Research Article

Multiple Attribute Decision Making Based on Hesitant Fuzzy Einstein Geometric Aggregation Operators

Xiaoqiang Zhou[1,2] and Qingguo Li[1]

[1] *College of Mathematics and Econometrics, Hunan University, Changsha 410082, China*
[2] *College of Mathematics, Hunan Institute of Science and Technology, Yueyang 414006, China*

Correspondence should be addressed to Qingguo Li; liqingguoli@aliyun.com

We first define an accuracy function of hesitant fuzzy elements (HFEs) and develop a new method to compare two HFEs. Then, based on Einstein operators, we give some new operational laws on HFEs and some desirable properties of these operations. We also develop several new hesitant fuzzy aggregation operators, including the hesitant fuzzy Einstein weighted geometric (HFEWG$_\varepsilon$) operator and the hesitant fuzzy Einstein ordered weighted geometric (HFEWG$_\varepsilon$) operator, which are the extensions of the weighted geometric operator and the ordered weighted geometric (OWG) operator with hesitant fuzzy information, respectively. Furthermore, we establish the connections between the proposed and the existing hesitant fuzzy aggregation operators and discuss various properties of the proposed operators. Finally, we apply the HFEWG$_\varepsilon$ operator to solve the hesitant fuzzy decision making problems.

1. Introduction

Atanassov [1, 2] introduced the concept of intuitionistic fuzzy set (IFS) characterized by a membership function and a non-membership function. It is more suitable to deal with fuzziness and uncertainty than the ordinary fuzzy set proposed by Zadeh [3] characterized by one membership function. Information aggregation is an important research topic in many applications such as fuzzy logic systems and multiattribute decision making as discussed by Chen and Hwang [4]. Research on aggregation operators with intuitionistic fuzzy information has received increasing attention as shown in the literature. Xu [5] developed some basic arithmetic aggregation operators based on intuitionistic fuzzy values (IFVs), such as the intuitionistic fuzzy weighted averaging operator and intuitionistic fuzzy ordered weighted averaging operator, while Xu and Yager [6] presented some basic geometric aggregation operators for aggregating IFVs, including the intuitionistic fuzzy weighted geometric operator and intuitionistic fuzzy ordered weighted geometric operator. Based on these basic aggregation operators proposed in [6] and [5], many generalized intuitionistic fuzzy aggregation operators have been investigated [5–30]. Recently, Torra and Narukawa [31] and Torra [32] proposed the hesitant fuzzy set (HFS), which is another generalization form of fuzzy set. The characteristic of HFS is that it allows membership degree to have a set of possible values. Therefore, HFS is a very useful tool in the situations where there are some difficulties in determining the membership of an element to a set. Lately, research on aggregation methods and multiple attribute decision making theories under hesitant fuzzy environment is very active, and a lot of results have been obtained for hesitant fuzzy information [33–43]. For example, Xia et al. [38] developed some confidence induced aggregation operators for hesitant fuzzy information. Xia et al. [37] gave several series of hesitant fuzzy aggregation operators with the help of quasiarithmetic means. Wei [35] explored several hesitant fuzzy prioritized aggregation operators and applied them to hesitant fuzzy decision making problems. Zhu et al. [43] investigated the geometric Bonferroni mean combining the Bonferroni mean and the geometric mean under hesitant fuzzy environment. Xia and Xu [36] presented some hesitant fuzzy operational

laws based on the relationship between the HFEs and the IFVs. They also proposed a series of aggregation operators, such as hesitant fuzzy weighted geometric (HFWG) operator and hesitant fuzzy ordered weighted geometric (HFOWG) operator. Furthermore, they applied the proposed aggregation operators to solve the multiple attribute decision making problems.

Note that all aggregation operators introduced previously are based on the algebraic product and algebraic sum of IFVs (or HFEs) to carry out the combination process. However, the algebraic operations include algebraic product and algebraic sum, which are not the unique operations that can be used to perform the intersection and union. There are many instances of various t-norms and t-conorms families which can be chosen to model the corresponding intersections and unions, among which Einstein product and Einstein sum are good alternatives for they typically give the same smooth approximation as algebraic product and algebraic sum, respectively. For intuitionistic fuzzy information, Wang and Liu [10, 11, 44] and Wei and Zhao [30] developed some new intuitionistic fuzzy aggregation operators with the help of Einstein operations. For hesitant fuzzy information, however, it seems that in the literature there is little investigation on aggregation techniques using the Einstein operations to aggregate hesitant fuzzy information. Therefore, it is necessary to develop some hesitant fuzzy information aggregation operators based on Einstein operations.

The remainder of this paper is structured as follows. In Section 2, we briefly review some basic concepts and operations related to IFS and HFS. we also define an accuracy function of HFEs to distinguish the two HFEs having the same score values, based on which we give the new comparison laws on HFEs. In Section 3, we present some new operations for HFEs and discuss some basic properties of the proposed operations. In Section 4, we develop some novel hesitant fuzzy geometric aggregation operators with the help of Einstein operations, such as the HFEWG$_\varepsilon$ operator and the HFEOWG$_\varepsilon$ operator, and we further study various properties of these operators. Section 5 gives an approach to solve the multiple attribute hesitant fuzzy decision making problems based on the HFEOWG$_\varepsilon$ operator. Finally, Section 6 concludes the paper.

2. Preliminaries

In this section, we briefly introduce Einstein operations and some notions of IFS and HFS. Meantime, we define an accuracy function of HFEs and redefine the comparison laws between two HFEs.

2.1. Einstein Operations. Since the appearance of fuzzy set theory, the set theoretical operators have played an important role and received more and more attention. It is well known that the t-norms and t-conorms are the general concepts including all types of the specific operators, and they satisfy the requirements of the conjunction and disjunction operators, respectively. There are various t-norms and t-conorms families that can be used to perform the corresponding intersections and unions. Einstein sum $\oplus_\varepsilon$ and Einstein product

$\otimes_\varepsilon$ are examples of t-conorms and t-norms, respectively. They are called Einstein operations and defined as [45]

$$x \otimes_\varepsilon y = \frac{x \cdot y}{1 + (1 - x) \cdot (1 - y)}, \qquad x \otimes_\varepsilon y = \frac{x + y}{1 + x \cdot y}, \tag{1}$$

$$\forall x, y \in [0, 1].$$

2.2. Intuitionistic Fuzzy Set. Atanassov [1, 2] generalized the concept of fuzzy set [3] and defined the concept of intuitionistic fuzzy set (IFS) as follows.

Definition 1. Let U be fixed an IFS A on U is given by;

$$A = \{\langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in U\}, \tag{2}$$

where $\mu_A : U \rightarrow [0, 1]$ and $\nu_A : U \rightarrow [0, 1]$, with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in U$. Xu [5] called $\tilde{a} = (\mu_{\tilde{a}}, \nu_{\tilde{a}})$ an IFV.

For IFVs, Wang and Liu [11] introduced some operations as follows.

Let $\lambda > 0$, $\tilde{a}_1 = (\mu_{\tilde{a}_1}, \nu_{\tilde{a}_1})$ and $\tilde{a}_2 = (\mu_{\tilde{a}_2}, \nu_{\tilde{a}_2})$ be two IFVs; then

(1) $\tilde{a}_1 \otimes_\varepsilon \tilde{a}_2 = \left(\dfrac{\mu_{\tilde{a}_1} + \mu_{\tilde{a}_2}}{1 + \mu_{\tilde{a}_1} \mu_{\tilde{a}_2}}, \dfrac{\nu_{\tilde{a}_1} \nu_{\tilde{a}_2}}{1 + \left(1 - \nu_{\tilde{a}_1}\right)\left(1 - \nu_{\tilde{a}_2}\right)} \right)$

(2) $\tilde{a}_1 \otimes_\varepsilon \tilde{a}_2 = \left(\dfrac{\mu_{\tilde{a}_1} \mu_{\tilde{a}_2}}{1 + \left(1 - \mu_{\tilde{a}_1}\right)\left(1 - \mu_{\tilde{a}_2}\right)}, \dfrac{\nu_{\tilde{a}_1} + \nu_{\tilde{a}_2}}{1 + \nu_{\tilde{a}_1} \nu_{\tilde{a}_2}} \right)$

(3) $\tilde{a}_1^{\wedge_\varepsilon \lambda} = \left(\dfrac{2\nu_{\tilde{a}_1}^\lambda}{\left(2 - \nu_{\tilde{a}_1}\right)^\lambda + \nu_{\tilde{a}_1}^\lambda}, \dfrac{\left(1 + \mu_{\tilde{a}_1}\right)^\lambda - \left(1 - \mu_{\tilde{a}_1}\right)^\lambda}{\left(1 + \mu_{\tilde{a}_1}\right)^\lambda + \left(1 - \mu_{\tilde{a}_1}\right)^\lambda} \right).$

$$\tag{3}$$

2.3. Hesitant Fuzzy Set. As another generalization of fuzzy set, HFS was first introduced by Torra and Narukawa [31, 32].

Definition 2. Let X be a reference set; an HFS on X is in terms of a function that when applied to X returns a subset of $[0, 1]$.

To be easily understood, Xia and Xu use the following mathematical symbol to express the HFS:

$$H = \left\{ \frac{h_H(x)}{x} \mid x \in X \right\}, \tag{4}$$

where $h_H(x)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees of the element $x \in X$ to the set H. For convenience, Xu and Xia [40] called $h_H(x)$ a hesitant fuzzy element (HFE).

Let h be an HFE, $h^- = \min\{\gamma \mid \gamma \in h\}$, and $h^+ = \max\{\gamma \mid \gamma \in h\}$. Torra and Narukawa [31, 32] define the IFV $A_{\text{env}}(h)$ as the envelope of h, where $A_{\text{env}}(h) = (h^-, 1 - h^+)$.

Let $\alpha > 0$, h_1 and h_2 be two HFEs. Xia and Xu [36] defined some operations as follows:

$$
\begin{aligned}
&(4)\ h_1 \bigoplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\}\\[2mm]
&(5)\ h_1 \bigotimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\}\\[2mm]
&(6)\ \alpha h = \bigcup_{\gamma \in h} \{\gamma^\alpha\}\\[2mm]
&(7)\ h^\alpha = \bigcup_{\gamma \in h} \left\{1 - (1 - \gamma)^\alpha\right\}.
\end{aligned}
\tag{5}
$$

In [36], Xia and Xu defined the score function of an HFE h to compare the HFEs and gave the comparison laws.

Definition 3. Let h be an HFE; $s(h) = (1/n(h)) \sum_{\gamma \in h} \gamma$ is called the score function of h, where $n(h)$ is the number of values of h. For two HFEs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

From Definition 3, it can be seen that all HFEs are regarded as the same if their score values are equal. In hesitant fuzzy decision making process, however, we usually need to compare two HFEs for reordering or ranking. In the case where two HFEs have the same score values, they can not be distinguished by Definition 3. Therefore, it is necessary to develop a new method to overcome the difficulty.

For an IFV, Hong and Choi [46] showed that the relation between the score function and the accuracy function is similar to the relation between mean and variance in statistics. From Definition 3, we know that the score value of HFE h is just the mean of the values in h. Motivated by the idea of Hong and Choi [46], we can define the accuracy function of HFE h by using the variance of the values in h.

Definition 4. Let h be an HFE; $k(h) = 1 - \sqrt{(1/n(h)) \sum_{\gamma \in h} (\gamma - s(h))^2}$ is called the accuracy function of h, where $n(h)$ is the number of values in h and $s(h)$ is the score function of h.

It is well known that an efficient estimator is a measure of the variance of an estimate's sampling distribution in statistics: the smaller the variance, the better the performance of the estimator. Motivated by this idea, it is meaningful and appropriate to stipulate that the higher the accuracy degree of HFE, the better the HFE. Therefore, in the following, we develop a new method to compare two HFEs, which is based on the score function and the accuracy function, defined as follows.

Definition 5. Let h_1 and h_2 be two HFEs and let $s(\cdot)$ and $k(\cdot)$ be the score function and accuracy function of HFEs, respectively. Then

(1) if $s(h_1) < s(h_2)$, then h_1 is smaller than h_2, denoted by $h_1 \prec h_2$;

(2) if $s(h_1) = s(h_2)$, then

(i) if $k(h_1) < k(h_2)$, then h_1 is smaller than h_2, denoted by $h_1 \prec h_2$;

(ii) if $k(h_1) = k(h_2)$, then h_1 and h_2 represent the same information, denoted by $h_1 \doteq h_2$. In particular, if $\gamma_1 = \gamma_2$ for any $\gamma_1 \in h_1$ and $\gamma_2 \in h_2$, then h_1 is equal to h_2, denoted by $h_1 = h_2$.

Example 6. Let $h_1 = \{0.5\}$, $h_2 = \{0.1, 0.9\}$, $h_3 = \{0.3, 0.7\}$, $h_4 = \{0.1, 0.3, 0.7, 0.9\}$, $h_5 = \{0.2, 0.4, 0.6, 0.8\}$, and $h_6 = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$; then $s(h_1) = s(h_2) = s(h_3) = s(h_4) = s(h_5) = s(h_6) = 0.5$, $k(h_1) = 1$, $k(h_2) = 0.6$, $k(h_3) = 0.8$, $k(h_4) = 0.6838$, $k(h_5) = 0.7764$, and $k(h_6) = 0.7418$. By Definition 5, we have $h_1 \succ h_3 \succ h_5 \succ h_6 \succ h_4 \succ h_2$.

3. Einstein Operations of Hesitant Fuzzy Sets

In this section, we will introduce the Einstein operations on HFEs and analyze some desirable properties of these operations. Motivated by the operational laws (1)–(3) on IFVs and based on the interconnection between HFEs and IFVs, we give some new operations of HFEs as follows.

Let $\alpha > 0$, h, h_1, and h_2 be three HFEs; then

$$
\begin{aligned}
&(8)\ h_1 \otimes_\varepsilon h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{\frac{\gamma_1 + \gamma_2}{1 + \gamma_1 \gamma_2}\right\},\\[3mm]
&(9)\ h_1 \otimes_\varepsilon h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{\frac{\gamma_1 \gamma_2}{1 + (1 - \gamma_1)(1 - \gamma_2)}\right\},\\[3mm]
&(10)\ h^{\wedge_\varepsilon \alpha} = \bigcup_{\gamma \in h} \left\{\frac{2\gamma^\alpha}{(2 - \gamma)^\alpha + \gamma^\alpha}\right\}.
\end{aligned}
\tag{6}
$$

Proposition 7. *Let $\alpha > 0$, $\alpha_1 > 0$, $\alpha_2 > 0$, h, h_1 and h_2 be three HFEs; then*

(1) $h_1 \otimes_\varepsilon h_2 = h_2 \otimes_\varepsilon h_1$,

(2) $(h_1 \otimes_\varepsilon h_2) \otimes_\varepsilon h_3 = h_1 \otimes_\varepsilon (h_2 \otimes_\varepsilon h_3)$,

(3) $(h_1 \otimes_\varepsilon h_2)^{\wedge_\varepsilon \alpha} = h_1^{\wedge_\varepsilon \alpha} \otimes_\varepsilon h_2^{\wedge_\varepsilon \alpha}$,

(4) $(h^{\wedge_\varepsilon \alpha_1})^{\wedge_\varepsilon \alpha_2} = h^{\wedge_\varepsilon (\alpha_1 \alpha_2)}$;

(5) $A_{env}(h^{\wedge_\varepsilon \alpha}) = (A_{env}(h))^{\wedge_\varepsilon \alpha}$,

(6) $A_{env}(h_1 \otimes_\varepsilon h_2) = A_{env}(h_1) \otimes_\varepsilon A_{env}(h_2)$.

Proof. (1) It is trivial.
(2) By the operational law (9), we have

$$
\begin{aligned}
&(h_1 \otimes_\varepsilon h_2) \otimes_\varepsilon h_3\\[2mm]
&= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \gamma_3 \in h_3} \Big\{\big((\gamma_1 \gamma_2 / (1 + (1 - \gamma_1)(1 - \gamma_2))) \gamma_3\big)\\[2mm]
&\qquad \times \big(1 + (1 - (\gamma_1 \gamma_2 / (1 + (1 - \gamma_1)(1 - \gamma_2))))\\[2mm]
&\qquad \times (1 - \gamma_3)\big)^{-1}\Big\}
\end{aligned}
$$

$$= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \gamma_3 \in h_3} \left\{ (\gamma_1 \gamma_2 \gamma_3) \times (1 + (1 - \gamma_1)(1 - \gamma_2) \right.$$
$$+ (1 - \gamma_1)(1 - \gamma_3) + (1 - \gamma_2)$$
$$\left. \times (1 - \gamma_3))^{-1} \right\}$$

$$= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \gamma_3 \in h_3} \left\{ (\gamma_1 (\gamma_2 \gamma_3 / (1 + (1 - \gamma_2)(1 - \gamma_3)))) \right.$$
$$\times (1 + (1 - \gamma_1)$$
$$\left. \times (1 - (\gamma_2 \gamma_3 / (1 + (1 - \gamma_2)(1 - \gamma_3)))))^{-1} \right\}$$

$$= h_1 \otimes_\varepsilon (h_2 \otimes_\varepsilon h_3).$$

$$(7)$$

(3) Let $h = h_1 \otimes_\varepsilon h_2$; then $h = h_1 \otimes_\varepsilon h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2 / (1 + (1 - \gamma_1)(1 - \gamma_2))\}$

$$(h_1 \otimes_\varepsilon h_2)^{\wedge_\varepsilon \alpha}$$

$$= h^{\wedge_\varepsilon \alpha} = \bigcup_{\gamma \in h} \left\{ \frac{2\gamma^\alpha}{(2 - \gamma)^\alpha + \gamma^\alpha} \right\}$$

$$= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \left(2(\gamma_1 \gamma_2 / (1 + (1 - \gamma_1)(1 - \gamma_2)))^\alpha \right) \right.$$
$$\times \left((2 - (\gamma_1 \gamma_2 / (1 + (1 - \gamma_1)(1 - \gamma_2))))^\alpha \right.$$
$$\left. \left. + (\gamma_1 \gamma_2 / (1 + (1 - \gamma_1)(1 - \gamma_2)))^\alpha \right)^{-1} \right\}$$

$$= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{2(\gamma_1 \gamma_2)^\alpha}{(4 - 2\gamma_1 - 2\gamma_2 + \gamma_1 \gamma_2)^\alpha + (\gamma_1 \gamma_2)^\alpha} \right\},$$

$$= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{2(\gamma_1 \gamma_2)^\alpha}{(2 - \gamma_1)^\alpha (2 - \gamma_2)^\alpha + (\gamma_1 \gamma_2)^\alpha} \right\}.$$

$$(8)$$

Since $h_1^{\wedge_\varepsilon \alpha} = \bigcup_{\gamma_1 \in h} \{2\gamma_1^\alpha / ((2 - \gamma_1)^\alpha + \gamma_1^\alpha)\}$ and $h_2^{\wedge_\varepsilon \alpha} = \bigcup_{\gamma_2 \in h} \{2\gamma_2^\alpha / ((2 - \gamma_2)^\alpha + \gamma_2^\alpha)\}$, then

$$h_1^{\wedge_\varepsilon \alpha} \otimes_\varepsilon h_2^{\wedge_\varepsilon \alpha}$$

$$= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \left((2\gamma_1^\alpha / ((2 - \gamma_1)^\alpha + \gamma_1^\alpha)) \right. \right.$$
$$\cdot (2\gamma_2^\alpha / ((2 - \gamma_2)^\alpha + \gamma_2^\alpha)))$$
$$\times (1 + (1 - (2\gamma_1^\alpha / ((2 - \gamma_1)^\alpha + \gamma_1^\alpha)))$$
$$\left. \times (1 - (2\gamma_2^\alpha / ((2 - \gamma_2)^\alpha + \gamma_2^\alpha))))^{-1} \right\}$$

$$= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{2(\gamma_1 \gamma_2)^\alpha}{(2 - \gamma_1)^\alpha (2 - \gamma_2)^\alpha + (\gamma_1 \gamma_2)^\alpha} \right\}.$$

$$(9)$$

Thus $(h_1 \otimes_\varepsilon h_2)^{\wedge_\varepsilon \alpha} = h_1^{\wedge_\varepsilon \alpha} \otimes_\varepsilon h_2^{\wedge_\varepsilon \alpha}$.

(4) Since $h^{\wedge_\varepsilon \alpha_1} = \bigcup_{\gamma \in h} \{2\gamma^{\alpha_1} / ((2 - \gamma)^{\alpha_1} + \gamma^{\alpha_1})\}$, then

$$\left(h^{\wedge_\varepsilon \alpha_1} \right)^{\wedge_\varepsilon \alpha_2}$$

$$= \bigcup_{\gamma \in h} \left\{ \left(2(2\gamma^{\alpha_1} / ((2 - \gamma)^{\alpha_1} + \gamma^{\alpha_1}))^{\alpha_2} \right) \right.$$
$$\times \left((2 - (2\gamma^{\alpha_1} / ((2 - \gamma)^{\alpha_1} + \gamma^{\alpha_1})))^{\alpha_2} \right.$$
$$\left. \left. + (2\gamma^{\alpha_1} / ((2 - \gamma)^{\alpha_1} + \gamma^{\alpha_1}))^{\alpha_2} \right)^{-1} \right\}$$

$$= \bigcup_{\gamma \in h} \left\{ \frac{2\gamma^{(\alpha_1 \alpha_2)}}{(2 - \gamma)^{(\alpha_1 \alpha_2)} + \gamma^{(\alpha_1 \alpha_2)}} \right\}$$

$$= h^{\wedge_\varepsilon (\alpha_1 \alpha_2)}.$$

$$(10)$$

(5) By the definition of the envelope of an HFE and the operation laws (3) and (10), we have

$$\left(A_{\mathrm{env}}(h) \right)^{\wedge_\varepsilon \alpha}$$

$$= \left(h^-, 1 - h^+ \right)^{\wedge_\varepsilon \alpha}$$

$$= \left(\frac{2(h^-)^\alpha}{(2 - h^-)^\alpha + (h^-)^\alpha}, \frac{[1 + (1 - h^+)]^\alpha - [1 - (1 - h^+)]^\alpha}{[1 + (1 - h^+)]^\alpha + [1 - (1 - h^+)]^\alpha} \right)$$

$$= \left(\frac{2(h^-)^\alpha}{(2 - h^-)^\alpha + (h^-)^\alpha}, \frac{(2 - h^+)^\alpha - (h^+)^\alpha}{(2 - h^+)^\alpha + (h^+)^\alpha} \right).$$

$$A_{\mathrm{env}}\left(h^{\wedge_\varepsilon \alpha} \right)$$

$$= A_{\mathrm{env}} \left(\bigcup_{\gamma \in h} \left\{ \frac{2\gamma^\alpha}{(2 - \gamma)^\alpha + \gamma^\alpha} \right\} \right)$$

$$= \left(\frac{2(h^-)^\alpha}{(2 - h^-)^\alpha + (h^-)^\alpha}, 1 - \frac{2(h^+)^\alpha}{(2 - h^+)^\alpha + (h^+)^\alpha} \right)$$

$$= \left(\frac{2(h^-)^\alpha}{(2 - h^-)^\alpha + (h^-)^\alpha}, \frac{(2 - h^+)^\alpha - (h^+)^\alpha}{(2 - h^+)^\alpha + (h^+)^\alpha} \right).$$

$$(11)$$

Thus, $A_{\mathrm{env}}(h^{\wedge_\varepsilon \alpha}) = (A_{\mathrm{env}}(h))^{\wedge_\varepsilon \alpha}$.

(6) By the definition of the envelope of an HFE and the operation laws (2) and (9), we have

$$A_{\mathrm{env}}(h_1) \otimes_\varepsilon A_{\mathrm{env}}(h_2)$$

$$= \left(h_1^-, 1 - h_1^+ \right) \otimes_\varepsilon \left(h_2^-, 1 - h_2^+ \right)$$

$$= \left(\frac{h_1^- h_2^-}{1 + (1 - h_1^-)(1 - h_2^-)}, \frac{(1 - h_1^+) + (1 - h_2^+)}{1 + (1 - h_1^+)(1 - h_2^+)} \right)$$

$A_{\mathrm{env}}\left(h_1\otimes_\varepsilon h_2\right)$

$$= A_{\mathrm{env}}\left(\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\left\{\frac{\gamma_1\gamma_2}{1+\left(1-\gamma_1\right)\left(1-\gamma_2\right)}\right\}\right)$$

$$= \left(\frac{h_1^- h_2^-}{1+\left(1-h_1^-\right)\left(1-h_2^-\right)},1-\frac{h_1^+ h_2^+}{1+\left(1-h_1^+\right)\left(1-h_2^+\right)}\right)$$

$$= \left(\frac{h_1^- h_2^-}{1+\left(1-h_1^-\right)\left(1-h_2^-\right)},\frac{\left(1-h_1^+\right)+\left(1-h_2^+\right)}{1+\left(1-h_1^+\right)\left(1-h_2^+\right)}\right).$$

(12)

Thus, $A_{\mathrm{env}}(h_1\otimes_\varepsilon h_2)=A_{\mathrm{env}}(h_1)\otimes_\varepsilon A_{\mathrm{env}}(h_2)$. $\qquad\square$

Remark 8. Let $\alpha_1>0$, $\alpha_2>0$, and h be an HFE. It is worth noting that $h^{\wedge_\varepsilon\alpha_1}\otimes_\varepsilon h^{\wedge_\varepsilon\alpha_2}\doteq h^{\wedge_\varepsilon(\alpha_1+\alpha_2)}$ does not hold necessarily in general. To illustrate that, an example is given as follows.

Example 9. Let $h=(0.3,0.5)$, $\alpha_1=\alpha_2=1$; then $h^{\wedge_\varepsilon\alpha_1}\otimes_\varepsilon h^{\wedge_\varepsilon\alpha_2}=h\otimes_\varepsilon h=\bigcup_{\gamma_i\in h,\gamma_j\in h,(i,j=1,2)}\{\gamma_i\gamma_j/(1+(1-\gamma_i)(1-\gamma_j))\}=(0.0604,0.1111,0.2)$, and $h^{\wedge_\varepsilon(\alpha_1+\alpha_2)}=h^{\wedge_\varepsilon 2}=\bigcup_{\gamma\in h}\{2\gamma^2/((2-\gamma)^2+\gamma^2)\}=(0.0604,0.2)$. Clearly, $s(h^{\wedge_\varepsilon\alpha_1}\otimes_\varepsilon h^{\wedge_\varepsilon\alpha_2})=0.1238<0.1302=s(h^{\wedge_\varepsilon(\alpha_1+\alpha_2)})$. Thus $h^{\wedge_\varepsilon\alpha_1}\otimes_\varepsilon h^{\wedge_\varepsilon\alpha_1}\prec h^{\wedge_\varepsilon(\alpha_1+\alpha_2)}$.

However, if the number of the values in h is only one, that is, HFE h is reduced to a fuzzy value, then the above result holds.

Proposition 10. *Let $\alpha_1>0$, $\alpha_2>0$, and h be an HFE, in which the number of the values is only one, that is, $h=\{\gamma\}$; then $h^{\wedge_\varepsilon\alpha_1}\otimes_\varepsilon h^{\wedge_\varepsilon\alpha_2}=h^{\wedge_\varepsilon(\alpha_1+\alpha_2)}$.*

Proof. Since $h^{\wedge_\varepsilon\alpha_1}=\bigcup_{\gamma\in h}\{2\gamma^{\alpha_1}/((2-\gamma)^{\alpha_1}+\gamma^{\alpha_1})\}$ and $h^{\wedge_\varepsilon\alpha_2}=\bigcup_{\gamma\in h}\{2\gamma^{\alpha_2}/((2-\gamma)^{\alpha_2}+\gamma^{\alpha_2})\}$, then

$h^{\wedge_\varepsilon\alpha_1}\otimes_\varepsilon h^{\wedge_\varepsilon\alpha_1}$

$$= \bigcup_{\gamma\in h}\left\{\left(\left(2\gamma^{\alpha_1}/\left(\left(2-\gamma\right)^{\alpha_1}+\gamma^{\alpha_1}\right)\right)\right.\right.$$

$$\left.\cdot\left(2\gamma^{\alpha_2}/\left(\left(2-\gamma\right)^{\alpha_2}+\gamma^{\alpha_2}\right)\right)\right)$$

$$\times\left(1+\left(1-\left(2\gamma^{\alpha_1}/\left(\left(2-\gamma\right)^{\alpha_1}+\gamma^{\alpha_1}\right)\right)\right)\right.$$

$$\left.\times\left(1-\left(2\gamma^{\alpha_2}/\left(\left(2-\gamma\right)^{\alpha_2}+\gamma^{\alpha_2}\right)\right)\right)\right)^{-1}\right\}$$

$$= \bigcup_{\gamma\in h}\left\{\left(2\gamma^{\alpha_1}\cdot 2\gamma^{\alpha_2}\right)\times\left(\left[\left(2-\gamma\right)^{\alpha_1}+\gamma^{\alpha_1}\right]\cdot\left[\left(2-\gamma\right)^{\alpha_2}+\gamma^{\alpha_2}\right]\right.\right.$$

$$+\left[\left(2-\gamma\right)^{\alpha_1}-\gamma^{\alpha_1}\right]$$

$$\left.\left.\cdot\left[\left(2-\gamma\right)^{\alpha_2}-\gamma^{\alpha_2}\right]\right)^{-1}\right\}$$

$$= \bigcup_{\gamma\in h}\left\{\frac{2\gamma^{\alpha_1+\alpha_2}}{\left(2-\gamma\right)^{\alpha_1+\alpha_2}+\gamma^{\alpha_1+\alpha_2}}\right\}$$

$$= h^{\wedge_\varepsilon(\alpha_1+\alpha_2)}.$$

(13)

$\square$

Proposition 10 shows that it is consistent with the result (iii) in Theorem 2 in the literature [11].

4. Hesitant Fuzzy Einstein Geometric Aggregation Operators

The weighted geometric operator [47] and the ordered weighted geometric operator [48] are two of the most common and basic aggregation operators. Since their appearance, they have received more and more attention. In this section, we extend them to aggregate hesitant fuzzy information using Einstein operations.

4.1. Hesitant Fuzzy Einstein Geometric Weighted Aggregation Operator. Based on the operational laws (5) and (7) on HFEs, Xia and Xu [36] developed some hesitant fuzzy aggregation operators as listed below.

Let h_j $(j=1,2,\ldots,n)$ be a collection of HFEs; then.

(1) the hesitant fuzzy weighted geometric (HFWG) operator

$$\mathrm{HFWG}\left(h_1,h_2,\ldots,h_n\right)=\bigotimes_{j=1}^n h_j^{\omega_j}$$

$$= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n}\left\{\prod_{j=1}^n\gamma_j^{\omega_j}\right\},$$

(14)

where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ is the weight vector of h_j $(j=1,2,\ldots,n)$ with $\omega_j\in[0,1]$ and $\sum_{j=1}^n\omega_j=1$.

(2) the hesitant fuzzy ordered weighted geometric (HFOWG) operator

$$\mathrm{HFOWG}\left(h_1,h_2,\ldots,h_n\right)$$

$$= \bigotimes_{j=1}^n w_j^{h_{\sigma(j)}}$$

$$= \bigcup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\ldots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{\prod_{j=1}^n\gamma_{\sigma(j)}^{w_j}\right\},$$

(15)

where $\sigma(1),\sigma(2),\ldots,\sigma(n)$ is a permutation of $1,2,\ldots,n$, such that $h_{\sigma(j-1)}>h_{\sigma(j)}$ for all $j=2,\ldots,n$ and $w=(w_1,w_2,\ldots,w_n)^T$ is aggregation-associated vector with $w_j\in[0,1]$ and $\sum_{j=1}^n w_j=1$.

For convenience, let H be the set of all HFEs. Based on the proposed Einstein operations on HFEs, we develop some new aggregation operators for HFEs and discuss their desirable properties.

Definition 11. Let h_j ($j = 1, 2, \ldots, n$) be a collection of HFEs. A hesitant fuzzy Einstein weighted geometric (HFEWG$_\varepsilon$) operator of dimension n is a mapping HFEWG$_\varepsilon : H^n \to H$ defined as follows:

$$
\begin{aligned}
&\text{HFEWG}_\varepsilon \left(h_1, h_2, \ldots, h_n\right) \\
&= \bigotimes_{\varepsilon}_{j=1}^{n} h_j^{\wedge_\varepsilon \omega_j} \\
&= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \left\{ \frac{2\prod_{j=1}^{n}\gamma_j^{\omega_j}}{\prod_{j=1}^{n}\left(2 - \gamma_j\right)^{\omega_j} + \prod_{j=1}^{n}\gamma_j^{\omega_j}} \right\},
\end{aligned} \tag{16}
$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of h_j ($j = 1, 2, \ldots, n$) and $w_j > 0$, $\sum_{j=1}^{n} w_j = 1$. In particular, when $w_j = 1/n$, $j = 1, 2, \ldots, n$, the HFEWG$_\varepsilon$ operator is reduced to the hesitant fuzzy Einstein geometric (HFEG$_\varepsilon$) operator:

$$
\begin{aligned}
&\text{HFEG}_\varepsilon \left(h_1, h_2, \ldots, h_n\right) \\
&= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \left\{ \frac{2\prod_{j=1}^{n}\gamma_j^{1/n}}{\prod_{j=1}^{n}\left(2 - \gamma_j\right)^{1/n} + \prod_{j=1}^{n}\gamma_j^{1/n}} \right\}.
\end{aligned} \tag{17}
$$

From Proposition 10, we easily get the following result.

Corollary 12. *If all h_j ($j = 1, 2, \ldots, n$) are equal and the number of values in h_j is only one, that is, $h_j = h = \{\gamma\}$ for all $j = 1, 2, \ldots, n$, then*

$$
HFEWG_\varepsilon \left(h_1, h_2, \ldots, h_n\right) = h. \tag{18}
$$

Note that the HFEWG$_\varepsilon$ operator is not idempotent in general; we give the following example to illustrate this case.

Example 13. Let $h_1 = h_2 = h_3 = h = (0.3, 0.7)$, $w = (0.4, 0.25, 0.35)^T$; then HFEWG$_\varepsilon(h_1, h_2, h_3) = \{0.3, 0.4137, 0.3782, 0.5126, 0.4323, 0.579, 0.5342, 0.7\}$. By Definition 3, we have $s(\text{HFEWG}_\varepsilon(h_1, h_2, h_3)) = 0.4812 < 0.5 = s(h)$. Hence HFEWG$_\varepsilon(h_1, h_2, h_3) \prec h$.

Lemma 14 (see [18, 49]). *Let $\gamma_j > 0$, $w_j > 0$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} w_j = 1$. Then*

$$
\prod_{j=1}^{n}\gamma_j^{w_j} \leq \sum_{j=1}^{n} w_j \gamma_j \tag{19}
$$

with equality if and only if $\gamma_1 = \gamma_2 = \cdots = \gamma_n$.

Theorem 15. *Let h_j ($j = 1, 2, \ldots, n$) be a collection of HFEs and $w = (w_1, w_2, \ldots, w_n)^T$ the weight vector of h_j ($j = 1, 2, \ldots, n$) with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Then*

$$
HFEWG_\varepsilon \left(h_1, h_2, \ldots, h_n\right) \succeq HFWG \left(h_1, h_2, \ldots, h_n\right), \tag{20}
$$

where the equality holds if only if all h_j ($j = 1, 2, \ldots, n$) are equal and the number of values in h_j is only one.

Proof. For any $\gamma_j \in h_j$ ($j = 1, 2, \ldots, n$), by Lemma 14, we have

$$
\prod_{j=1}^{n}\left(2 - \gamma_j\right)^{w_j} + \prod_{j=1}^{n}\gamma_j^{w_j} \leq \sum_{j=1}^{n} w_j \left(2 - \gamma_j\right) + \sum_{j=1}^{n} w_j \gamma_j = 2. \tag{21}
$$

Then

$$
\frac{2\prod_{j=1}^{n}\gamma_j^{\omega_j}}{\prod_{j=1}^{n}\left(2 - \gamma_j\right)^{\omega_j} + \prod_{j=1}^{n}\gamma_j^{\omega_j}} \geq \prod_{j=1}^{n}\gamma_j^{\omega_j}. \tag{22}
$$

It follows that $s(\otimes_{\varepsilon}_{j=1}^{n} h_j^{\wedge_\varepsilon \omega_j}) \geq s(\otimes_{\varepsilon}_{j=1}^{n} h_j^{\omega_j})$, which completes the proof of Theorem 15. $\square$

Theorem 15 tells us the result that the HFEWG$_\varepsilon$ operator shows the decision maker's more optimistic attitude than the HFWA operator proposed by Xia and Xu [36] (i.e., (15)) in aggregation process. To illustrate that, we give an example adopted from Example 1 in [36] as follows.

Example 16. Let $h_1 = (0.2, 0.3, 0.5)$, $h_2 = (0.4, 0.6)$ be two HFEs, and let $w = (0.7, 0.3)^T$ be the weight vector of h_j ($j = 1, 2$); then by Definition 11, we have

$$
\begin{aligned}
&\text{HFEWG}_\varepsilon \left(h_1, h_2\right) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{2\prod_{j=1}^{2}\gamma_j^{\omega_j}}{\prod_{j=1}^{2}\left(2 - \gamma_j\right)^{\omega_j} + \prod_{j=1}^{2}\gamma_j^{\omega_j}} \right\} \\
&= \{0.2482, 0.2856, 0.3276, 0.3744, \\
&\quad\quad 0.4683, 0.5288\}.
\end{aligned} \tag{23}
$$

However, Xia and Xu [36] used the HFWG operator to aggregate the h_j ($j = 1, 2$) and got

$$
\begin{aligned}
&\text{HFEG} \left(h_1, h_2\right) \\
&= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \prod_{j=1}^{2} \gamma_j^{w_j} \right\} \\
&= \{0.2462, 0.2781, 0.3270, 0.3693, 0.4676, 0.5281\}.
\end{aligned} \tag{24}
$$

It is clear that $s(\text{HFEWG}_\varepsilon(h_1, h_2)) = 0.3722 > 0.3694 = s(\text{HFEG}(h_1, h_2))$. Thus HFEWG$_\varepsilon(h_1, h_2) \succ$ HFEG(h_1, h_2).

Based on Definition 11 and the proposed operational laws, we can obtain the following properties on HFEWG$_\varepsilon$ operator.

Theorem 17. *Let $\alpha > 0$, h_j ($j = 1, 2, \ldots, n$), be a collection of HFEs and $w = (w_1, w_2, \ldots, w_n)^T$ the weight vector of h_j ($j = 1, 2, \ldots, n$) with $w_j \in [0, 1]$ and $\sum_{i=1}^{n} w_j = 1$. Then*

$$
\begin{aligned}
&HFEWG_\varepsilon \left(h_1^{\wedge_\varepsilon \alpha}, h_2^{\wedge_\varepsilon \alpha}, \ldots, h_n^{\wedge_\varepsilon \alpha}\right) \\
&= \left(HFEWG_\varepsilon \left(h_1, h_2, \ldots, h_n\right)\right)^{\wedge_\varepsilon \alpha}.
\end{aligned} \tag{25}
$$

Proof. Since $h_j^{\wedge_\varepsilon \alpha} = \bigcup_{\gamma \in h_j}\{2\gamma_j^\alpha/((2-\gamma_j)^\alpha + \gamma_j^\alpha)\}$ for all $j = 1, 2, \ldots, n$, by the definition of HFEWG_ε, we have

$$\text{HFEWG}_\varepsilon\left(h_1^{\wedge_\varepsilon \alpha}, h_2^{\wedge_\varepsilon \alpha}, \ldots, h_n^{\wedge_\varepsilon \alpha}\right)$$

$$= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \left\{ \left(2\prod_{j=1}^n (2\gamma_j^\alpha/((2-\gamma_j)^\alpha + \gamma_j^\alpha))^{\omega_j} \right) \right.$$

$$\times \left(\prod_{j=1}^n (2 - (2\gamma_j^\alpha/((2-\gamma_j)^\alpha + \gamma_j^\alpha)))^{\omega_j} \right.$$

$$\left. \left. + \prod_{j=1}^n (2\gamma_j^\alpha/((2-\gamma_j)^\alpha + \gamma_j^\alpha))^{\omega_j} \right)^{-1} \right\}$$

$$= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \left\{ \frac{2\prod_{j=1}^n \gamma_j^{\alpha\omega_j}}{\prod_{j=1}^n (2-\gamma_j)^{\alpha\omega_j} + \prod_{j=1}^n \gamma_j^{\alpha\omega_j}} \right\}.$$

(26)

Since $\text{HFEWG}_\varepsilon(h_1, h_2, \ldots, h_n) = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n}\{2\prod_{j=1}^n \gamma_j^{\omega_j}/(\prod_{j=1}^n (2-\gamma_j)^{\omega_j} + \prod_{j=1}^n \gamma_j^{\omega_j})\}$, then

$$\left(\text{HFEWG}_\varepsilon\left(h_1, h_2, \ldots, h_n\right)\right)^{\wedge_\varepsilon \alpha}$$

$$= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \left\{ \left(2\left(2\prod_{j=1}^n \gamma_j^{\omega_j}/\left(\prod_{j=1}^n (2-\gamma_j)^{\omega_j} \right.\right.\right.\right.$$

$$\left.\left.\left.\left. + \prod_{j=1}^n \gamma_j^{\omega_j} \right) \right)^\alpha \right) \right.$$

$$\times \left(\left(2 - \left(2\prod_{j=1}^n \gamma_j^{\omega_j}/\left(\prod_{j=1}^n (2-\gamma_j)^{\omega_j} \right.\right.\right.\right.$$

$$\left.\left.\left.\left. + \prod_{j=1}^n \gamma_j^{\omega_j} \right) \right) \right)^\alpha \right.$$

$$\left. + \left(2\prod_{j=1}^n \gamma_j^{\omega_j}/\left(\prod_{j=1}^n (2-\gamma_j)^{\omega_j} \right.\right.\right.$$

$$\left.\left.\left. + \prod_{j=1}^n \gamma_j^{\omega_j} \right) \right)^\alpha \right)^{-1} \right\}$$

$$= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \left\{ \frac{2(\prod_{j=1}^n \gamma_j^{\omega_j})^\alpha}{(\prod_{j=1}^n (2-\gamma_j)^{\omega_j})^\alpha + (\prod_{j=1}^n \gamma_j^{\alpha\omega_j})^\alpha} \right\}$$

$$= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \left\{ \frac{2\prod_{j=1}^n \gamma_j^{\alpha\omega_j}}{\prod_{j=1}^n (2-\gamma_j)^{\alpha\omega_j} + \prod_{j=1}^n \gamma_j^{\alpha\omega_j}} \right\}.$$

(27)

□

Theorem 18. *Let h be an HFE, h_j ($j = 1, 2, \ldots, n$) a collection of HFEs, and $w = (w_1, w_2, \ldots, w_n)^T$ the weight vector of h_j ($j = 1, 2, \ldots, n$) with $w_j \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$. Then*

$$\text{HFEWG}_\varepsilon\left(h_1 \otimes_\varepsilon h, h_2 \otimes_\varepsilon h, \ldots, h_n \otimes_\varepsilon h\right)$$

$$= \text{HFEWG}_\varepsilon\left(h_1, h_2, \ldots, h_n\right) \otimes_\varepsilon h.$$

(28)

Proof. By the definition of HFEWG_ε and Einstein product operator of HFEs, we have

$$\text{HFEWG}_\varepsilon\left(h_1, h_2, \ldots, h_n\right) \otimes_\varepsilon h$$

$$= \bigcup_{\gamma \in h, \gamma_j \in h_j, j=1,\ldots,n} \left\{ \frac{\left(2\prod_{j=1}^n \gamma_j^{\omega_j}/\left(\prod_{j=1}^n (2-\gamma_j)^{\omega_j} + \prod_{j=1}^n \gamma_j^{\omega_j}\right)\right) \cdot \gamma}{1 + \left(1 - \left(2\prod_{j=1}^n \gamma_j^{\omega_j}/\left(\prod_{j=1}^n (2-\gamma_j)^{\omega_j} + \prod_{j=1}^n \gamma_j^{\omega_j}\right)\right)\right)(1-\gamma)} \right\}$$

$$= \bigcup_{\gamma \in h, \gamma_j \in h_j, j=1,\ldots,n} \left\{ \frac{2\gamma\prod_{j=1}^n \gamma_j^{\omega_j}}{(2-\gamma)\prod_{j=1}^n (2-\gamma_j)^{\omega_j} + \gamma\prod_{j=1}^n \gamma_j^{\omega_j}} \right\}.$$

(29)

Since $h_j \otimes_\varepsilon h = \bigcup_{\gamma_j \in h_j, \gamma \in h}\{\gamma_j\gamma/(1 + (1-\gamma_j)(1-\gamma))\}$ for all $j = 1, 2, \ldots, n$, by the definition of HFEWG_ε, we have

$$\text{HFEWG}_\varepsilon\left(h_1 \otimes_\varepsilon h, h_2 \otimes_\varepsilon h, \ldots, h_n \oplus_\varepsilon h\right)$$

$$= \bigcup_{\gamma \in h, \gamma_j \in h_j, j=1,\ldots,n} \left\{ \left(2\prod_{j=1}^n (\gamma_j\gamma/(1 + (1-\gamma_j)(1-\gamma)))^{\omega_j} \right) \right.$$

$$\times \left(\prod_{j=1}^n (2 - (\gamma_j\gamma/(1 + (1-\gamma_j) \right.$$

$$\left. \times (1-\gamma))))^{\omega_j} \right.$$

$$\left. + \prod_{j=1}^n (\gamma_j\gamma/(1 + (1-\gamma_j) \right.$$

$$\left. \left. \times (1-\gamma)))^{w_j} \right)^{-1} \right\}$$

$$= \bigcup_{\gamma \in h, \gamma_j \in h_j, j=1,\ldots,n} \left\{ \frac{2\prod_{j=1}^n (\gamma_j\gamma)^{\omega_j}}{\prod_{j=1}^n ((2-\gamma_j)(2-\gamma))^{\omega_j} + \prod_{j=1}^n (\gamma_j\gamma)^{\omega_j}} \right\}$$

$$
= \bigcup_{\gamma \in h, \gamma_j \in h_j, j=1,\ldots,n} \left\{ \left(2\prod_{j=1}^{n} \gamma^{\omega_j} \cdot \prod_{j=1}^{n} \gamma_j^{\omega_j} \right) \right.
$$

$$
\times \left(\prod_{j=1}^{n} (2-\gamma)^{\omega_j} \cdot \prod_{j=1}^{n} (2-\gamma_j)^{\omega_j} + \prod_{j=1}^{n} \gamma^{\omega_j} \right.
$$

$$
\left. \left. \cdot \prod_{j=1}^{n} \gamma_j^{\omega_j} \right)^{-1} \right\}
$$

$$
= \bigcup_{\gamma \in h, \gamma_j \in h_j, j=1,\ldots,n} \left\{ \left(2\gamma^{\sum_{j=1}^{n}\omega_j} \cdot \prod_{j=1}^{n} \gamma_j^{\omega_j} \right) \right.
$$

$$
\times \left((2-\gamma)^{\sum_{j=1}^{n}\omega_j} \cdot \prod_{j=1}^{n} (2-\gamma_j)^{\omega_j} \right.
$$

$$
\left. \left. + \gamma^{\sum_{j=1}^{n}\omega_j} \cdot \prod_{j=1}^{n} \gamma_j^{\omega_j} \right)^{-1} \right\}
$$

$$
= \bigcup_{\gamma \in h, \gamma_j \in h_j, j=1,\ldots,n} \left\{ \frac{2\gamma \prod_{j=1}^{n} \gamma_j^{\omega_j}}{(2-\gamma)\prod_{j=1}^{n}(2-\gamma_j)^{\omega_j} + \gamma\prod_{j=1}^{n}\gamma_j^{\omega_j}} \right\}.
$$

$$
\tag{30}
$$

$\square$

Based on Theorems 17 and 18, the following property can be obtained easily.

Theorem 19. *Let* $\alpha > 0$, *h be an HFE, let h_j ($j = 1, 2, \ldots, n$) be a collection of HFEs, and let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of h_j ($j = 1, 2, \ldots, n$) with $w_j \in [0,1]$ and $\sum_{i=1}^{n} w_j = 1$. Then*

$$
HFEWG_\varepsilon \left(h_1^{\wedge_\varepsilon \alpha} \otimes_\varepsilon h, h_1^{\wedge_\varepsilon \alpha} \otimes_\varepsilon h, \ldots, h_n^{\wedge_\varepsilon \alpha} \otimes_\varepsilon h \right)
$$

$$
= \left(HFEWG_\varepsilon \left(h_1, h_2, \ldots, h_n \right) \otimes_\varepsilon h \right)^{\wedge_\varepsilon \alpha}.
$$

$$
\tag{31}
$$

Theorem 20. *Let h_j and h_j' ($j = 1, 2, \ldots, n$) be two collections of HFEs and $w = (w_1, w_2, \ldots, w_n)^T$ the weight vector of h_j ($j = 1, 2, \ldots, n$) with $w_j \in [0,1]$ and $\sum_{i=1}^{n} w_j = 1$. Then*

$$
HFEWG_\varepsilon \left(h_1 \otimes_\varepsilon h_1', h_2 \otimes_\varepsilon h_2', \ldots, h_n \otimes_\varepsilon h_n' \right)
$$

$$
= HFEWG_\varepsilon \left(h_1, h_2, \ldots, h_n \right) \otimes_\varepsilon HFEWG_\varepsilon \left(h_1', h_2', \ldots, h_n' \right).
$$

$$
\tag{32}
$$

Proof. By the definition of $HFEWG_\varepsilon$ and Einstein product operator of HFEs, we have

$$
HFEWG_\varepsilon \left(h_1, h_2, \ldots, h_n \right) \otimes_\varepsilon HFEWG_\varepsilon \left(h_1', h_2', \ldots, h_n' \right)
$$

$$
= \bigcup_{\gamma_j \in h_j, \gamma_j' \in h_j', j=1,\ldots,n} \left\{ \frac{\left(2\prod_{j=1}^{n}\gamma_j^{\omega_j} / \left(\prod_{j=1}^{n}(2-\gamma_j)^{\omega_j} + \prod_{j=1}^{n}\gamma_j^{\omega_j} \right) \right) \cdot \left(2\prod_{j=1}^{n}\gamma_j'^{\omega_j} / \left(\prod_{j=1}^{n}(2-\gamma_j')^{\omega_j} + \prod_{j=1}^{n}\gamma_j'^{\omega_j} \right) \right)}{1 + \left(1 - \left(2\prod_{j=1}^{n}\gamma_j^{\omega_j} / \left(\prod_{j=1}^{n}(2-\gamma_j)^{\omega_j} + \prod_{j=1}^{n}\gamma_j^{\omega_j} \right) \right) \right) \left(1 - \left(2\prod_{j=1}^{n}\gamma_j'^{\omega_j} / \left(\prod_{j=1}^{n}(2-\gamma_j')^{\omega_j} + \prod_{j=1}^{n}\gamma_j'^{\omega_j} \right) \right) \right)} \right\}
$$

$$
= \bigcup_{\gamma_j \in h_j, \gamma_j' \in h_j', j=1,\ldots,n} \left\{ \frac{2\prod_{j=1}^{n}\gamma_j^{\omega_j} \cdot \prod_{j=1}^{n}\gamma_j'^{\omega_j}}{\prod_{j=1}^{n}(2-\gamma_j)^{\omega_j} \cdot \prod_{j=1}^{n}(2-\gamma_j')^{\omega_j} + \prod_{j=1}^{n}\gamma_j^{\omega_j} \cdot \prod_{j=1}^{n}\gamma_j'^{\omega_j}} \right\}.
$$

$$
\tag{33}
$$

Since $h_j \otimes_\varepsilon h_j' = \bigcup_{\gamma_j \in h_j, \gamma_j' \in h_j'} \{ \gamma_j \gamma_j' / (1 + (1-\gamma_j)(1-\gamma_j')) \}$ for all $j = 1, 2, \ldots, n$, by the definition of $HFEWG_\varepsilon$, we have

$$
HFEWG_\varepsilon \left(h_1 \bigotimes_\varepsilon h_1', h_2 \bigotimes_\varepsilon h_2', \ldots, h_n \bigotimes_\varepsilon h_n' \right)
$$

$$
= \bigcup_{\gamma_j \in h_j, \gamma_j' \in h_j', j=1,\ldots,n} \left\{ \left(2\prod_{j=1}^{n} \left(\gamma_j \gamma_j' / \left(1 + \left(1-\gamma_j \right) \left(1-\gamma_j' \right) \right) \right)^{\omega_j} \right) \right.
$$

$$
\times \left(\prod_{j=1}^{n} \left(2 - \left(\gamma_j \gamma_j' / \left(1 + \left(1-\gamma_j \right) \right. \right. \right. \right.
$$

$$
\left. \left. \left. \times \left(1-\gamma_j' \right) \right) \right) \right)^{\omega_j}
$$

$$
+ \prod_{j=1}^{n} \left(\gamma_j \gamma_j' / \left(1 + \left(1-\gamma_j \right) \right. \right.
$$

$$
\left. \left. \times \left(1-\gamma_j' \right) \right) \right)^{\omega_j} \right)^{-1} \right\}
$$

$$
= \bigcup_{\gamma_j \in h_j, \gamma_j' \in h_j', j=1,\ldots,n} \left\{ \left(2\prod_{j=1}^{n} \left(\gamma_j \gamma_j' \right)^{\omega_j} \right) \right.
$$

$$
\times \left(\prod_{j=1}^{n} \left[\left(2-\gamma_j \right) \left(2-\gamma_j' \right) \right]^{\omega_j} \right.
$$

$$
\left. \left. + \prod_{j=1}^{n} \left(\gamma_j \gamma_j' \right)^{\omega_j} \right)^{-1} \right\}
$$

$$
= \bigcup_{\gamma_j \in h_j, \gamma_j' \in h_j', j=1,\ldots,n} \left\{ \left(2\prod_{j=1}^{n} \gamma_j^{\omega_j} \cdot \prod_{j=1}^{n} \gamma_j'^{\omega_j} \right) \right.
$$

$$\times \left(\prod_{j=1}^{n} \left(2 - \gamma_j\right)^{\omega_j} \cdot \prod_{j=1}^{n} \left(2 - \gamma_j'\right)^{\omega_j} \right.$$

$$\left. + \prod_{j=1}^{n} \gamma_j^{\omega_j} \cdot \prod_{j=1}^{n} {\gamma_j'}^{\omega_j} \right)^{-1} \right\}.$$

(34)

□

Theorem 21. *Let h_j $(j = 1, 2, \ldots, n)$ be a collection of HFEs, $h_{\min}^- = \min_j\{h_j^- \mid h_j^- = \min\{\gamma_j \in h_j\}\}$, and $h_{\max}^+ = \max_j\{h_j^+ \mid h_j^+ = \max\{\gamma_j \in h_j\}\}$, and let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of h_j $(j = 1, 2, \ldots, n)$ with $w_j \in [0, 1]$ and $\sum_{i=1}^{n} w_j = 1$. Then*

$$h_{\min}^- \preceq HFEWG_\varepsilon\left(h_1, h_2, \ldots, h_n\right) \preceq h_{\max}^+, \tag{35}$$

where the equality holds if only if all h_j $(j = 1, 2, \ldots, n)$ are equal and the number of values in h_j is only one.

Proof. Let $f(t) = (2 - t)/t$, $t \in [0, 1]$. Then $f'(t) = -2/t^2 < 0$. Hence $f(t)$ is a decreasing function. Since $h_{\min}^- \leq h_j^- \leq \gamma_j \leq h_j^+ \leq h_{\max}^+$ for any $\gamma_j \in h_j$ $(j = 1, 2, \ldots, n)$, then $f(h_{\max}^+) \leq f(\gamma_j) \leq f(h_{\min}^-)$; that is, $(2 - h_{\max}^+)/h_{\max}^+ \leq (2 - \gamma_j)/\gamma_j \leq (2 - h_{\min}^-)/h_{\min}^-$. Then for any $\gamma_j \in h_j$ $(j = 1, 2, \ldots, n)$, we have

$$\prod_{j=1}^{n} \left(\frac{2 - h_{\max}^+}{h_{\max}^+} \right)^{w_j}$$

$$\leq \prod_{j=1}^{n} \left(\frac{2 - \gamma_j}{\gamma_j} \right)^{w_j} \leq \prod_{j=1}^{n} \left(\frac{1 - h_{\min}^-}{1 + h_{\min}^-} \right)^{w_j}$$

$$\Longleftrightarrow \left(\frac{2 - h_{\max}^+}{h_{\max}^+} \right)^{\sum_{j=1}^{n} w_j} \leq \prod_{j=1}^{n} \left(\frac{2 - \gamma_j}{\gamma_j} \right)^{w_j}$$

$$\leq \left(\frac{1 - h_{\min}^-}{1 + h_{\min}^-} \right)^{\sum_{j=1}^{n} w_j} \Longleftrightarrow \left(\frac{2 - h_{\max}^+}{h_{\max}^+} \right)$$

$$\leq \prod_{j=1}^{n} \left(\frac{2 - \gamma_j}{\gamma_j} \right)^{w_j} \leq \left(\frac{1 - h_{\min}^-}{1 + h_{\min}^-} \right) \Longleftrightarrow \frac{2}{h_{\max}^+}$$

$$\leq \prod_{j=1}^{n} \left(\frac{2 - \gamma_j}{\gamma_j} \right)^{w_j} + 1 \leq \frac{2}{h_{\min}^-} \Longleftrightarrow \frac{h_{\min}^-}{2}$$

$$\leq \frac{1}{\prod_{j=1}^{n} \left(\left(2 - \gamma_j\right)/\gamma_j\right)^{w_j} + 1} \leq \left(\frac{h_{\max}^+}{2} \right) \Longleftrightarrow h_{\min}^-$$

$$\leq \frac{2}{\prod_{j=1}^{n} \left(\left(2 - \gamma_j\right)/\gamma_j\right)^{w_j} + 1} \leq h_{\max}^+ \Longleftrightarrow h_{\min}^-$$

$$\leq \frac{2 \prod_{j=1}^{n} \gamma_j^{\omega_j}}{\prod_{j=1}^{n} \left(2 - \gamma_j\right)^{\omega_j} + \prod_{j=1}^{n} \gamma_j^{\omega_j}} \leq h_{\max}^+.$$

(36)

It follows that $h_{\min}^- \leq s(HFEWG_\varepsilon(h_1, h_2, \ldots, h_n)) \leq h_{\max}^+$. Thus we have $h_{\min}^- \preceq HFEWG_\varepsilon(h_1, h_2, \ldots, h_n) \preceq h_{\max}^+$. □

Remark 22. Let h_j and h_j' $(j = 1, 2, \ldots, n)$ be two collections of HFEs, and $h_j \prec h_j'$ for all j; then $HFEWG_\varepsilon(h_1, h_2, \ldots, h_n) \prec HFEWG_\varepsilon(h_1', h_2', \ldots, h_n')$ does not hold necessarily in general. To illustrate that, an example is given as follows.

Example 23. Let $h_1 = (0.45, 0.6)$, $h_2 = (0.6, 0.7)$, $h_3 = (0.5, 0.6)$, $h_1' = (0.2, 0.9)$, $h_2' = (0.45, 0.95)$, $h_3' = (0.35, 0.8)$, and $w = (0.5, 0.3, 0.2)^T$; then $HFEWG_\varepsilon(h_1, h_2, h_3) = \{0.5024, 0.5215, 0.5286, 0.5483, 0.5791, 0.6, 0.6077, 0.6291\}$ and $HFEWG_\varepsilon(h_1', h_2', h_3') = \{0.2778, 0.3372, 0.3835, 0.4595, 0.6088, 0.7099, 0.7833, 0.8947\}$. By Definition 3, we have $s(HFEWG_\varepsilon(h_1, h_2, h_3)) = 0.5646$ and $s(HFEWG_\varepsilon(h_1', h_2', h_3')) = 0.5568$. It follows that $HFEWG_\varepsilon(h_1, h_2, h_3) \succ HFEWG_\varepsilon(h_1', h_2', h_3')$. Clearly, $h_j \prec h_j'$ for $j = 1, 2, 3$, but $HFEWG_\varepsilon(h_1, h_2, h_3) \succ HFEWG_\varepsilon(h_1', h_2', h_3')$.

4.2. Hesitant Fuzzy Einstein Ordered Weighted Averaging Operator. Similar to the HFOWG operator introduced by Xia and Xu [36] (i.e., (15)), in what follows, we develop an $(HFEOWG_\varepsilon)$ operator, which is an extension of OWA operator proposed by Yager [50].

Definition 24. For a collection of the HFEs h_j $(j = 1, 2, \ldots, n)$, a hesitant fuzzy Einstein ordered weighted averaging $(HFEOWG_\varepsilon)$ operator is a mapping $HFEWG_\varepsilon : H^n \to H$ such that

$$HFEOWG_\varepsilon\left(h_1, h_2, \ldots, h_n\right)$$

$$= \bigotimes_{j=1}^{n} {}_{\varepsilon} h_{\sigma(j)}^{\wedge_\varepsilon w_j}$$

$$= \bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \ldots, \gamma_{\sigma(n)} \in h_{\sigma(n)}} \left\{ \left(2 \prod_{j=1}^{n} \gamma_{\sigma(j)}^{w_j} \right) \right.$$

$$\times \left(\prod_{j=1}^{n} \left(2 - \gamma_{\sigma(j)}\right)^{w_j} \right.$$

$$\left. \left. + \prod_{j=1}^{n} \gamma_{\sigma(j)}^{w_j} \right)^{-1} \right\},$$

(37)

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$, such that $h_{\sigma(j-1)} \succ h_{\sigma(j)}$ for all $j = 2, \ldots, n$ and $w = (w_1, w_2, \ldots, w_n)^T$ is aggregation-associated vector with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. In particular, if $w = (1/n, 1/n, \ldots, 1/n)^T$, then the $HFEOWG_\varepsilon$ operator is reduced to the $HFEA_\varepsilon$ operator of dimension n (i.e., (17)).

Note that the HFEOWG$_\varepsilon$ weights can be obtained similar to the OWA weights. Several methods have been introduced to determine the OWA weights in [20, 21, 50–53].

Similar to the HFEWG$_\varepsilon$ operator, the HFEOWG$_\varepsilon$ operator has the following properties.

Theorem 25. *Let h_j $(j = 1, 2, \ldots, n)$ be a collection of HFEs and $w = (w_1, w_2, \ldots, w_n)^T$ the weight vector of h_j $(j = 1, 2, \ldots, n)$ with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Then*

$$HFEOWG_\varepsilon (h_1, h_2, \ldots, h_n) \succeq HFOWG (h_1, h_2, \ldots, h_n),$$
$$(38)$$

where the equality holds if only if all h_j $(j = 1, 2, \ldots, n)$ are equal and the number of values in h_j is only one.

From Theorem 25, we can conclude that the values obtained by the HFEOWG$_\varepsilon$ operator are not less than the ones obtained by the HFOWA operator proposed by Xia and Xu [36]. To illustrate that, let us consider the following example.

Example 26. Let $h_1 = (0.1, 0.4, 0.7)$, $h_2 = (0.3, 0.5)$, and $h_3 = (0.2, 0.6)$ be three HFEs and suppose that $w = (0.2, 0.45, 0.35)^T$ is the associated vector of the aggregation operator.

By Definitions 3 and 4, we calculate the score values and the accuracy values of h_1, h_2, and h_3 as follows, respectively:
$s(h_1) = s(h_2) = s(h_3) = 0.5$, $k(h_1) = 0.7551$, $k(h_2) = 0.9$, $k(h_3) = 0.8$.

According to Definition 5, we have $h_2 \prec h_3 \prec h_1$. Then $h_{\sigma(1)} = h_2$, $h_{\sigma(2)} = h_3$, $h_{\sigma(3)} = h_1$.

By the definition of HFEOWG$_\varepsilon$, we have

$$HFEOWG_\varepsilon (h_1, h_2, h_3)$$

$$= \bigotimes_{j=1}^{3} {}_\varepsilon h_{\sigma(j)}^{\wedge_\varepsilon w_j}$$

$$= \bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \gamma_{\sigma(3)} \in h_{\sigma(3)}} \left\{ \frac{2\prod_{j=1}^{3} \gamma_{\sigma(j)}^{w_j}}{\prod_{j=1}^{3} \left(2 - \gamma_{\sigma(j)}\right)^{w_j} + \prod_{j=1}^{3} \gamma_{\sigma(j)}^{w_j}} \right\}$$

$$= \{0.1716, 0.2787, 0.3495, 0.2939, 0.4582, 0.5598, 0.1926,$$
$$0.3106, 0.3877, 0.3272, 0.5047, 0.6125\}.$$
$$(39)$$

If we use the HFOWA operator, which was given by Xia and Xu [36] (i.e., (15)), to aggregate the HFEs h_j $(i = 1, 2, 3)$, then we have

$$HFOWG (h_1, h_2, \ldots, h_n)$$

$$= \bigotimes_{j=1}^{3} h_{\sigma(j)}^{w_j} = \bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \gamma_{\sigma(3)} \in h_{\sigma(3)}} \left\{ \prod_{j=1}^{3} \gamma_{\sigma(j)}^{w_j} \right\}$$

$$= \{0.1702, 0.2764, 0.3363, 0.2790, 0.4532, 0.5513,$$
$$0.1885, 0.3062, 0.3724, 0.3090, 0.5020, 0.6106\}.$$
$$(40)$$

Clearly, $s(HFEOWG_\varepsilon(h_1, h_2, h_3)) = 0.3706 > 0.3629 = s(HFOWG(h_1, h_2, \ldots, h_n))$. By Definition 3, we have $HFEOWG_\varepsilon(h_1, h_2, h_3) \succ HFOWG(h_1, h_2, h_3)$.

Theorem 27. *Let $\alpha > 0$, h be an HFE, let h_j and h_j' $(j = 1, 2, \ldots, n)$ be two collection of HFEs, and let $w = (w_1, w_2, \ldots, w_n)^T$ be an aggregation-associated vector with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Then*

(1) $HFEWG_\varepsilon(h_1^{\wedge_\varepsilon \alpha}, h_2^{\wedge_\varepsilon \alpha}, \ldots, h_n^{\wedge_\varepsilon \alpha}) = (HFEWG_\varepsilon(h_1, h_2, \ldots, h_n))^{\wedge_\varepsilon \alpha}$,

(2) $HFEWG_\varepsilon(h_1 \otimes_\varepsilon h, h_2 \otimes_\varepsilon h, \ldots, h_n \otimes_\varepsilon h) = HFEWG_\varepsilon(h_1, h_2, \ldots, h_n) \otimes_\varepsilon h$,

(3) $HFEWG_\varepsilon(h_1^{\wedge_\varepsilon \alpha} \otimes_\varepsilon h, h_1^{\wedge_\varepsilon \alpha} \otimes_\varepsilon h, \ldots, h_n^{\wedge_\varepsilon \alpha} \otimes_\varepsilon h) = (HFEWG_\varepsilon(h_1, h_2, \ldots, h_n) \otimes_\varepsilon h)^{\wedge_\varepsilon \alpha}$,

(4) $HFEWG_\varepsilon(h_1 \otimes_\varepsilon h_1', h_2 \otimes_\varepsilon h_2', \ldots, h_n \otimes_\varepsilon h_n') = HFEWG_\varepsilon(h_1, h_2, \ldots, h_n) \otimes_\varepsilon HFEWG_\varepsilon(h_1', h_2', \ldots, h_n')$.

Theorem 28. *Let h_j $(j = 1, 2, \ldots, n)$ be a collection of HFEs and let $w = (w_1, w_2, \ldots, w_n)^T$ be an aggregation-associated vector with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Then*

$$h_{\min}^- \preceq HFEOWG_\varepsilon (h_1, h_2, \ldots, h_n) \preceq h_{\max}^+, \quad (41)$$

where $h_{\min}^- = \min_j \{h_j^- \mid h_j^- = \min\{\gamma_j \in h_j\}\}$ and $h_{\max}^+ = \max_j \{h_j^+ \mid h_j^+ = \max\{\gamma_j \in h_j\}\}$.

Besides the above properties, we can get the following desirable results on the HFOWG$_\varepsilon$ operator.

Theorem 29. *Let h_j $(j = 1, 2, \ldots, n)$ be a collection of HFEs, and let $w = (w_1, w_2, \ldots, w_n)^T$ be an aggregation-associated vector with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Then*

$$HFEOWG_\varepsilon (h_1, h_2, \ldots, h_n) = HFEOWG_\varepsilon (h_1', h_2', \ldots, h_n'),$$
$$(42)$$

where $(h_1', h_2', \ldots, h_n')$ is any permutation of $(h_1, h_2, \ldots, h_n)$.

Proof. Let $HFEOWG_\varepsilon(h_1, h_2, \ldots, h_n) = \otimes_{\varepsilon j=1}^{n} h_{\sigma(j)}^{\wedge_\varepsilon w_j}$ and $HFEOWG_\varepsilon(h_1', h_2', \ldots, h_n') = \otimes_{\varepsilon j=1}^{n} h_{\sigma(j)}'^{\wedge_\varepsilon w_j}$. Since $(h_1', h_2', \ldots, h_n')$ is any permutation of $(h_1, h_2, \ldots, h_n)$, then we have $h_{\sigma(j)} = h_{\sigma(j)}'$ $(j = 1, 2, \ldots, n)$. Thus $HFEOWG_\varepsilon(h_1, h_2, \ldots, h_n) = HFEOWG_\varepsilon(h_1', h_2', \ldots, h_n')$. $\square$

Theorem 30. *Let h_j $(j = 1, 2, \ldots, n)$ be a collection of HFEs, and let $w = (w_1, w_2, \ldots, w_n)^T$ be an aggregation-associated vector with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Then*

(1) *if $w = (0, 0, \ldots, 1)$, then $HFOWG_\varepsilon(h_1, h_2, \ldots, h_n) = \min\{h_1, h_2, \ldots, h_n\}$;*

(2) *if* $w = (1, 0, \ldots, 0)$, *then* $HFOWG_\varepsilon(h_1, h_2, \ldots, h_n) = \max\{h_1, h_2, \ldots, h_n\}$;

(3) *if* $w_j = 1$ *and* $w_i = 0$ $(i \neq j)$, *then* $HFOWG_\varepsilon(h_1, h_2, \ldots, h_n) = h_{\sigma(j)}$, *where* $h_{\sigma(j)}$ *is the jth largest of* h_i $(i = 1, 2, \ldots, n)$.

5. An Application in Hesitant Fuzzy Decision Making

In this section, we apply the $HFEWG_\varepsilon$ and $HFEOWG_\varepsilon$ operators to multiple attribute decision making with hesitant fuzzy information.

For hesitant fuzzy multiple attribute decision making problems, let $Y = \{Y_1, Y_2, \ldots, Y_m\}$ be a discrete set of alternatives, let $A = \{A_1, A_2, \ldots, A_n\}$ be a collection of attributes, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of A_j $(j = 1, 2, \ldots, n)$ with $\omega_j \geq 0$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^n \omega_j = 1$. If the decision makers provide several values for the alternative Y_i $(i = 1, 2, \ldots, m)$ under the attribute A_j $(j = 1, 2, \ldots, n)$ with anonymity, these values can be considered as an HFE h_{ij}. In the case where two decision makers provide the same value, the value emerges only once in h_{ij}. Suppose that the decision matrix $H = (h_{ij})_{m \times n}$ is the hesitant fuzzy decision matrix, where h_{ij} $(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$ are in the form of HFEs.

To get the best alternative, we can utilize the $HFEWG_\varepsilon$ operator or the $HFEOWG_\varepsilon$ operator; that is,

$$h_i = HFEWG_\varepsilon(h_{i1}, h_{i2}, \ldots, h_{in})$$

$$= \bigcup_{\gamma_{i1} \in h_{i1}, \gamma_{i2} \in h_{i2}, \ldots, \gamma_{in} \in h_{in}} \left\{ \frac{2 \prod_{j=1}^n \gamma_{ij}^{\omega_j}}{\prod_{j=1}^n (2 - \gamma_{ij})^{\omega_j} + \prod_{j=1}^n \gamma_{ij}^{\omega_j}} \right\} \quad (43)$$

or

$$h_i = HFEOWG_\varepsilon(h_{i1}, h_{i2}, \ldots, h_{in})$$

$$= \bigcup_{\gamma_{i\sigma(j)} \in h_{i\sigma(j)}, j = 1, 2, \ldots, n} \left\{ \frac{2 \prod_{j=1}^n \gamma_{i\sigma(j)}^{w_j}}{\prod_{j=1}^n (2 - \gamma_{i\sigma(j)})^{w_j} + \prod_{j=1}^n \gamma_{i\sigma(j)}^{w_j}} \right\} \quad (44)$$

to derive the overall value h_i of the alternatives Y_i $(i = 1, 2, \ldots, m)$, where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector related to the $HFEOWA_\varepsilon$ operator, such that $w_j \geq 0$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^n w_j = 1$, which can be obtained by the normal distribution based method [20].

Then by Definition 3, we compute the scores $s(h_i)$ $(i = 1, 2, \ldots, m)$ of the overall values h_i $(i = 1, 2, \ldots, m)$ and use the scores $s(h_i)$ $(i = 1, 2, \ldots, m)$ to rank the alternatives $Y = \{Y_1, Y_2, \ldots, Y_m\}$ and then select the best one (note that if there is no difference between the two scores h_i and h_j, then we need to compute the accuracy degrees $k(h_i)$ and $k(h_j)$ of the overall values h_i and h_j by Definition 4, respectively, and then rank the alternatives Y_i and Y_j in accordance with Definition 5).

In the following, an example on multiple attribute decision making problem involving a customer buying a car,

which is adopted from Herrera and Martinez [54], is given to illustrate the proposed method using the $HFEOWG_\varepsilon$ operator.

Example 31. Consider that a customer wants to buy a car, which will be chosen from five types Y_i $(i = 1, 2, \ldots, 5)$. In the process of choosing one of the cars, four factors are considered: A_1 is the consumption petrol, A_2 is the price, A_3 is the degree of comfort, and A_4 is the safety factor. Suppose that the characteristic information of the alternatives Y_i $(i = 1, 2, \ldots, 5)$ can be represented by HFEs h_{ij} $(i = 1, 2, \ldots, 5; j = 1, 2, \ldots, 4)$, and the hesitant fuzzy decision matrix is given in Table 1.

To use $HFEOWG_\varepsilon$ operator, we first reorder the h_{ij} $(j = 1, 2, \ldots, 4)$ for each alternative Y_i $(i = 1, 2, \ldots, 5)$. According to Definitions 3 and 4, we compute the score values and accuracy degrees of $s(h_{ij})$ $(i = 1, 2, \ldots, 5; j = 1, 2, \ldots, 4)$ as follows:

$$s(h_{11}) = 0.45, \quad s(h_{12}) = 0.75, \quad s(h_{13}) = 0.3,$$
$$s(h_{14}) = 0.3, \quad k(h_{13}) = 0.9184, \quad k(h_{14}) = 0.9;$$
$$s(h_{21}) = 0.5, \quad s(h_{22}) = 0.7, \quad s(h_{23}) = 0.7,$$
$$s(h_{24}) = 0.5, \quad k(h_{21}) = 0.7551, \quad k(h_{24}) = 0.8129,$$
$$k(h_{22}) = 0.8367, \quad k(h_{23}) = 0.9;$$
$$s(h_{31}) = 0.85, \quad s(h_{32}) = 0.4, \quad s(h_{33}) = 0.35,$$
$$s(h_{34}) = 0.4, \quad k(h_{32}) = 0.8367, \quad k(h_{34}) = 0.7764;$$
$$s(h_{41}) = 0.6, \quad s(h_{42}) = 0.6, \quad s(h_{43}) = 0.3,$$
$$s(h_{44}) = 0.4, \quad k(h_{41}) = 0.772, \quad k(h_{42}) = 0.8367;$$
$$s(h_{51}) = 0.5, \quad s(h_{52}) = 0.3, \quad s(h_{53}) = 0.5,$$
$$s(h_{54}) = 0.35, \quad k(h_{51}) = 0.8367, \quad k(h_{53}) = 0.8129. \quad (45)$$

Then by Definition 5, we have

$$h_{1\sigma(1)} = h_{12}, \quad h_{1\sigma(2)} = h_{11}, \quad h_{1\sigma(3)} = h_{13}, \quad h_{1\sigma(4)} = h_{14};$$
$$h_{2\sigma(1)} = h_{23}, \quad h_{2\sigma(2)} = h_{22}, \quad h_{2\sigma(3)} = h_{24}, \quad h_{2\sigma(4)} = h_{21};$$
$$h_{3\sigma(1)} = h_{31}, \quad h_{3\sigma(2)} = h_{32}, \quad h_{3\sigma(3)} = h_{34}, \quad h_{3\sigma(4)} = h_{33};$$
$$h_{4\sigma(1)} = h_{42}, \quad h_{4\sigma(2)} = h_{41}, \quad h_{4\sigma(3)} = h_{44}, \quad h_{4\sigma(4)} = h_{43};$$
$$h_{5\sigma(1)} = h_{51}, \quad h_{5\sigma(2)} = h_{53}, \quad h_{5\sigma(3)} = h_{54}, \quad h_{5\sigma(4)} = h_{52}. \quad (46)$$

Suppose that $w = (0.1835, 0.3165, 0.3165, 0.1835)^T$ is the weighted vector related to the $HFEOWA_\varepsilon$ operator and it is derived by the normal distribution based method [20]. Then we utilize the $HFEOWA_\varepsilon$ operator to obtain the hesitant

TABLE 1: Hesitant fuzzy decision making matrix.

	A_1	A_2	A_3	A_4
Y_1	$\{0.4, 0.5\}$	$\{0.7, 0.8\}$	$\{0.2, 0.3, 0.4\}$	$\{0.2, 0.4\}$
Y_2	$\{0.2, 0.5, 0.8\}$	$\{0.5, 0.7, 0.9\}$	$\{0.6, 0.8\}$	$\{0.2, 0.5, 0.6, 0.7\}$
Y_3	$\{0.8, 0.9\}$	$\{0.2, 0.4, 0.6\}$	$\{0.2, 0.3, 0.4, 0.5\}$	$\{0.1, 0.3, 0.5, 0.7\}$
Y_4	$\{0.3, 0.4, 0.6, 0.8, 0.9\}$	$\{0.4, 0.6, 0.8\}$	$\{0.1, 0.2, 0.4, 0.5\}$	$\{0.2, 0.3, 0.5, 0.6\}$
Y_5	$\{0.3, 0.5, 0.7\}$	$\{0.2, 0.3, 0.4\}$	$\{0.2, 0.5, 0.6, 0.7\}$	$\{0.1, 0.3, 0.4, 0.6\}$

fuzzy elements h_i ($i = 1, 2, 3, 4, 5$) for the alternatives X_i ($i = 1, 2, 3, 4, 5$). Take alternative X_1 for an example; we have

$$h_1 = \text{HFEOWG}_\varepsilon (h_{11}, h_{12}, \ldots, h_{14})$$

$$= \bigcup_{\gamma_{1\sigma(j)} \in h_{1\sigma(j)}, j=1,2,3,4} \left\{ \frac{2\prod_{j=1}^4 \gamma_{1\sigma(j)}^{w_j}}{\prod_{j=1}^4 \left(2 - \gamma_{1\sigma(j)}\right)^{w_j} + \prod_{j=1}^4 \gamma_{1\sigma(j)}^{w_j}} \right\}$$

$$= \{0.3220, 0.3642, 0.3635, 0.4099, 0.3974, 0.4470,$$

$$0.3473, 0.3921, 0.3914, 0.4403, 0.4272, 0.4794,$$

$$0.3327, 0.3760, 0.3753, 0.4228, 0.4101, 0.4607,$$

$$0.3587, 0.4046, 0.4039, 0.4539, 0.4405, 0.4938\} .$$
(47)

The results can be obtained similarly for the other alternatives; here we will not list them for vast amounts of data. By Definition 3, the score values $s(h_i)$ of h_i ($i = 1, 2, 3, 4, 5$) can be computed as follows:

$$s(h_1) = 0.4048, \quad s(h_2) = 0.5758, \quad s(h_3) = 0.4311,$$
$$s(h_4) = 0.4479, \quad s(h_5) = 0.3620.$$
(48)

According to the scores $s(h_i)$ of the overall hesitant fuzzy values h_i ($i = 1, 2, 3, 4, 5$), we can rank all the alternatives X_i: $X_2 \succ X_4 \succ X_3 \succ X_1 \succ X_5$. Thus the optimal alternative is X_2.

If we use the HFWG operator introduced by Xia and Xu [36] to aggregate the hesitant fuzzy values, then

$$s(h_1) = 0.3960, \quad s(h_2) = 0.5630, \quad s(h_3) = 0.4164,$$
$$s(h_4) = 0.4344, \quad s(h_5) = 0.3548.$$
(49)

By Definition 5, we have $X_2 \succ X_4 \succ X_3 \succ X_1 \succ X_5$.

Note that the rankings are the same in such two cases, but the overall values of alternatives by the HFEOWG$_\varepsilon$ operator are not smaller than the ones by the HFOWG operator. It shows that the attitude of the decision maker using the proposed HFEOWG$_\varepsilon$ operator is more optimistic than the one using the HFOWG operator introduced by Xia and Xu [36] in aggregation process. Therefore, according to the decision makers' optimistic (or pessimistic) attitudes, the different hesitant fuzzy aggregation operators can be used to aggregate the hesitant fuzzy information in decision making process.

6. Conclusions

The purpose of multicriteria decision making is to select the optimal alternative from several alternatives or to get their ranking by aggregating the performances of each alternative under some attributes, which is the pervasive phenomenon in modern life. Hesitancy is the most common problem in decision making, for which hesitant fuzzy set can be considered as a suitable means allowing several possible degrees for an element to a set. Therefore, the hesitant fuzzy multiple attribute decision making problems have received more and more attention. In this paper, an accuracy function of HFEs has been defined for distinguishing between the two HFEs having the same score values, and a new order relation between two HFEs has been provided. Some Einstein operations on HFEs and their basic properties have been presented. With the help of the proposed operations, several new hesitant fuzzy aggregation operators including the HFEWG$_\varepsilon$ operator and HFEOWG$_\varepsilon$ operator have been developed, which are extensions of the weighted geometric operator and the OWG operator with hesitant fuzzy information, respectively. Moreover, some desirable properties of the proposed operators have been discussed and the relationships between the proposed operators and the existing hesitant fuzzy aggregation operators introduced by Xia and Xu [36] have been established. Finally, based on the HFEOWG$_\varepsilon$ operator, an approach of hesitant fuzzy decision making has been given and a practical example has been presented to demonstrate its practicality and effectiveness.

Conflict of Interests

The authors declared that there is no conflict of interests regarding the publication of this paper.

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