almost everywhere smooth, and satisfy the following conditions (S_1):

$$\int_{-\infty}^{0} K(x)\, dx = \frac{1}{2}, \qquad \int_{-\infty}^{+\infty} K(x)\, dx = 1, \qquad \int_{-\infty}^{0} |s| K(s)\, ds > 0; \qquad |K(x)| \leq k \exp(-\rho x) \quad \text{in } R, \tag{6}$$

where k and ρ are positive constants.

Under the above assumptions, Zhang [11] studied the existence, uniqueness, and stability of traveling wave solutions to the model equation (5) for three typical classes of kernel functions. Lv and Wang [12] studied the existence and uniqueness of traveling waves of the same equation for five classes of oscillatory kernels and recently, and Magpantay and Zou [13] studied the model equation (4). The kernel function in the feedback channel was assumed to be nonnegative (pure excitation), and the kernel function in the synaptic coupling that was considered in their paper [13] included types (A), (B), and (C) and pure inhibition type. Recently, some similar equations with different fire rate functions [14–16] were proposed and investigated to model the pattern formation in neuronal networks. These equations take the form:

$$u_t + u = \int_R K(x-y) f(u(y,t))\, dy, \tag{7}$$

where

$$f(u(y,t)) = 2\exp\left(\frac{-r}{(u(y,t)-\theta)^2}\right) H(u(y,t)-\theta), \tag{8}$$

and the coupling function is

$$K(x,y) = \exp(-b|x-y|)(b\sin|x-y| + \cos(x-y)) \tag{9}$$

which is an oscillatory function. It was showed [17] that the oscillatory nature of the coupling function is likely to lead to novel behavior. Consequently, it is reasonable and meaningful to proceed with the study of the model equation (5) with oscillatory kernel functions. Motivated by their pioneer works [11–13, 18–20]; in this paper, we aim to study the existence and uniqueness of the wave front solutions of IDE (5) with more general kernel functions.

1.2. Mathematical Assumption of the Kernel Function. In addition to the basic assumptions for kernel functions, we assume that the kernel function $K(x)$ on $(-\infty, 0)$ satisfies one of the following conditions:

(L_1) $K(x) \geq 0$ for all $x \in (-\infty, 0)$.

(L_2) $K(x) \geq 0$ for $x \in (-M_1, 0) \cup (-M_3, -M_2) \cup (-M_5, -M_4) \cup \cdots \cup (-M_{2n+1}, -M_{2n})$, $K(x) \leq 0$ for

$x \in (-M_2, -M_1) \cup (-M_4, -M_3) \cup \cdots \cup (-M_{2n}, -M_{2n-1}) \cup (-\infty, -M_{2n+1})$, where $0 < M_1 < M_2 < \cdots < M_{2n+1} < \infty$ and

$$\int_{-M_{2i}}^{-M_{2i-2}} |s| K(s)\, ds \ge 0,$$
$$\frac{\alpha}{2} - \alpha \int_{-M_{2i}}^{0} K(t)\, dt < \theta, \tag{10}$$

where $i = 1, 2, \ldots, n, n+1$ and $M_0 = 0$, $M_{2n+1} = \infty$.

(L_3) $K(x) \ge 0$ for $x \in (-M_1, 0) \cup (-M_3, -M_2) \cup (-M_5, -M_4) \cup \cdots \cup (-M_{2n-1}, -M_{2n-2}) \cup (-\infty, -M_{2n})$, $K(x) \le 0$ for $x \in (-M_2, -M_1) \cup (-M_4, -M_3) \cup \cdots \cup (-M_{2n}, -M_{2n-1})$, where $0 < M_1 < M_2 < \cdots < M_{2n} < \infty$ and

$$\int_{-M_{2i}}^{0} |s| K(s)\, ds \ge 0,$$
$$\frac{\alpha}{2} - \alpha \int_{-M_{2i}}^{0} K(t)\, dt < \theta, \tag{11}$$

where $i = 1, 2, \ldots, n$.

(L_4) $K(x) \ge 0$ for $x \in (-\infty, -M)$ and $K(x) \le 0$ for $x \in (-M, 0)$, where $0 < M < \infty$.

Assume that the kernel function $K(x)$ on $(0, \infty)$ satisfies one of the following conditions:

(R_1) $K(x) \ge 0$ for all $x \in (0, +\infty)$.

(R_2) $K(x) \ge 0$ for $x \in (N_1, N_2) \cup (N_3, N_4) \cup \cdots \cup (N_{2n+1}, +\infty)$, $K(x) \le 0$ for $x \in [0, N_1] \cup [N_2, N_3] \cup \cdots \cup [N_{2n}, N_{2n+1}]$, where $0 < N_1 < N_2 < \cdots < N_{2n+1} < \infty$, and

$$\alpha \int_0^{N_{2i-1}} K(s)\, ds > \theta - \frac{\alpha}{2}, \quad i = 1, 2, \ldots n, n+1. \tag{12}$$

(R_3) $K(x) \ge 0$ for $x \in (N_1, N_2) \cup (N_3, N_4) \cup \cdots \cup (N_{2n-1}, N_{2n})$, $K(x) \le 0$ for $x \in [0, N_1] \cup [N_2, N_3] \cup \cdots \cup [N_{2n}, +\infty)$, where $0 < N_1 < N_2 < \cdots < N_{2n} < \infty$, and

$$\alpha \int_0^{N_{2i-1}} K(s)\, ds > \theta - \frac{\alpha}{2}, \quad i = 1, 2, \ldots n. \tag{13}$$

(R_4) $K(x) \ge 0$ for $x \in (N, +\infty)$ and $K(x) \le 0$ for $x \in (0, N]$, where $0 < N < \infty$.

(R_5) $K(x) \ge 0$ for $x \in [0, N_1] \cup [N_2, N_3] \cup \cdots \cup [N_{2n-2}, N_{2n-1}]$, $K(x) \le 0$ for $x \in (N_1, N_2) \cup (N_3, N_4) \cup \cdots \cup (N_{2n+1}, +\infty)$, where $0 < N_1 < N_2 < \cdots < N_{2n+1} < \infty$, and

$$\alpha \int_0^{N_{2i}} K(s)\, ds > \theta - \frac{\alpha}{2}, \quad i = 1, 2, \ldots n-1. \tag{14}$$

(R_6) $K(x) \ge 0$ for $x \in [0, N_1] \cup [N_2, N_3] \cup \cdots \cup [N_{2n}, +\infty)$, $K(x) \le 0$ for $x \in (N_1, N_2) \cup (N_3, N_4) \cup \cdots \cup (N_{2n-1}, N_{2n})$, where $0 < N_1 < N_2 < \cdots < N_{2n} < \infty$, and

$$\alpha \int_0^{N_{2i}} K(s)\, ds > \theta - \frac{\alpha}{2}, \quad i = 1, 2, \ldots n. \tag{15}$$

Obviously, the kernel functions satisfying one of the conditions (R_i) ($i = 1, 2, \ldots, 6$) on $(0, +\infty)$ and one of the conditions (L_j) ($j = 1, 2, 3, 4$) on $(-\infty, 0)$ form a very general class of functions including almost all the classes of the kernel functions found in previous literatures. For example, if the kernel function $K(x)$ satisfies (L_4) and (R_3) with $n = 1$ (see Section 3), then it is an upside down Mexican hat kernel functions actually; if the kernel function $K(x)$ satisfies (L_3) and (R_5) with $n = 1$, then it is the case (A) in [12].

We prove the existence and uniqueness of the traveling wave solution of the model equation (5) for more general classes of kernel functions including not only the kernel functions studied in [11, 12] but also the oscillating kernel functions. The main idea in this paper is employing the speed index functions (the main idea in [11, 19] and other pioneering works).

2. Preliminary Analysis

It is well known that the traveling wave solutions of an equation are the solutions of the form $u(x, t) = U(z)$, where $z = x + \mu t$ is the moving coordinate and μ is a constant which represents the speed of the traveling wave. There are two kinds of traveling waves which possess some important and practical meanings in neural network. These are traveling wave fronts and traveling pulses. In this paper, we mainly focus on the traveling wave fronts.

To study the traveling wave solutions of the integral-differential equation (5), we suppose that $u(x, t) = U(x+\mu t) = U(z)$, and then we have

$$\mu U' + U = \alpha \int_R K(x - y) H\left(U\left(y + \mu t - \frac{\mu}{c}|x - y|\right) - \theta\right) dy. \tag{16}$$

It is easy to see that by transformation of variables the integral IDE (16) can be written as

$$\mu U' + U = \alpha \int_R K(z - y) H\left(U\left(y - \frac{\mu}{c}|z - y|\right) - \theta\right) dy. \tag{17}$$

Let $t = y - (\mu/c)|z - y|$ and $\mu < c$ in the first term of the right side of the above equation, and then we get

$$\mu U' + U = \alpha \int_R K\left(\frac{z - t}{1 + \operatorname{sgn}(z - t)(\mu/c)}\right) H(U(t) - \theta)\, dt. \tag{18}$$

Obviously, the nonlinear terms are on the left side of (18). From the special property of Heaviside step function, we know that (18) can be simplified if some properties of function $U(x)$ are known. Let $R[U,\theta] = \{x \mid U(x) > \theta\}$, and then (18) can be rewritten as

$$\mu U' + U = \alpha \int_{R[U,\theta]} K\left(\frac{z-t}{1+\operatorname{sgn}(z-t)(\mu/c)}\right) dt. \quad (19)$$

Firstly, we study the equilibria of (19). Suppose that U is a constant, and then $R[U,\theta] = \mathfrak{R}$ if $U > \theta$ and $R[U,\theta] = \emptyset$ if $U < \theta$. So the constant solutions of (5) are 0 and α if $0 < \theta < \alpha$. According to the property of traveling wave front, we know that if $U(z)$ is a traveling wave front of (5), $U_+ = \lim_{z\to+\infty} U(z)$ and $U_- = \lim_{z\to-\infty} U(z)$, $U_+ > U_-$, and then U_+, U_- are two different constant solutions of (5) and (18), respectively. So $U_+ = \alpha$ and $U_- = 0$.

If $U(z)$ is a traveling wave front of (5) satisfying $(0) = \theta$, $U(z) < \theta$ for $z \in (-\infty, 0)$, and $U(z) > \theta$ for $z \in (0,+\infty)$, then (19) can be reduced to

$$\mu U' + U = \alpha \int_0^{+\infty} K\left(\frac{z-t}{1+\operatorname{sgn}(z-t)(\mu/c)}\right) dt, \quad (20)$$

which can be further rewritten as

$$\mu U' + U = \alpha \int_{-\infty}^{cz/(c+\operatorname{sgn}(z)\mu)} K(t)\, dt. \quad (21)$$

Obviously, (21) is a linear ordinary differential equation having two equilibria 0 and α, which means that the wave front of (5) satisfying $U(0) = \theta$, $U(z) < \theta$ for $z \in (-\infty, 0)$, $\theta < U(z)$ for $z \in (0,+\infty)$ must have the limits as $\lim_{z\to+\infty} U(z) = \alpha$ and $\lim_{z\to-\infty} U(z) = 0$. Solving (21), we get the solution satisfying the limit conditions mentioned above as follows:

$$\begin{aligned} U(z) = {} & \alpha \int_{-\infty}^{cz/(c+\operatorname{sgn}(z)\mu)} K(s)\, ds \\ & - \alpha \int_{-\infty}^{z} \exp\left(\frac{s-z}{\mu}\right) K(s) \frac{c}{c+\operatorname{sgn}(s)\mu} ds, \end{aligned} \quad (22)$$

$$U' = \frac{\alpha}{\mu} \int_{-\infty}^{z} \exp\left(\frac{s-z}{\mu}\right) K(s) \frac{c}{c+\operatorname{sgn}(s)\mu} ds. \quad (23)$$

Notice that the function (22) we obtained is just a solution of (21), but not necessarily a solution of (16); that is to say, it is not necessarily a traveling wave front of (5). The function (22) could be a traveling wave front of (5) only if it satisfies $U(0) = \theta$, $U(z) < \theta$ for $z \in (-\infty, 0)$, $\theta < U(z)$ for $z \in (0,+\infty)$. We investigate these problems to study the existence and uniqueness of the wave solution of IDE (5) under some conditions in the following sections.

3. Existence and Uniqueness of the Wave Front

In this section, we study the existence and uniqueness of the wave front solution to model equation (5); that is to say, we study the solution which tends from 0 to α as z goes from $-\infty$ to ∞. For this equation, Zhang [11] studied the existence and uniqueness of the traveling wave front for the case that the kernel function $K(x)$ is a nonnegative function, Mexican hat function, or upside down Mexican hat function. Lv and Wang [12] also studied the same equation with five kinds of kernel functions. We prove the existence and uniqueness of the traveling wave front of this equation for more general kernel functions under less constraints.

Based on the analysis above, we know that function (22) is a wave front satisfying $U(0) = \theta$ only if $U(z) \le \theta$ for $z \in (-\infty, 0)$ and $U(z) \ge \theta$ for $z \in (0,+\infty)$. Substituting $z = 0$ into (22) and letting $U(0) = \theta$, we get

$$\frac{\alpha}{2} - \alpha \int_{-\infty}^{0} \exp\left(\frac{s}{\mu}\right) K\left(\frac{c}{c+\operatorname{sgn}(s)\mu} s\right) \frac{c}{c+\operatorname{sgn}(s)\mu} ds = \theta. \quad (24)$$

So it is one of the necessary conditions, under which $U(z)$ might be a traveling wave solution of the model equation (5) and $U(0) = \theta$, and that there exists a wave speed μ satisfying (24). Following the definition in [11], we also name (24) as wave speed equation. We now prove that, for a more general kind of kernel functions $K(x)$, there is a unique μ^* satisfying the speed equation; that is,

$$\alpha \int_{-\infty}^{0} \exp\left(\frac{c-\mu}{c\mu} s\right) K(s)\, ds = \frac{\alpha}{2} - \theta, \quad 0 < \mu < c. \quad (25)$$

3.1. Existence and Uniqueness of the Wave Speed. At first, we give the following lemma to prove the existence and uniqueness of the wave speed, that is to say, to prove that there exists a unique wave speed μ^* such that $U(0) = \theta$, which is one of the necessary conditions under which function (22) might be a traveling wave front of the IDE (5). Actually, $U(0) = \theta$ is equivalent to (24), which implies that we can prove the existence and uniqueness of the wave speed by proving that the wave speed equation (24) has a unique solution. Denote $f(A) = \alpha \int_{-\infty}^{0} \exp(s/A) K(s) ds$, and then we have the following Lemma.

Lemma 1. *Suppose that the positive parameters α and θ satisfy the condition $0 < 2\theta < \alpha$. If the function $K(x)$ satisfies the condition (S_1) and one of (L_i) $(i = 1,2,3,4)$, then there exists a unique $A_0 \in (0,\infty)$ such that $f(A_0) = \alpha/2 - \theta$.*

Proof. It is easy to see that $\lim_{A\to o^+} f(A) = 0$ and $\lim_{A\to+\infty} f(A) = \alpha/2$. So we have

$$\lim_{A\to o^+} f(A) < \frac{\alpha}{2} - \theta < \lim_{A\to+\infty} f(A). \quad (26)$$

Since $f(A)$ is a continuous function on $(0,+\infty)$ with respect to A, there exists $A_0 \in (0,\infty)$ such that $f(A_0) = \alpha/2 - \theta$. We now prove the uniqueness of A_0 when $K(x)$ satisfies each of the first three cases on $(-\infty, 0)$ by proving that $f'(A) > 0$. □

Case 1. Obviously, as $K(x)$ satisfies (L_1), that is, $K(x) \ge 0$ for all $x \in (-\infty, 0)$, we have

$$\int_{-\infty}^{0} |s| \exp\left(\frac{s}{A}\right) K(s)\, ds > 0. \quad (27)$$

Case 2. As $K(x)$ satisfies (L_2); that is, there exist $0 = M_0 < M_1 < M_2 < \cdots < M_{2n+1} < M_{2n+2} = \infty$, such that $K(x) \geq 0$ for $x \in (-M_1, 0) \cup (-M_3, -M_2) \cup (-M_5, -M_4) \cup \cdots \cup (-M_{2n+1}, -M_{2n})$, $K(x) \leq 0$ for $x \in (-M_2, -M_1) \cup (-M_4, -M_3) \cup \cdots \cup (-M_{2n}, -M_{2n-1}) \cup (-\infty, -M_{2n+1})$, and for any i, $i = 1, 2, \ldots, n, n+1$,

$$\int_{-M_{2i}}^{-M_{2i-2}} |s| K(s)\, ds \geq 0, \qquad \frac{\alpha}{2} - \alpha \int_{-M_{2i}}^{0} K(t)\, dt < \theta, \tag{28}$$

we have

$$\begin{aligned}
&\int_{-\infty}^{0} |s| \exp\left(\frac{s}{A}\right) K(s)\, ds \\
&\quad = \int_{-M_2}^{0} |s| \exp\left(\frac{s}{A}\right) K(s)\, ds \\
&\qquad + \int_{-M_4}^{-M_2} |s| \exp\left(\frac{s}{A}\right) K(s)\, ds + \cdots \\
&\qquad + \int_{-M_{2n}}^{-M_{2n-2}} |s| K(s)\, ds + \int_{-\infty}^{-M_{2n}} |s| \exp\left(\frac{s}{A}\right) K(s)\, ds \\
&\quad > \exp\left(\frac{-M_1}{A}\right) \int_{-M_2}^{0} |s| K(s)\, ds \\
&\qquad + \exp\left(\frac{-M_3}{A}\right) \int_{-M_4}^{-M_2} |s| K(s)\, ds + \cdots \\
&\qquad + \exp\left(\frac{-M_{2n-1}}{A}\right) \int_{-M_{2n}}^{-M_{2n-2}} |s| K(s)\, ds \\
&\qquad + \exp\left(\frac{-M_{2n+1}}{A}\right) \int_{-\infty}^{-M_{2n-2}} |s| K(s)\, ds > 0.
\end{aligned} \tag{29}$$

Case 3. As $K(x)$ satisfies (L_3); that is, there exist $0 < M_1 < M_2 < \cdots < M_{2n} < \infty$, such that $K(x) \geq 0$ for $x \in (-M_1, 0) \cup (-M_3, -M_2) \cup (-M_5, -M_4) \cup \cdots \cup (-M_{2n-1}, -M_{2n-2}) \cup (-\infty, -M_{2n})$, $K(x) \leq 0$ for $x \in (-M_2, -M_1) \cup (-M_4, -M_3) \cup \cdots \cup (-M_{2n}, -M_{2n-1})$, and for any i, $i = 1, 2, \ldots, n$,

$$\int_{-M_{2i}}^{0} |s| K(s)\, ds \geq 0, \qquad \frac{\alpha}{2} - \alpha \int_{-M_{2i}}^{0} K(t)\, dt < \theta, \tag{30}$$

we have

$$\begin{aligned}
&\int_{-\infty}^{0} |s| \exp\left(\frac{s}{A}\right) K(s)\, ds \\
&\quad = \int_{-M_2}^{0} |s| \exp\left(\frac{s}{A}\right) K(s)\, ds \\
&\qquad + \int_{-M_4}^{-M_2} |s| \exp\left(\frac{s}{A}\right) K(s)\, ds + \cdots \\
&\qquad + \int_{-M_{2n}}^{-M_{2n-2}} |s| K(s)\, ds \\
&\qquad + \int_{-\infty}^{-M_{2n}} |s| \exp\left(\frac{s}{A}\right) K(s)\, ds \\
&\quad \geq \exp\left(\frac{-M_1}{A}\right) \int_{-M_2}^{0} |s| K(s)\, ds \\
&\qquad + \exp\left(\frac{-M_3}{A}\right) \int_{-M_4}^{-M_2} |s| K(s)\, ds + \cdots \\
&\qquad + \exp\left(\frac{-M_{2n-1}}{A}\right) \int_{-M_{2n}}^{-M_{2n-2}} |s| K(s)\, ds \\
&\qquad + \exp\left(\frac{-M_{2n+1}}{A}\right) \int_{-\infty}^{-M_{2n}} |s| K(s)\, ds \\
&\quad \geq \exp\left(\frac{-M_3}{A}\right) \int_{-M_4}^{0} |s| K(s)\, ds \\
&\qquad + \exp\left(\frac{-M_5}{A}\right) \int_{-M_6}^{-M_4} |s| K(s)\, ds + \cdots \\
&\qquad + \exp\left(\frac{-M_{2n-1}}{A}\right) \int_{-M_{2n}}^{-M_{2n-2}} |s| K(s)\, ds \\
&\qquad + \exp\left(\frac{-M_{2n+1}}{A}\right) \int_{-\infty}^{-M_{2n}} |s| K(s)\, ds \\
&\quad \geq \exp\left(\frac{-M_{2n+1}}{A}\right) \int_{-\infty}^{0} |s| K(s)\, ds > 0.
\end{aligned} \tag{31}$$

Thus, as $K(x)$ satisfies one of the (L_i), $i = 1, 2, 3$,

$$f'(A) = \frac{\alpha}{A^2} \int_{-\infty}^{0} |s| \exp\left(\frac{s}{A}\right) K(s)\, ds > 0, \tag{32}$$

which implies that $f(A)$ is continuous and monotonically increasing function. Consequently, there exists a unique $A_0 \in (0, \infty)$ such that $f(A) = \alpha/2 - \theta$ if $K(x)$ satisfies one of (L_i) $i = 1, 2, 3$.

However, for the case that $K(x)$ satisfies (L_4); that is, there exists for $0 < M < \infty$, such that $K(x) \geq 0$ for $x \in (-\infty, -M)$ and $K(x) \leq 0$ for $x \in (-M, 0)$, we have

$$\begin{aligned}
f'(A) &= \frac{\alpha}{A^2} \int_{-\infty}^{0} |s| \exp\left(\frac{s}{A}\right) K(s)\, ds \\
&> \frac{\alpha}{A^2} M \int_{-\infty}^{0} \exp\left(\frac{s}{A}\right) K(s)\, ds = \frac{M}{A^2} f(A).
\end{aligned} \tag{33}$$

Consequently, if there exists $A_0 \in (0, \infty)$ such that $f(A_0) = \alpha/2 - \theta > 0$, then $f'(A_0) > 0$. Thus function $y = f(A)$ must be monotonically increasing in the neighborhood of the root of equation $f(A) = \alpha/2 - \theta$, which implies that $f(A) = \alpha/2 - \theta$ has at most one root. Since we have proved the existence of the root of equation $f(A) = \alpha/2 - \theta$, we know that there exists a unique $A_0 \in (0, \infty)$ such that $f(A) = \alpha/2 - \theta$ if $K(x)$ satisfies (L_4).

Theorem 2 (see [10]). *Suppose that the positive parameters α and θ satisfy the condition $0 < 2\theta < \alpha$. In addition to the basic assumptions for kernel functions, if $K(x)$ satisfies one of the assumptions L_i, ($i = 1, 2, 3, 4$), then there exists a unique $\mu^* \in (0, c)$ such that $\alpha \int_{-\infty}^{0} \exp(((c-\mu)/c\mu)s)K(s)ds = \alpha/2-\theta$.*

Proof. Let $A = c\mu/(c-\mu)$, then, for any $A \in (0, \infty)$, there is a unique $\mu = c/(Ac+1)$ for $c > 0$. Clearly, $\mu \in (0, c)$ for any $A \in (0, \infty)$ and $c > 0$. From Lemma 1, we know that there exists a unique $A_0 \in (0, \infty)$ such that $f(A_0) = \alpha/2 - \theta$; that is, there exists a unique $\mu^* = c/(A_0c+1) \in (0, c)$ such that $f(A_0) = f(c\mu_*/(c-\mu_*)) = \alpha/2 - \theta$; that is, $\alpha \int_{-\infty}^{0} \exp(((c - \mu^*)/c\mu^*)s)K(s)ds = \alpha/2 - \theta$. □

From Theorem 2, we see that there exists a unique μ^* such that function (22) satisfies $U(0) = \theta$ if the kernel function $K(x)$ satisfies one of the assumptions L_i, $i = 1, 2, 3, 4$. However, it does not follow that function (22) is a traveling wave front solution to IDE (5). We now prove that function (22) satisfies $U(z) < \theta$ on $(-\infty, 0)$ and $U(z) > \theta$ on $(0, +\infty)$ if $K(x)$ satisfies the assumptions of Theorem 2 and one of the assumptions R_i ($i = 1, 2, \ldots, 6$), from which the conclusion that function (22) is a traveling wave front solution of the IDE (5) will be derived obviously.

3.2. Existence and Uniqueness of the Wave Front. In this subsection, we first present two lemmas to prove that function (22) satisfies $U(z) < \theta$ on $(-\infty, 0)$ and $U(z) > \theta$ on $(0, +\infty)$. Then we obtain the existence and uniqueness of the wave front.

It follows from (21) and function (22) that

$$U'(z) = \frac{\alpha}{\mu} \exp\left(\frac{-z}{\mu}\right) \times \int_{-\infty}^{z} \exp\left(\frac{s}{\mu}\right) K\left(\frac{c}{c+\operatorname{sgn}(s)\mu}s\right) \frac{c}{c+\operatorname{sgn}(s)\mu} ds. \quad (34)$$

Lemma 3. *Function* (22) *satisfies $U(z) < \theta$ on $(-\infty, 0)$ if $K(x)$ satisfies one of the assumptions (L_i) ($i = 1, 2, 3, 4$).*

Proof. From (34), we have

$$\begin{aligned} U'(z) &= \frac{\alpha}{\mu} \exp\left(\frac{-z}{\mu}\right) \int_{-\infty}^{z} \exp\left(\frac{s}{\mu}\right) K\left(\frac{c}{c-\mu}s\right) \frac{c}{c-\mu} ds \\ &= \frac{\alpha}{\mu} \exp\left(\frac{-z}{\mu}\right) \int_{-\infty}^{(c/(c-\mu))z} \exp\left(\frac{c-\mu}{c\mu}s\right) K(s)\, ds \end{aligned} \quad (35)$$

for $z \le 0$. Denote

$$\varphi(z) = \int_{-\infty}^{z} \exp\left(\frac{c-\mu}{c\mu}s\right) K(s)\, ds. \quad (36)$$

Obviously, $U'(z)$ has the same sign as $\varphi((c/(c-\mu))z)$ for $z \in (-\infty, 0)$. Consequently, taking into account that $U'(0) = (1/\mu)(\alpha/2 - \theta) > 0$, we get $\varphi(0) > 0$. □

Case 1. As $K(x)$ satisfies (L_1), it is easy to see that $U'(z) \ge 0$ for all $z \in (-\infty, 0)$. So $U(z)$ is a continuous and monotonically increasing function on $(-\infty, 0)$. Consequently, $U(z) < \theta$ on $(-\infty, 0)$ since $U(0) = \theta$ and $U'(0) = (1/\mu)(\alpha/2 - \theta) > 0$.

Case 2. As $K(x)$ satisfies (L_2), it is obvious that $\varphi(z) \le 0$ for $z \in (-\infty, -M_{2n+1})$ and $\varphi(z)$ is increasing on $(-M_{2i+1}, -M_{2i})$ for any i, $i = 0, 1, 2, \ldots, n$, and decreasing on $(-M_{2i}, -M_{2i-1})$ for any i, $i = 1, 2, \ldots, n$, because of the properties of function $K(x)$. Consequently, $\varphi(z)$ may only change its sign from negative to positive on $(-M_{2i+1}, -M_{2i})$ and from positive to negative on $(-M_{2i}, -M_{2i-1})$ for some i, $i = 1, 2, \ldots, n$. Recall that $U'(((c-\mu)/c)z)$ has the same sign as $\varphi(z)$, so the local maximal points of function $U(((c-\mu)/c)z)$ should locate on $(-M_{2i}, -M_{2i-1})$ and the local minimal points should locate on $(-M_{2i+1}, -M_{2i})$ if any. We will know that $U(z) \le \theta$ on $(-\infty, 0)$ if we prove that the local maximum of function $U(z)$ is no more than θ. Suppose that there exists a point $z_i \in (-M_{2i}, -M_{2i-1})$ such that $\varphi(z_i) = 0$, and so $U'(((c-\mu)/c)z_i) = 0$. If $U(((c-\mu)/c)z_i)$ attains a local maximum, then

$$\begin{aligned} U\left(\frac{c-\mu}{c}z_i\right) &= \alpha \int_{-\infty}^{z_i} K(t)\, dt = \frac{\alpha}{2} - \alpha \int_{z_i}^{0} K(t)\, dt \\ &\le \frac{\alpha}{2} - \alpha \int_{-M_{2i}}^{0} K(t)\, dt < \theta. \end{aligned} \quad (37)$$

Consequently, $U(z) < \theta$ on $(-\infty, 0)$ if $K(x)$ satisfies (L_2).

Case 3. As $K(x)$ satisfies (L_3), by the similar analysis as in Case 2 above, we get $U(z) < \theta$ on $(-\infty, 0)$.

Case 4. As $K(x)$ satisfies (L_4), it is easy to see that $\varphi(z) \ge 0$ for $z \in (-\infty, -M)$, and $\varphi(z) \ge \varphi(0) > 0$ for $z \in [-M, 0]$. Consequently, $U'(z) \ge 0$ on $(-\infty, 0)$. Thus $U(z) < \theta$ on $(-\infty, 0)$ follows from $U'(0) > 0$ and $U(0) = \theta$.

From the analysis above, we conclude that $U(z) < \theta$ on $(-\infty, 0)$ if $K(x)$ satisfies one of (L_i) ($i = 1, \ldots, 4$).

Lemma 4. *Function* (22) *satisfies $U(z) > \theta$ on $(0, +\infty)$ if $K(x)$ satisfies one of the assumptions (R_i) ($i = 1, 2, \ldots, 6$).*

Proof. For $z \in (0, +\infty)$, it follows from (34) that

$$\begin{aligned} U'(z) &= \frac{\alpha}{\mu} \exp\left(\frac{-z}{\mu}\right) \left[\int_{-\infty}^{0} \exp\left(\frac{s}{\mu}\right) K\left(\frac{c}{c-\mu}s\right) \frac{c}{c-\mu} ds \right. \\ &\quad \left. + \int_{0}^{z} \exp\left(\frac{s}{\mu}\right) K\left(\frac{c}{c+\mu}s\right) \frac{c}{c+\mu} ds\right] \\ &= \frac{\alpha}{\mu} \exp\left(\frac{-z}{\mu}\right) \\ &\quad \times \left[\varphi(0) + \int_{0}^{(c/(c+\mu))z} \exp\left(\frac{c+\mu}{c\mu}s\right) K(s)\, ds\right]. \end{aligned} \quad (38)$$

Denote

$$\psi(z) = \int_{0}^{(c/(c+\mu))z} \exp\left(\frac{c+\mu}{c\mu}s\right) K(s)\, ds. \quad (39)$$

Then

$$U'(z) = \frac{\alpha}{\mu}\exp\left(\frac{-z}{\mu}\right)[\varphi(0) + \psi(z)]. \quad (40)$$

Recall that $\varphi(0) > 0$. □

Case 1. As $K(x)$ satisfies (R_1), it is easy to see that $\psi(z) \geq 0$ and thus $U'(z) > 0$ on $(0, +\infty)$. Consequently, $U(z) > \theta$ on $(0, +\infty)$ in consideration of $U(0) = \theta$.

Case 2. As $K(x)$ satisfies (R_2), it is easy to see that $y = \psi(z)$ is decreasing on $(((c+\mu)/c)N_{2i}, ((c+\mu)/c)N_{2i+1})$ but increasing on $(((c+\mu)/c)N_{2i-1}, ((c+\mu)/c)N_{2i})$ for any i, $i = 1, 2, \ldots, n$, and $(((c+\mu)/c)N_{2n+1}, \infty)$. So $U'(z)$ may change its sign from negative to positive on $(((c+\mu)/c)N_{2i-1}, ((c+\mu)/c)N_{2i})$ for some i, $i = 1, 2, \ldots, n$ or $(((c+\mu)/c)N_{2n+1}, \infty)$ and from positive to negative on $(((c+\mu)/c)N_{2i}, ((c+\mu)/c)N_{2i+1})$ for some i, $i = 0, 1, 2, \ldots, n$. Consequently, the local maximal points of function $U(z)$ should locate on $(((c+\mu)/c)N_{2i}, ((c+\mu)/c)N_{2i+1})$ for some, i, $i = 0, 1, 2, \ldots, n$, and the local minimal points should locate on $(((c+\mu)/c)N_{2i-1}, ((c+\mu)/c)N_{2i})$ for some i, $i = 1, 2, \ldots, n$, if any. Recall that $\lim_{z\to+\infty} U(z) = \alpha > \theta$, so if we can prove that the local minimum is more than θ, then we have $U(z) > \theta$ on $(0, +\infty)$. If there exists a point $z_i \in (N_{2i+1}, N_{2i+2})$ $(i = 1, 2, \ldots, n)$ such that $U'(((c+\mu)/c)z_i) = 0$, which means that $U(((c+\mu)/c)z_i)$ attains a local minimum and

$$\begin{aligned} U\left(\frac{c+\mu}{c}z_i\right) &= \alpha\int_{-\infty}^{z_i} K(t)\,dt \\ &\geq \alpha\int_{-\infty}^{N_{2i-1}} K(t)\,dt = \frac{\alpha}{2} + \alpha\int_0^{N_{2i-1}} K(t)\,dt \\ &> \theta. \end{aligned} \quad (41)$$

Consequently, $U(z) > \theta$ on $(0, +\infty)$ if $K(x)$ satisfies (R_2).

Case 3. As $K(x)$ satisfies (R_3), similar to the proof in Case 2 we can prove that $U(z) > \theta$ on $(0, +\infty)$.

Case 4. As $K(x)$ satisfies (R_4), it is easy to see that $\psi(z) \geq 0$ for $z \in (0, N]$ and the function $y = \psi(z)$ is monotonically decreasing on $(N, +\infty)$. So $U'(z)$ at most changes its sign once from positive to negative, that is to say, that $U(z)$ keeps increasing on $(0, \infty)$ or there exists $z_0 \in (N, +\infty)$ such that $U(z)$ is increasing on $(0, z_0)$ but decreasing on (z_0, ∞). Clearly, $U(z) > \theta$ on $(0, \infty)$ if $U(z)$ keeps increasing on $(0, \infty)$. $U(z) > \theta$ on $(0, \infty)$ if $U(z)$ is increasing on $(0, z_0)$ and decreasing on (z_0, ∞) because $\lim_{z\to+\infty} U(z) = \alpha > \theta$. Consequently, $U(z) > \theta$ on $(0, +\infty)$ if $K(x)$ satisfies (R_4).

Case 5. As $K(x)$ satisfies (R_5), $\psi(z)$ is increasing on $(((c+\mu)/c)N_{2i}, ((c+\mu)/c)N_{2i+1})$ but decreasing on $(((c+\mu)/c)N_{2i+1}, ((c+\mu)/c)N_{2i+2})$ for any i, $i = 0, 1, 2, \ldots, n$. So $U'(z)$ may change its sign from negative to positive on $(((c+\mu)/c)N_{2i}, ((c+\mu)/c)N_{2i+1})$ for some i, $i = 1, 2, \ldots, n$, and from positive to negative on $(((c+\mu)/c)N_{2i+1}, ((c+\mu)/c)N_{2i+2})$ for some i, $i = 0, 1, 2, \ldots, n$, or $(((c+\mu)/c)N_{2n+1}, \infty)$. Consequently, the local minimal points of function $U(z)$ should locate on $(((c+\mu)/c)N_{2i}, ((c+\mu)/c)N_{2i+1})$ $(i = 0, 1, 2, \ldots, n)$ and the local maximal points should locate on $(((c+\mu)/c)N_{2i-1}, ((c+\mu)/c)N_{2i})$ for some i, $i = 1, 2, \ldots, n$, or $(((c+\mu)/c)N_{2n}, \infty)$ if any. If we can prove that the local minimum is greater than θ, then we have $U(z) \geq \theta$ on $(0, \infty)$ because $\lim_{z\to+\infty} U(z) = \alpha > \theta$. If there exists a point $z_i \in (N_{2i}, N_{2i+1})$ $(i = 0, 1, 2, \ldots, n)$ such that $U'(((c+\mu)/c)z_i) = 0$, $U(((c+\mu)/c)z_i)$ attains a local minimum and

$$U\left(\frac{c+\mu}{c}z_i\right) = \alpha\int_{-\infty}^{z_i} K(t)\,dt \geq \alpha\int_{-\infty}^{N_{2i}} K(t)\,dt > \theta. \quad (42)$$

Consequently, $U(z) > \theta$ on $(0, +\infty)$ if $K(x)$ satisfies (R_5).

Case 6. As $K(x)$ satisfies (R_6), $\psi(z)$ is increasing on $(((c+\mu)/c)N_{2i}, ((c+\mu)/c)N_{2i+1})$ for any i, $i = 0, 1, 2, \ldots, n-1$, and $(((c+\mu)/c)N_{2n}, \infty)$ but decreasing on $(((c+\mu)/c)N_{2i-1}, ((c+\mu)/c)N_{2i})$ for any i, $i = 1, 2, \ldots, n$. So $U'(z)$ may change its sign from negative to positive on $(((c+\mu)/c)N_{2i}, ((c+\mu)/c)N_{2i+1})$ for some i, $i = 1, 2, \ldots, n-1$, or $(((c+\mu)/c)N_{2n}, \infty)$ and from positive to negative on $(((c+\mu)/c)N_{2i-1}, ((c+\mu)/c)N_{2i})$ for some i, $i = 1, 2, \ldots, n$. Consequently, the local maximal points of function $U(z)$ should locate on $(((c+\mu)/c)N_{2i-1}, ((c+\mu)/c)N_{2i})$ for some i, $i = 1, 2, \ldots, n$, and the local minimal points should locate on $(((c+\mu)/c)N_{2i}, ((c+\mu)/c)N_{2i+1})$ for some i, $i = 0, 1, 2, \ldots, n$, or $(((c+\mu)/c)N_{2n}, +\infty)$ if any. If we can prove that the local minimum is more than θ, then we have $U(z) > \theta$ on $(0, +\infty)$. If there exists a point $z_i \in (N_{2i}, N_{2i+1})$ for some i, $i = 1, 2, \ldots, n-1$, or $z_n \in (N_{2n}, +\infty)$ such that $U'(((c-\mu)/c)z_i) = 0$, $U(((c-\mu)/c)z_i)$ attains a local minimum and

$$U\left(\frac{c-\mu}{c}z_i\right) = \alpha\int_{-\infty}^{z_i} K(t)\,dt \geq \alpha\int_{-\infty}^{N_{2i}} K(t)\,dt > \theta. \quad (43)$$

Consequently, $U(z) > \theta$ on $(0, +\infty)$ if $K(x)$ satisfies (R_6).

In conclusion, $U(z) > \theta$ on $(0, +\infty)$ if $K(x)$ satisfies one of (R_i) $(i = 1, 2, \ldots, 6)$.

It follows from the above two lemmas that the function (22) satisfies $U(z) < \theta$ on $(-\infty, 0)$ and $U(z) > \theta$ on $(0, +\infty)$ if $K(x)$ satisfies one of the assumptions L_i $(i = 1, 2, 3, 4)$ on $(-\infty, 0)$ and one of R_j $(j = 1, 2, \ldots, 6)$ on $(0, +\infty)$. We sum up all these results in the following theorem.

Theorem 5 (see [10]). *Suppose that the positive parameters α and θ satisfy the condition $0 < 2\theta < \alpha$ and $K(x)$ satisfies the assumption (S_1), one of L_i $(i = 1, 2, 3, 4)$ on $(-\infty, 0)$ and one of R_j $(j = 1, 2, \ldots, 6)$ on $(0, +\infty)$. Then (5) has a unique traveling wave front solution $U(z) = U(x + \mu t)$ satisfying the phase conditions:*

$$\begin{gathered} U(0) = \theta, \quad U(z) < \theta \quad \text{on } (-\infty, 0), \\ U(z) > \theta \quad \text{on } (0, +\infty). \end{gathered} \quad (44)$$

The unique traveling wave front could be expressed as

$$\begin{aligned} U(z) &= \alpha\int_{-\infty}^{cz/(c+\operatorname{sgn}(z)\mu)} K(s)\,ds \\ &\quad - \alpha\int_{-\infty}^{z} \exp\left(\frac{s-z}{\mu}\right) K(s)\frac{c}{c+\operatorname{sgn}(s)\mu}\,ds. \end{aligned} \quad (45)$$

The wave speed μ $(0 < \mu < c)$ is determined by the speed equation

$$\alpha \int_{-\infty}^{0} \exp\left(\frac{c-\mu}{c\mu}s\right) K(s)\, ds = \frac{\alpha}{2} - \theta. \tag{46}$$

The wave front solution $U(z)$ also satisfies the boundary conditions:

$$\lim_{z\to-\infty} U(z) = 0, \qquad \lim_{z\to+\infty} U(z) = \alpha, \qquad \lim_{z\to\pm\infty} U'(z) = 0. \tag{47}$$

From this theorem, we see that, not only for the classical nonnegative kernel function, Mexican hat kernel function, and upside down Mexican hat kernel function but also for some types of kernel function oscillating n-times, the IDE (5) has a unique traveling front wave solution. It is well known that, in this biophysically motivated nonlinear nonlocal firing rate model equation, pure excitation, lateral inhibition, and lateral excitation are modeled by nonnegative kernel function, Mexican hat kernel function and upside down Mexican hat kernel function, respectively. The kernel function with n-times oscillations may hopefully be applied to more complicated and more accurate reaction cases.

4. Conclusion and Some Discussions

In this paper, we have investigated the existence and uniqueness of the wave front solution of the integral-differential model equation (5) arising from neuronal networks. At first, we reduced the nonlinear-integral differential equation (5) into a simpler solvable linear differential equation (21) by using the special property of Heaviside gain function and the hypothesis of the wave front solution that $U(0) = \theta, U(z) < \theta$ for $z \in (-\infty, 0)$, and $U(z) > \theta$ for $z \in (0, +\infty)$. Next, we checked that the solution (22) which we got easily from (21) was the unique wave front solution of (5) satisfying the hypothesis $U(0) = \theta, U(z) < \theta$ for $z \in (-\infty, 0)$, and $U(z) > \theta$ for $z \in (0, +\infty)$. The uniqueness was obtained by proving the uniqueness of the solution of the speed equation $U(0) = \theta$, that is, the uniqueness of the wave speed.

The difficulties and key points of this work were how to test that solution (22) satisfies the phase conditions: $U(0) = \theta$, $U(z) < \theta$ for $z \in (-\infty, 0)$ and $U(z) > \theta$ for $z \in (0, +\infty)$ if the kernel function $K(x)$ is not of class (A) or (B) because the function (22) may not be an increasing function any more. We achieved it by proving the maximum values of (22) on $(-\infty, 0)$ if any were less than θ and the minimum values of (22) on $(0, +\infty)$ if any were greater than θ, using the idea motivated by Lv and Wang [12]. It is worth pointing out that we simplified the problem and generalized the kernel function $K(x)$ to a very general class of functions, including not only the well-known three typical classes of kernels and five classes of oscillatory kernels proposed in [12], but also oscillating n-times kernels. We separated the whole problem into some simpler ones by noticing that the existence and uniqueness of the wave speed and the phase condition $U(z) < \theta$ for $z \in (-\infty, 0)$ only depended on the conditions of kernel function $k(x)$ on $(-\infty, 0)$; however, the phase condition $U(z) > \theta$ for $z \in (0, +\infty)$ depended on the conditions of kernel function $k(x)$ on $(0, +\infty)$.

As we mentioned in the first section, the results of this work could be employed to investigate some more complicated model equation. For instance, the solution of model equation (2), for the case when $I(x, t)$ is irrelevant to variables, can be obtained by a simple translation of the solution of (5) and was well applied to investigate the model equation (4) [10]. The results we obtained might be applied to approximate the solution of the equation with oscillating infinity times kernels. Particularly, we believe that the idea and method we used in this paper can be applied to investigate other nonlinear equations, which is our main research interest at present.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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