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Research Article

Fixed-Point Theorems for Mean Nonexpansive Mappings in Banach Spaces

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We define a mean nonexpansive mapping T on X in the sense that $\|Tx - Ty\| \le a\|x-y\| + b\|x - Ty\|$, $a, b \ge 0, a + b \le 1$. It is proved that mean nonexpansive mapping has approximate fixed-point sequence, and, under some suitable conditions, we get some existence and uniqueness theorems of fixed point.

1. Introduction

Let X be a Banach space, C a nonempty bounded closed convex subset of X, and $T : C \to C$ a nonexpansive mapping; that is,

$$\|Tx - Ty\| \le \|x - y\| \quad \forall\ x, y \in C. \tag{1}$$

We say that X has the fixed-point property if every nonexpansive mapping defined on a nonempty bounded closed convex subset of X has a fixed point. In 1965, Kirk [1] proved that if X is a reflexive Banach space with normal structure, then X has the fixed-point property.

Let C be a nonempty subset of real Banach space X and T a mapping from C to C. T is called mean nonexpansive if for each $x, y \in C$,

$$\|Tx - Ty\| \le a\|x - y\| + b\|x - Ty\|, \quad a, b \ge 0, \quad a + b \le 1. \tag{2}$$

In 1975, Zhang [2] introduced this definition and proved that T has a fixed point in C, where C is a weakly compact closed convex subset and has normal structure. For more information about mean nonexpansive mapping, one can refer to [3–5].

2. Main Results

Lemma 1. *Let T be a mean nonexpansive mapping of the Banach space X. If T is continuous and $a + b < 1$, then T has a unique fixed point.*

Proof. The proof is similar to the proof of the Banach contractive theorem. □

If we let $b > 0$ and $a + b \le 1 - b$ as in Lemma 1, then the condition that T is continuous may not be needed. Firstly, we recall the following two lemmas.

Lemma 2. *Let C be a nonempty subset of Banach space X and T a mean nonexpansive self-mapping on C with $a + 2b \le 1$ and $b > 0$. Let K be a nonempty subset of C; one defines $\phi(x) = \|x - Tx\|$ for any $x \in K$; if the set $\phi(K)$ is bounded, then K is also bounded.*

Proof. Let $M = \sup_{x\in K}\|x - Tx\| < \infty$ and set $x_0 \in K$ as fixed; then for any $x \in K$, we have

$$\begin{aligned}\|x - x_0\| &\le \phi(x) + \phi(x_0) + \|Tx - Tx_0\| \\ &\le a\|x - x_0\| + b\|x - Tx_0\| \\ &\quad + \phi(x) + \phi(x_0) \\ &\le a\|x - x_0\| + b\|x - x_0\|\end{aligned}$$

$$\begin{aligned} &+ (1+b)\phi(x_0) + \phi(x) \\ &\leq (1-b)\|x-x_0\| + (2+b)M. \end{aligned} \tag{3}$$

This implies that

$$\|x-x_0\| \leq \frac{2+b}{b}M. \tag{4}$$

Hence, K is bounded. The proof is complete. □

Lemma 3. *Let C be a nonempty subset of Banach space X and T a mean nonexpansive self-mapping on C. If $a+2b \leq 1$ and $b > 0$, then for any $x \in C$, one has the following inequality:*

$$\begin{aligned} \left\|T^n x - T^{n+1}x\right\| &\leq d + \left(\frac{a}{1-b}\right)^k \\ &\quad \times \left\{\left\|T^{n-k}x - T^{n+1}x\right\| - (1+k)d\right\}, \end{aligned} \tag{5}$$

where $d = \|x - Tx\|$ and n, k are two positive integers such that $2a/(1-b-a) \leq k \leq n$.

Proof. By the definition of mean nonexpansive mapping, we have that

$$\left\|T^n x - T^{n-1}x\right\| \leq a\left\|T^{n-1}x - T^{n-2}x\right\| \leq \left\|T^{n-1}x - T^{n-2}x\right\|; \tag{6}$$

this implies that

$$\left\|T^{r+1}x - T^r x\right\| \leq \|x - Tx\| = d, \tag{7}$$

where r is an integer.

When $k = 0$, the result is obvious. Suppose that (5) is true for $k = l < n$; that is,

$$\begin{aligned} \left\|T^n x - T^{n+1}x\right\| &\leq d + \left(\frac{a}{1-b}\right)^l \\ &\quad \times \left\{\left\|T^{n-l}x - T^{n+1}x\right\| - (1+l)d\right\}. \end{aligned} \tag{8}$$

By the inequality (2) and (7), we have

$$\begin{aligned} \left\|T^{n+1}x - T^{n-l}x\right\| &\leq a\left\|T^n x - T^{n-l-1}x\right\| \\ &\quad + b\left\|T^n x - T^{n-l}x\right\| \\ &\leq a\left\|T^{n+1}x - T^{n-l-1}x\right\| \\ &\quad + b\left\|T^{n+1}x - T^{n-l}x\right\| \\ &\quad + (a+b)d. \end{aligned} \tag{9}$$

This implies from $a + 2b \leq 1$ and $b > 0$ that

$$\left\|T^{n-l}x - T^{n+1}x\right\| \leq \frac{a}{1-b}\left\|T^{n-l-1}x - T^{n+1}x\right\| + d, \tag{10}$$

which follows that

$$\begin{aligned} \left\|T^n x - T^{n+1}x\right\| &\leq d + \left(\frac{a}{1-b}\right)^{l+1} \\ &\quad \times \left\{\left\|T^{n-l-1}x - T^{n+1}x\right\| - (2+l)d\right\}. \end{aligned} \tag{11}$$

By induction, this completes the proof. □

Theorem 4. *Let C be a nonempty closed subset of Banach space X and T a mean nonexpansive self-mapping on C. If $a+2b \leq 1$ and $b > 0$, then T has a unique fixed point.*

Proof. For any $x \in C$, set $\alpha_n = \|T^n x - T^{n-1}x\|$; by the definition of mean nonexpansive mapping, we have that

$$\begin{aligned} \alpha_n = \left\|T^n x - T^{n-1}x\right\| &\leq a\left\|T^{n-1}x - T^{n-2}x\right\| \\ &\leq \left\|T^{n-1}x - T^{n-2}x\right\| = \alpha_{n-1}. \end{aligned} \tag{12}$$

Thus, the sequence $\{\alpha_n\}$ is nonincreasing and bounded below, so $\lim_{n\to\infty}\alpha_n$ exists.

Suppose that $\lim_{n\to\infty}\alpha_n = \alpha > 0$; then we have by Lemma 2 that the set $K = \{T^n x : n = 1, 2, \ldots\}$ is bounded, so there exists a positive number A such that

$$\left\|T^p x - T^q x\right\| \leq A, \tag{13}$$

where p and q are two integers. Since $\alpha > 0$ and $b > 0$, for any $\epsilon > 0$, there exists an integer N_0 such that $N_0 \geq 2a/(1-b-a)$: and

$$\epsilon < \left(\frac{a}{1-b}\right)^{N_0}\left[(N_0+1)\alpha - A\right]. \tag{14}$$

Since $\lim_{n\to\infty}\alpha_n = \alpha > 0$, there exists an integer N such that for $n \geq N_0 + N$; we have $0 \leq \alpha_n - \alpha < \epsilon$. Setting $n = N_0 + N$ and $y = T^{N-1}x$, then $\alpha_n = \|T^{N_0}y - T^{N_0+1}y\|$ and $\alpha_N = \|y - Ty\|$. Thus, from (14) and Lemma 3, we have that

$$\begin{aligned} \alpha_n &= \left\|T^{N_0}y - T^{N_0+1}y\right\| \\ &\leq \|y - Ty\| + \left(\frac{a}{1-b}\right)^{N_0} \\ &\quad \times \left\{\left\|y - T^{N_0+1}y\right\| - (N_0+1)\|y - Ty\|\right\} \\ &\leq \alpha_N + \left(\frac{a}{1-b}\right)^{N_0}\left\{A - (N_0+1)\alpha_N\right\} \\ &\leq \alpha + \epsilon - \left(\frac{a}{1-b}\right)^{N_0}\left\{(N_0+1)\alpha_N - A\right\} < \alpha; \end{aligned} \tag{15}$$

on the other hand, by condition (12), we have that $\alpha_n \geq \alpha$, which is a contradiction, so $\alpha = 0$. We next show that $\lim_{n\to\infty}T^n x$ exists. In fact, since $\lim_{n\to\infty}\|T^n x - T^{n+1}x\| = 0$, we have

$$\begin{aligned} \left\|T^n x - T^m x\right\| &\leq \left\|T^n x - T^{n+1}x\right\| + \left\|T^m x - T^{m+1}x\right\| \\ &\quad + \left\|T^{m+1}x - T^{n+1}x\right\| \\ &\leq \left\|T^n x - T^{n+1}x\right\| + \left\|T^m x - T^{m+1}x\right\| \\ &\quad + a\left\|T^n x - T^m x\right\| \\ &\quad + b\left\|T^{n+1}x - T^n x\right\| \\ &\quad + b\left\|T^m x - T^n x\right\|. \end{aligned} \tag{16}$$

This implies that

$$\left\|T^n x - T^m x\right\| \leq \frac{1+b}{1-a-b}\left\|T^n x - T^{n+1} x\right\| + \frac{1}{1-a-b}\left\|T^m x - T^{m+1} x\right\| \longrightarrow 0. \tag{17}$$

That is, the sequence $\{T^n x\}$ is a Cauchy sequence in X. Since X is complete, thus there exists $x^* \in C$ such that $\lim_{n\to\infty} T^n x = x^*$.

Finally, we prove that $x^* = Tx^*$. We have from $\lim_{n\to\infty}\|T^n x - T^{n+1}x\| = 0$ and (2) that

$$\left\|T^n x - Tx^*\right\| \leq a\left\|T^{n-1}x - x^*\right\| + b\left\|T^{n-1}x - Tx^*\right\|; \tag{18}$$

it follows as $n \to \infty$ that $\|x^* - Tx^*\| \leq b\|x^* - Tx^*\|$, which implies by $0 < b < 1$ that $x^* = Tx^*$ and the proof is complete. □

We now consider the approximate fixed-point sequence. A sequence $\{x_n\}$ is called an approximate fixed-point sequence for T if $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. It is easy to prove that if C is a nonempty bounded closed convex subset of Banach space X, and T is a nonexpansive mapping from C to C, then T has an approximate fixed point sequence in C. For mean nonexpansive mapping, we have the same result. Firstly, we give the following lemma.

Lemma 5 (see [6]). *Let s a real number and $\{u_i\}$ be a sequence in Banach space X. Then, for any positive integer N,*

$$(1-s)s^{N-1}\sum_{i=1}^{N}u_i = \left(1-s^N\right)u_N - s^{N-1}\sum_{i=1}^{N-1}\left(s^{-i}-1\right)\left(u_{i+1}-su_i\right). \tag{19}$$

If X is the real line and $u_i = 1$ for all i, one has the special case

$$N(1-s)s^{N-1} = 1 - s^N - (1-s)s^{N-1} \times \sum_{i=1}^{N-1}\left(s^{-i}-1\right). \tag{20}$$

Theorem 6. *Let C be a nonempty bounded closed subset of Banach space X and T a mean nonexpansive self-mapping on C. Let $x_1 \in C$ be fixed and the sequence $\{x_n\}$ defined by*

$$x_{n+1} = (1-t)x_n + tTx_n. \tag{21}$$

If $1/2 \leq t \leq 1$, then $\{x_n - Tx_n\}$ converges strongly to 0 as $n \to \infty$.

Proof. Since T is a mean nonexpansive mapping, from (21) and $t \geq 1/2$, we get that

$$\begin{aligned}
\left\|x_{n+1} - Tx_{n+1}\right\| &= \left\|(1-t)(x_n - Tx_n) + Tx_n - Tx_{n+1}\right\| \\
&\leq (1-t)\left\|x_n - Tx_n\right\| + a\left\|x_n - x_{n+1}\right\| + b\left\|x_{n+1} - Tx_n\right\| \\
&= (1-t)\left\|x_n - Tx_n\right\| + at\left\|x_n - Tx_n\right\| + b(1-t)\left\|x_n - Tx_n\right\| \\
&\leq (1-t)\left\|x_n - Tx_n\right\| + (a+b)t\left\|x_n - Tx_n\right\| \\
&\leq \left\|x_n - Tx_n\right\|.
\end{aligned} \tag{22}$$

Thus, the sequence $\{\|x_n - Tx_n\|\}$ is nonincreasing and bounded below, so $\lim_{n\to\infty}\|x_n - Tx_n\|$ exists. Suppose that $\lim_{n\to\infty}\|x_n - Tx_n\| = r > 0$. That is, for any $\epsilon > 0$, there exists an integer m such that

$$r \leq \left\|x_{m+i} - Tx_{m+i}\right\| \leq (1+\epsilon)r \quad \forall\, i \in R^+. \tag{23}$$

Since $\{x_n\}$ is bounded and $1/2 \leq t \leq 1$, there exists an integer N such that

$$(N-1)tr \leq \delta(M) + 1 \leq Ntr, \tag{24}$$

where $\delta(M) := \sup\{\|x_i - x_j\| : 0 < i, j < \infty\}$.

Now setting $s = 1-t$ and $u_i = x_{m+i} - Tx_{m+i}$ for all positive integers i, we get from (21) and (23) that

$$\begin{aligned}
\left\|u_{i+1} - (1-t)u_i\right\| &= \left\|Tx_{m+i+1} - x_{m+i+1} - (1-t)(Tx_{m+i} - x_{m+i})\right\| \\
&= \left\|T\left((1-t)x_{m+i} + tx_{m+i}\right) - Tx_{m+i}\right\| \\
&\leq at\left\|Tx_{m+i} - x_{m+i}\right\| + b(1-t)\left\|Tx_{m+i} - x_{m+i}\right\| \\
&\leq t\left\|Tx_{m+i} - x_{m+i}\right\| \leq t(1+\epsilon)r, \\
x_{m+N+1} - x_{m+1} &= \sum_{i=1}^{N}\left(x_{m+i+1} - x_{m+i}\right) \\
&= \sum_{i=1}^{N}t\left(Tx_{m+i} - x_{m+i}\right) = t\sum_{i=1}^{N}u_i.
\end{aligned} \tag{25}$$

Hence, by Lemma 5, (19), (20), and (23), we get that

$$\begin{aligned}
&(1-t)^{N-1}\left\|x_{m+N+1} - x_{m+1}\right\| \\
&= \left\|t(1-t)^{N-1}\sum_{i=1}^{N}u_i\right\| \\
&\geq \left(1-(1-t)^N\right)\left\|u_N\right\| - (1-t)^{N-1}
\end{aligned}$$

$$\begin{aligned}
&\times \sum_{i=1}^{N-1} \left((1-t)^{-i}-1\right) \|u_{i+1}-(1-t)u_i\| \\
&= \left(1-(1-t)^N\right) r - t(1-t)^{N-1} \\
&\times \sum_{i=1}^{N-1} \left((1-t)^{-i}-1\right)(1+\varepsilon) r \\
&= \Bigg(1-(1-t)^N - t(1-t)^{N-1} \\
&\times \sum_{i=1}^{N-1} \left((1-t)^{-i}-1\right)\Bigg) r \\
&- \varepsilon r t(1-t)^{N-1} \sum_{i=1}^{N-1} \left((1-t)^{-i}-1\right) \\
&= Nt(1-t)^{N-1} r \\
&- \varepsilon r \left(1-(1-t)^N - Nt(1-t)^{N-1}\right) \\
&\geq Nt(1-t)^{N-1} r - \varepsilon r.
\end{aligned} \tag{26}$$

This implies from (24) that

$$\begin{aligned}
\|x_{m+N+1}-x_{m+1}\| &\geq Ntr - \epsilon r\left(1+\frac{t}{1-t}\right)^{N-1} \\
&\geq \delta(M) + 1 - \epsilon r\left(1+\frac{t}{1-t}\right)^{N-1}.
\end{aligned} \tag{27}$$

Since $\ln(1+y) \leq y$ for $y \in (-1,\infty)$, we have

$$\begin{aligned}
\left(1+\frac{t}{1-t}\right)^{N-1} &= \exp\left\{(N-1)\ln^{(1+t/(1-t))}\right\} \\
&\leq \exp\left\{(N-1)\frac{t}{1-t}\right\} \\
&\leq \exp\left\{(1-t)^{-1}(\delta(M)+1)y^{-1}\right\}.
\end{aligned} \tag{28}$$

Hence, we have

$$\begin{aligned}
&\delta(M) + 1 - \epsilon r \exp\left\{(1-t)^{-1}(\delta(M)+1)y^{-1}\right\} \\
&\quad \leq \|x_{m+N+1}-x_{m+1}\| \leq \delta(M).
\end{aligned} \tag{29}$$

Since ϵ is an arbitrary positive number, it follows that $\delta(M) + 1 \leq \delta(M)$. This contradiction completes the proof. □

Corollary 7. *Let C be a bounded closed convex subset of Banach space X and T a mean nonexpansive self-mapping on C. Then, T has an approximate fixed-point sequence in C.*

Proof. For any $x \in C$, define $T_1(x) = (1/2)x + (1/2)Tx$, and let $x_{n+1} = T_1^n(x)$, where $n = 0, 1, \ldots$; then the sequence $\{x_n\}$ may be written as $x_{n+1} = (1/2)x_n + (1/2)Tx_n$, so the conditions of Theorem 6 are satisfied. Hence, we have that the sequence $\{x_n\}$ is an approximate fixed-point sequence. The proof is complete. □

Next, we consider the Opial condition. Related to the problem of existence of a fixed point for mapping and its approximation, in 1967, Opial [7] introduced the following inequality.

Definition 8. Let X be a Banach space; X satisfies Opial's condition if for each x in X and each sequence $\{x_n\}$ weakly convergent to x

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{30}$$

holds for $y \neq x$.

This definition is motivated by the fact that this property implies that asymptotic center of sequence coincides with its weak limit, which of course fails in L^p for $p \geq 1$ (and more generally in Orlicz spaces L^Φ; see [8, 9]).

Opial's condition is connected to the following fixed-point property.

Theorem 9. *Let X be a real reflexive Banach space which satisfies Opial's condition, C a nonempty bounded closed convex subset of X, and $T : C \to C$ a mean nonexpansive. Then T has a fixed point.*

Proof. Let T be mean nonexpansive. By Corollary 7, we have that T has an approximate fixed-point sequence in C; that is, there exists a sequence $\{x_n\}$ of C such that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{31}$$

Since C is a weakly compact convex subset of X, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ weakly convergent to $x_0 \in C$.

Now, we show that $x_0 = Tx_0$. Suppose, by way of contradiction, that $x_0 \neq Tx_0$; then

$$\begin{aligned}
\liminf_{n\to\infty} \|x_n - x_0\| &< \liminf_{n\to\infty} \|x_n - Tx_0\| \\
&= \liminf_{n\to\infty} \|Tx_n - Tx_0\|.
\end{aligned} \tag{32}$$

Since T is mean nonexpansive, we have from (2) that

$$\begin{aligned}
\liminf_{n\to\infty} \{\|x_n - Tx_0\|\} &= \liminf_{n\to\infty} \{\|Tx_n - Tx_0\|\} \\
&\leq \liminf_{n\to\infty} \{a\|x_n - x_0\| \\
&\quad + b\|x_n - Tx_0\|\} \\
&\leq \liminf_{n\to\infty} \{a\|x_n - x_0\| \\
&\quad + (1-a)\|x_n - Tx_0\|\}.
\end{aligned} \tag{33}$$

It follows that

$$\liminf_{n\to\infty} \left\| x_n - Tx_0 \right\| \leq \liminf_{n\to\infty} \left\| x_n - x_0 \right\|. \tag{34}$$

This is a contradiction. Therefore, x_0 ia a fixed point of T and the proof is complete. □

In fact, spaces which satisfy Opial's condition not only have the fixed point property, but also satisfy the so-called demiclosedness principle for the mean nonexpansive mapping.

Corollary 10. *If X is a reflexive Banach space which satisfies Opial's condition, let C be a nonempty closed convex subset of X and suppose that $T : C \to X$ is mean nonexpansive. For any sequence $\{x_n\}$ in C with $x_n \rightharpoonup x_0$ and $(x_n - Tx_n) \to 0$, then $x_0 = Tx_0$.*

Proof. The proof of this corollary is the same as in Theorem 9. □

In order to understand the connection between nonexpansive mapping and mean nonexpansive mapping better, we have the following remark.

Remark 11. It is easy to see that the nonexpansive mappings and contractive mappings both are uniformly continuous and mean nonexpansive; the converse does not hold. Examples will be given to support our point of view.

(1) Let T be the unit interval defined by

$$T(x) = \begin{cases} \dfrac{x}{5} & \text{if } x \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{x}{6} & \text{if } x \in \left[\dfrac{1}{2}, 1\right), \end{cases} \tag{35}$$

and the norm is the ordinary Euclidean distance on the line. Here, T is discontinuous at $x = 1/2$; consequently, T is neither nonexpansive mapping nor contractive mapping. Now, we prove that T is mean nonexpansive.

Case 1 ($x, y \in [0, 1/2)$). By the definition of T,

$$\begin{aligned} \|Tx - Ty\| &= \frac{1}{4}\left\|\frac{4}{5}x - \frac{4}{5}y\right\| \\ &= \frac{1}{4}\left\| x - \frac{y}{5} + \frac{y}{5} - \frac{x}{5} \right. \\ &\qquad \left. - \left(y - x + x - \frac{y}{5}\right)\right\| \\ &\leq \frac{1}{4}\|x - y\| + \frac{1}{2}\|x - Ty\| \\ &\quad + \frac{1}{4}\|Tx - Ty\|. \end{aligned} \tag{36}$$

This implies that $\|Tx - Ty\| \leq (1/3)\|x - y\| + (2/3)\|x - Ty\|$.

Case 2 ($x \in [0, 1/2)$ and $y \in [1/2, 1)$). In this case, we have

$$\begin{aligned} \|Tx - Ty\| &= \left\|\frac{x}{5} - \frac{y}{6}\right\| \\ &= \left\|\frac{x}{5} - \frac{Tx}{5} + \frac{Tx}{5} - \frac{Ty}{5} + \frac{Ty}{5} - \frac{y}{6}\right\| \\ &\leq \frac{1}{5}\|x - Tx\| + \frac{1}{5}\|Tx - Ty\| \\ &\quad + \frac{1}{5}\|y - Ty\| \\ &\leq \frac{2}{5}\|x - Ty\| + \frac{2}{5}\|Ty - Tx\| \\ &\quad + \frac{1}{5}\|y - x\|. \end{aligned} \tag{37}$$

This implies that $\|Tx - Ty\| \leq (1/3)\|x - y\| + (2/3)\|x - Ty\|$.

Case 3 ($y \in [0, 1/2)$ and $x \in [1/2, 1)$). The proof is the same as in Case 2.

Case 4 ($x, y \in [1/2, 1)$). The proof is the same as in Case 1.

Hence, T is mean nonexpansive by taking $a = 1/3$, $b = 2/3$.

(2) Mean nonexpansive mappings, however, may be continuous only at their fixed points. For example, the map T defined by

$$T(x) = \begin{cases} \dfrac{1 - x}{3} & \text{if } x \in [0, 1] \text{ and } x \text{ is irrational}, \\ \dfrac{1 + x}{5} & \text{if } x \in [0, 1] \text{ and } x \text{ is rational} \end{cases} \tag{38}$$

is a mean nonexpansive mapping on unit interval by taking $a = 1/3$, $b = 2/3$ and is continuous only at its fixed point $x_0 = 1/4$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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