

Research Article

Some Connections between the Spherical and Parabolic Bases on the Cone Expressed in terms of the Macdonald Function

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Computing the matrix elements of the linear operator, which transforms the spherical basis of $SO(3, 1)$ -representation space into the hyperbolic basis, very recently, Shilin and Choi (2013) presented an integral formula involving the product of two Legendre functions of the first kind expressed in terms of ${}_4F_3$ -hypergeometric function and, using the general Mehler-Fock transform, another integral formula for the Legendre function of the first kind. In the sequel, we investigate the pairwise connections between the spherical, hyperbolic, and parabolic bases. Using the above connections, we give an interesting series involving the Gauss hypergeometric functions expressed in terms of the Macdonald function.

1. Introduction and Preliminaries

For completeness and an easier reference, we begin by just recalling some parts of [1, Section 1]. Let Λ be the cone in the Euclidean space \mathbb{R}^4 defined by

$$\Lambda := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 - x_2^2 - x_3^2 - x_4^2 = 0\}, \quad (1)$$

where \mathbb{R} denotes the set of real numbers. Let $SO(3, 1)$ be a multiplicative group consisting of all 4×4 matrices g in \mathbb{R} which satisfies the following two properties:

$$\det g = 1, \quad g^T I_{1,3} g = I_{1,3}, \quad (2)$$

where g^T denotes (as usual) the transpose of the matrix g and $I_{1,3}$ is a 4×4 matrix given by

$$I_{1,3} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (3)$$

Remark 1. $SO(3, 1)$ is called the special pseudoorthogonal group. $SO(3, 1)$ is a group of linear operators preserving the quadratic form $x_1^2 - x_2^2 - x_3^2 - x_4^2$, that is, the special classical Lorentz group. Similarly, for the group $SO(2, 1)$, see [2–4].

For a $\sigma \in \mathbb{C}$ where \mathbb{C} is the set of complex numbers, \mathfrak{D}_σ denotes the linear space consisting of infinitely differentiable σ -homogeneous functions on Λ . The representation of $SO(3, 1)$ in \mathfrak{D}_σ is a homomorphism $g \mapsto T_\sigma(g)$, where the operator $T_\sigma(g)$ acts as $f(x) \mapsto f(g^{-1}x)$ in the space \mathfrak{D}_σ .

Let γ_1 be the intersection of Λ and the hyperplane $x_1 = 1$, γ_2 the intersection of Λ and the pair of hyperplanes $x_4 = \pm 1$, and γ_3 the intersection of Λ and the hyperplane $x_1 + x_4 = 0$. In other words, γ_1 is a sphere with radius $\sqrt{2}$, γ_2 is a two-sheet hyperboloid, and γ_3 is a paraboloid. It is seen that, for $i \in \{1, 2, 3\}$, γ_i are two-parameter manifolds on Λ explicitly given by

$$\gamma_1 = \{(1, \sin \alpha_1 \sin \beta_1, \sin \alpha_1 \cos \beta_1, \cos \alpha_1) \mid \alpha_1 \in [0, \pi], \beta_1 \in [0, 2\pi)\},$$

$$\gamma_2 = \{(\cosh \alpha_2, \sinh \alpha_2 \sin \beta_2, \sinh \alpha_2 \cos \beta_2, \pm 1) \mid \alpha_2 > 0, \beta_2 \in [0, 2\pi]\},$$

$$\gamma_3 = \left\{ \left(\frac{1 + \alpha_3^2}{2}, \alpha_3 \sin \beta_3, \alpha_3 \cos \beta_3, \frac{1 - \alpha_3^2}{2} \right) \mid \alpha_3 > 0, \beta_3 \in [0, 2\pi] \right\}. \tag{4}$$

Let H_i denote subgroups acting transitively on γ_i ($i = 1, 2, 3$). It is noted, in particular, that $H_1 \simeq SO(3)$ and $H_2 \simeq SO(2, 1)$, where $SO(n)$ is the group of rotations of n -dimensional Euclidean space \mathbb{R}^n (for more details of this group, see [5, Chapter IX]). In order to describe the group H_3 , in detail, on the linear space of all diagonal matrices $a := \text{diag}(a_1, a_2)$, we introduce a scalar product as $a \cdot b := \text{tr}(ab^T)$. Then γ_3 is a homogeneous space of the subgroup H_3 consisting of all matrices

$$\begin{pmatrix} 1 + \frac{1}{2}|a|^2 & a & \frac{1}{2}|a|^2 \\ a^T & \text{diag}(1, 1) & a^T \\ -\frac{1}{2}|a|^2 & -a & 1 - \frac{1}{2}|a|^2 \end{pmatrix}, \tag{5}$$

where $|a|^2$ denotes the square of the length of the matrix a . The H_1 -invariant measure on γ_1 can be written as

$$(dx)_{\gamma_1} = \frac{dx_{\tau(2)} dx_{\tau(3)}}{|x_{\tau(4)}|}, \tag{6}$$

where τ is an arbitrary permutation of the set $\{2, 3, 4\}$. The H_2 -invariant measure on γ_2 is

$$(dx)_{\gamma_2} = \frac{dx_{\tau(1)} dx_{\tau(2)}}{|x_{\tau(3)}|} \quad (\tau \in \mathbf{S}_3), \tag{7}$$

where \mathbf{S}_3 is the symmetric group. The H_3 -invariant measure on γ_3 is

$$(dx)_{\gamma_3} = dx_2 dx_3. \tag{8}$$

We define the bilinear functionals $F_j : \mathfrak{D}_\sigma \times \mathfrak{D}_{\hat{\sigma}} \rightarrow \mathbb{C}$ ($j = 1, 2, 3$) given by

$$(u, v) \mapsto \int_{\gamma_j} u(x) v(x) (dx)_{\gamma_j}. \tag{9}$$

Then we observe the invariant property for the functionals F_j asserted by the following lemma.

Lemma 2. *If $\hat{\sigma} = -\sigma - 2$, then, for any $i, j \in \{1, 2, 3\}$, $F_i = F_j$.*

Proof. Choose $\tau = \text{id}$ in (6) and $\tau = (1\ 2\ 3)$ in (7) and write the measures $(dx)_{\gamma_j}$ ($j \in \{1, 2\}$) in (α_j, β_j) -coordinate system, respectively:

$$(dx)_{\gamma_1} = \left| \frac{\partial(x_2, x_3)}{\partial(\alpha_1, \beta_1)} \right| \frac{d\alpha_1 d\beta_1}{|x_4|} = \sin \alpha_1 d\alpha_1 d\beta_1, \tag{10}$$

$$(dx)_{\gamma_2} = \left| \frac{\partial(x_2, x_3)}{\partial(\alpha_2, \beta_2)} \right| \frac{d\alpha_2 d\beta_2}{|x_1|} = \sinh \alpha_2 d\alpha_2 d\beta_2.$$

Here id and $\partial(x_2, x_3)/\partial(\alpha_j, \beta_j)$, respectively, denote the identical permutation and the corresponding Jacobian determinant.

Let $u \in \mathfrak{D}_\sigma$ and $v \in \mathfrak{D}_{-\sigma-2}$. Then

$$F_2(u, v) = \int_{\gamma_2} [\text{sech } \alpha_2]^2 u(1, \tanh \alpha_2 \sin \beta_2, \tanh \alpha_2 \cos \beta_2, \pm \text{sech } \alpha_2) \cdot v(1, \tanh \alpha_2 \sin \beta_2, \tanh \alpha_2 \cos \beta_2, \pm \text{sech } \alpha_2) \times \sinh \alpha_2 d\alpha_2 d\beta_2. \tag{11}$$

Setting $\cos \varphi := \pm \text{sech } \alpha_2$, we obtain $d\alpha_2 = \pm \cosh \alpha_2 d\varphi$ and $\sinh \alpha_2 = (1/2) \sin 2\varphi$. We therefore find that

$$F_2(u, v) = \int_{\gamma_1} u(1, \sin \varphi \sin \beta_2, \sin \varphi \cos \beta_2, \cos \varphi) \cdot v(1, \sin \varphi \sin \beta_2, \sin \varphi \cos \beta_2, \cos \varphi) \sin \varphi d\varphi d\beta_2, \tag{12}$$

which implies $F_2 = F_1$. In the same way,

$$F_3(u, v) = 4 \int_{\gamma_3} (1 + \alpha_3^2)^{-2} u \left(1, \frac{2\alpha_3 \sin \beta_3}{1 + \alpha_3^2}, \frac{2\alpha_3 \cos \beta_3}{1 + \alpha_3^2}, \frac{1 - \alpha_3^2}{1 + \alpha_3^2} \right) \cdot v \left(1, \frac{2\alpha_3 \sin \beta_3}{1 + \alpha_3^2}, \frac{2\alpha_3 \cos \beta_3}{1 + \alpha_3^2}, \frac{1 - \alpha_3^2}{1 + \alpha_3^2} \right) \alpha_3 d\alpha_3 d\beta_3. \tag{13}$$

Setting here $\cos \varphi := (1 - \alpha_3^2)/(1 + \alpha_3^2)$, we obtain $\sin \varphi = 2\alpha_3^2/(1 + \alpha_3^2)$ and $d\alpha_3 = ((1 + \alpha_3^2)d\varphi)/2$. It means that

$$F_3(u, v) = \int_{\gamma_1} u(1, \sin \varphi \sin \beta_3, \sin \varphi \cos \beta_3, \cos \varphi) \cdot v(1, \sin \varphi \sin \beta_3, \sin \varphi \cos \beta_3, \cos \varphi) \sin \varphi d\varphi d\beta_3. \tag{14}$$

We thus see that $F_1 = F_3$.

Since the relation $=$ defined on the set $\{F_1, F_2, F_3\}$ is transitive, we have $F_2 = F_3$. The proof is complete. \square

In [5, Chapter IX], Vilenkin constructed the canonical basis E_K^I on a sphere. Here, continuing this canonical basis from the sphere γ_1 to the cone Λ via σ -homogeneity, we obtain the basis consisting of f_{p_1, q_1} :

$$\begin{aligned}
 & f_{p_1, q_1}(x) \\
 &= x_1^{\sigma - |q_1|} C_{p_1 - |q_1|}^{|q_1| + (1/2)} \left(\frac{x_4}{x_1} \right) (x_3 + \mathbf{i}x_2 \operatorname{sign} q_1)^{|q_1|} \quad (15) \\
 & (\mathbf{i} = \sqrt{-1}; p_1, q_1 \in \mathbb{Z}; p_1 \geq 0, |q_1| \leq p_1),
 \end{aligned}$$

where \mathbb{Z} denotes the set of integers and $C_m^p(t)$ are the Gegenbauer polynomials (see, e.g., [6, Chapter 17]; see also [5, page 458, Equation (3)]). We will call it the spherical basis. Similarly, the hyperbolic basis consists of functions of the following forms:

$$\begin{aligned}
 & f_{p_2, q_2, \pm}^*(x) \\
 &= (x_4)_{\pm}^{\sigma} (x_2^2 + x_3^2)^{-|q_2|/2} P_{-(1/2) + \mathbf{i}p_2}^{-|q_2|} \left(\frac{x_1}{(x_4)_{\pm}} \right) \quad (16) \\
 & \times (x_3 + \mathbf{i}x_2 \operatorname{sign} q_2)^{|q_2|} \quad (p_2 > 0, q_2 \in \mathbb{Z}),
 \end{aligned}$$

where $P_{\nu}^{\mu}(z)$ are the Legendre functions of the first kind (see, e.g., [7, pages 194–228]; see also [8, 9]) and

$$(x)_{\pm}^{\mu} = \begin{cases} 0, & \text{if sign } x = \mp 1 \text{ or } x = 0, \\ |x|^{\mu}, & \text{if sign } x = \pm 1. \end{cases} \quad (17)$$

About the nonhomogeneous *hyperbolic* part of the hyperbolic basis, see, for instance, [5, Chapter X]. Finally, the parabolic basis consists of the functions

$$\begin{aligned}
 & f_{p_3, q_3}^{**}(x) \\
 &:= (x_1 + x_4)^{\sigma} (x_2^2 + x_3^2)^{-|q_3|/2} J_{|q_3|} \left(\frac{\sqrt{x_2^2 + x_3^2} p_3}{x_1 + x_4} \right) \quad (18) \\
 & \times (x_2 + \mathbf{i}x_3 \operatorname{sign} q_3)^{|q_3|},
 \end{aligned}$$

where $p_3 \geq 0, q_3 \in \mathbb{Z}$, and $J_{\nu}(z)$ are the Bessel functions. The main idea of this basis is described in [5, Chapters IV and XI]; the multiplier $(x_1 + x_4)^{\sigma}$ is its σ -homogeneous part.

Computing the matrix elements of the linear operator, which transforms the spherical basis of $SO(3, 1)$ -representation space into the hyperbolic basis, very recently, Shilin and Choi [1] presented an integral formula involving the product of two Legendre functions of the first kind expressed in terms of ${}_4F_3$ -hypergeometric function and, using the general Mehler-Fock transform (see, e.g., [10, 11]), another integral formula for the Legendre function of the first kind. In the sequel, using the matrix elements of the linear operators, which transform the spherical basis into the parabolic basis and the hyperbolic basis into the parabolic basis, respectively, here, in this paper, we also give certain connections between the spherical and parabolic bases on

the cone and an interesting series involving the Gauss hypergeometric functions, both expressed in terms of the Macdonald function.

2. Description of the Connection between the Spherical and Hyperbolic Bases in terms of ${}_4F_3$ Function

Using

$$\int_0^{2\pi} \exp(\mathbf{i}(p + q)z) dz = 2\pi \delta_{p, -q} \quad (19)$$

and the orthogonality relation (see, e.g., [6, page 281, Equations (27) and (28)]; see also [5, page 462, Equations (4) and (5)])

$$\begin{aligned}
 & \int_{-1}^1 C_p^q(z) C_{\bar{p}}^q(z) (1 - z^2)^{q - (1/2)} dz \\
 &= \frac{\pi \Gamma(2q + p) \delta_{p, \bar{p}}}{2^{2q-1} p! (p + q) [\Gamma(q)]^2}, \quad (20)
 \end{aligned}$$

where $\delta_{i, j}$ is the Kronecker symbol, we find from the decomposition

$$f_{p_2, q_2, \pm}^*(x) = \sum_{p_1=0}^{\infty} \sum_{l=-|p_1|}^{|p_1|} c_{(p_2, q_2, \pm), (p_1, l)} f_{p_1, l}(x) \quad (21)$$

that

$$\begin{aligned}
 & c_{(p_2, q_2, \pm), (p_1, l)} \\
 &= \frac{2^{2l-1} (p_1 - l)! (p_1 + (1/2)) [\Gamma(l + (1/2))]^2}{\pi^2 \Gamma(2l + 1)} \quad (22) \\
 & \times F_j(f_{p_2, q_2, \pm}^*, f_{p_1, -l}),
 \end{aligned}$$

where Γ is the familiar Gamma function (see, e.g., [12, Section 1.1]). Shilin and Choi [1] then observed that the numbers $F_j(f_{p_2, q_2, \pm}^*, f_{p_1, q_1})$ give a relationship between the spherical and hyperbolic bases asserted by the following theorem.

Theorem 3. *Each of the following formulas holds true:*

$$F_2(f_{p_2, q_2, +}^*, f_{p_1, q_1}) = 0 \quad (q_1 \neq -q_2) \quad (23)$$

and, for $\Re(\sigma) < -(1/2)$,

$$\begin{aligned}
 F_2(f_{p_2, -q_1, +}^*, f_{p_1, q_1}) &= 2^{p_1 - |q_1| - \sigma - 1} \sqrt{\pi} \\
 &\cdot \frac{\Gamma((1/2) + p_1) \Gamma(((p_1 - |q_1| - \sigma + ip_2)/2) - (1/4)) \Gamma(((p_1 - |q_1| - \sigma - ip_2)/2) - (1/4))}{\Gamma(1 + p_1 - |q_1|) \Gamma(p_1 - \sigma) \Gamma((1/2) + |q_1|)} \\
 &\cdot {}_4F_3 \left[\begin{matrix} \frac{|q_1| - p_1}{2}, \frac{1 + |q_1| - p_1}{2}, \frac{1 + \sigma - p_1}{2}, 1 - \frac{\sigma + p_1}{2}; \\ \frac{1}{2} - p_1, \frac{5}{4} + \frac{\sigma + |q_1| - p_1 + ip_2}{2}, \frac{5}{4} + \frac{\sigma + |q_1| - p_1 - ip_2}{2}; \end{matrix} 4 \right], \tag{24}
 \end{aligned}$$

provided that one of the numerator parameters in ${}_4F_3$ is zero or a negative integer, and ${}_pF_q$ are the generalized hypergeometric functions (see, e.g., [12, Section 1.5]).

Here we provide a more general result than the one in Theorem 3 for all matrix elements $c_{(p_2, q_2, \pm), (p_1, q_1)}$ instead of $c_{(p_2, q_2, +), (p_1, q_1)}$. We omit the corresponding proof of Theorem 3 to reformulate Theorem 3 for a general case asserted by Theorem 1*.

Theorem 1*. Let

$$\Psi_{\iota, p_1, q_1} := \begin{cases} 1, & \text{if } \iota = + \text{ or } p_1 - |q_1| \text{ is even,} \\ -1, & \text{if } \iota = -, p_1 - |q_1| \text{ is odd.} \end{cases} \tag{25}$$

Then each of the following formulas holds true:

$$F_2(f_{p_2, q_2, \pm}^*, f_{p_1, q_1}) = 0 \quad (q_1 \neq -q_2) \tag{26}$$

and, for $\Re(\sigma) < -(1/2)$,

$$\begin{aligned}
 F_2(f_{p_2, -q_1, \iota}^*, f_{p_1, q_1}) &= \Psi_{\iota, p_1, q_1} 2^{p_1 - |q_1| - \sigma - 1} \sqrt{\pi} \\
 &\cdot \frac{\Gamma((1/2) + p_1) \Gamma(((p_1 - |q_1| - \sigma + ip_2)/2) - (1/4)) \Gamma(((p_1 - |q_1| - \sigma - ip_2)/2) - (1/4))}{\Gamma(1 + p_1 - |q_1|) \Gamma(p_1 - \sigma) \Gamma((1/2) + |q_1|)} \\
 &\cdot {}_4F_3 \left[\begin{matrix} \frac{|q_1| - p_1}{2}, \frac{1 + |q_1| - p_1}{2}, \frac{1 + \sigma - p_1}{2}, 1 - \frac{\sigma + p_1}{2}; \\ \frac{1}{2} - p_1, \frac{5}{4} + \frac{\sigma + |q_1| - p_1 + ip_2}{2}, \frac{5}{4} + \frac{\sigma + |q_1| - p_1 - ip_2}{2}; \end{matrix} 4 \right], \tag{27}
 \end{aligned}$$

provided that one of the numerator parameters in ${}_4F_3$ is zero or a negative integer, and ${}_pF_q$ are the generalized hypergeometric functions (see, e.g., [12, Section 1.5]).

3. Description of the Connection between the Spherical and Parabolic Bases in terms of the Macdonald Function

Using the same method used in Section 2, from the decomposition

$$f_{p_3, q_3}^{**}(x) = \sum_{p_1=0}^{\infty} \sum_{q_1=-|p_1|}^{|p_1|} d_{(p_3, q_3), (p_1, q_1)} f_{p_1, q_1}(x), \tag{28}$$

we derive

$$\begin{aligned}
 &d_{(p_3, q_3), (p_1, q_1)} \\
 &= \frac{2^{2|q_1| - 1} (p_1 - |q_1|)! (p_1 + (1/2)) [\Gamma(|q_1| + (1/2))]^2}{\pi^2 \Gamma(2|q_1| + 1)} \\
 &\times F_j(f_{p_3, q_3}^{**}, f_{p_1, -|q_1|}). \tag{29}
 \end{aligned}$$

Theorem 4. The following formula holds true: for $\Re(\sigma) < -(1/4)$,

$$\begin{aligned}
 F_3(f_{p_3, q_3}^{**}, f_{|q_3|, -q_3}) & \\
 &= 4\pi p_3^{|q_3| - \sigma - 1} [\Gamma(|q_3| - \sigma)]^{-1} K_{\sigma+1}(p_3), \tag{30}
 \end{aligned}$$

where $K_{\sigma+1}(p_3)$ is the Macdonald function (the modified Bessel function of the second kind) (see, e.g., [13, page 675]).

Proof. We have

$$\begin{aligned}
 F_3(f_{p_3, q_3}^{**}, f_{p_1, -q_3}) & \\
 &= 2^{|q_3| - \sigma + 1} \pi \\
 &\cdot \int_0^{+\infty} \alpha_3^{|q_3| + 1} (1 + \alpha_3^2)^{\sigma - |q_3|} J_{|q_3|}(p_3 \alpha_3) \\
 &\times C_{p_1 - |q_3|}^{|q_3| + (1/2)} \left(\frac{1 - \alpha_3^2}{1 + \alpha_3^2} \right) d\alpha_3. \tag{31}
 \end{aligned}$$

Using the explicit representation (see, e.g., [14, page 175, Equation (18)])

$$C_n^\nu(x) = \frac{1}{\Gamma(\nu)} \sum_{j=0}^{[n/2]} \frac{(-1)^j \Gamma(n + \nu - j)}{j!(n - 2j)!} (2x)^{n-2j}, \quad (32)$$

[u] being the greatest integer less than or equal to u, we obtain

$$\begin{aligned} &F_3(f_{p_3, q_3}^{**}, f_{p_1, -q_3}) \\ &= 2^{p_1 \sigma + 1} \pi \left[\Gamma\left(|q_3| + \frac{1}{2}\right) \right]^{-1} \\ &\times \sum_{j=0}^{[(p_1 - |q_3|)/2]} \frac{(-1)^j 2^{-2j} \Gamma(p_1 - j + (1/2))}{j!(p_1 - |q_3| - 2j)!} \\ &\cdot \int_0^{+\infty} \alpha_3^{|q_3|+1} (1 + \alpha_3^2)^{\sigma - p_1 + 2j} \\ &\times (1 - \alpha_3^2)^{p_1 - |q_3| - 2j} J_{|q_3|}(p_3 \alpha_3) d\alpha_3. \end{aligned} \quad (33)$$

In case of $p_1 = |q_3|$, we can use the following known formula (see [15, page 179, Entry 2.12.4-28]):

$$\int_0^{+\infty} \frac{x^{\nu+1}}{(x^2 + z^2)^\rho} J_\nu(cx) dx = \frac{c^{\rho-1} z^{\nu-\rho+1}}{2^{\rho-1} \Gamma(\rho)} K_{\nu-\rho+1}, \quad (34)$$

which holds for $c > 0$, $\Re(z) > 0$, and $-1 < \Re(\nu) < 2\Re(\rho) - (1/2)$. □

Corollary 5. *The following formula holds true: for $\Re(\sigma) < -(1/4)$,*

$$\begin{aligned} &d_{(p_3, q_3), (|q_3|, -q_3)} \\ &= \left(\left(2^{2|q_3|+1} p_3^{|q_3|-\sigma-1} \left(|q_3| + \frac{1}{2} \right) \left[\Gamma\left(|q_3| + \frac{1}{2}\right) \right]^2 \right) \right. \\ &\quad \times \left. (\pi \Gamma(|q_3| - \sigma) \Gamma(2|q_3| + 1))^{-1} \right) \\ &\quad \times K_{\sigma+1}(p_3). \end{aligned} \quad (35)$$

Further we will deal only with the matrix elements $d_{(p_3, q_3), (p_1, -q_3)}$ in case of $p_1 = |q_3|$, but let us pay attention to general case. Let $\Re(\sigma) < -(1/4)$. As

$$\begin{aligned} &p_1 - 2j - \frac{|q_3|}{2} - k - \frac{1}{4} \\ &= (p_1 - |q_3| - 2j - k) + \frac{|q_3|}{2} - \frac{1}{4} \geq -\frac{1}{4}, \end{aligned} \quad (36)$$

by using the binomial theorem for the factor $(1 - \alpha_3^2)^{p_1 - |q_3| - 2j}$ in (33) and formula (34), apparently, we can represent numbers $F_3(f_{p_3, q_3}^{**}, f_{p_1, -q_3})$ as a sum whose terms contain a product of the Macdonald function and the generalized hypergeometric function. Without loss of generality, we

demonstrate it, for example, under condition $p_1 - |q_3| = 4$. In this case,

$$F_3(f_{p_3, q_3}^{**}, f_{p_1, -q_3}) = \frac{4\pi p_3^{p_1 - \sigma - 1}}{\Gamma(|q_3| + (1/2))} \sum_{j=0}^{[(p_1 - |q_3|)/2]} \Xi_j \cdot \Theta_j, \quad (37)$$

where

$$\begin{aligned} \Xi_j &= \frac{(-1)^j p_3^{-2j} \Gamma(p_1 - j + (1/2))}{j!(p_1 - |q_3| - 2j)! \Gamma(p_1 - \sigma - 2j)}, \\ \Theta_j &= \sum_{k=1}^{p_1 - |q_3| - 2j} (-1)^k \binom{p_1 - |q_3| - 2j}{k} K_{\sigma - (4-2j) + 2k+1}(p_3). \end{aligned} \quad (38)$$

As

$$\begin{aligned} \Gamma(p_1 - j + (1/2)) &= \frac{(-1)^j \Gamma(p_1 + (1/2))}{((1/2) - p_1)_j}, \\ \Gamma(p_1 - \sigma - 2j) &= \frac{2^{-2j} \Gamma(p_1 - \sigma)}{((1 + \sigma - p_1)/2)_j (1 - ((\sigma + p_1)/2))_j}, \\ \sum_{k=0}^n \binom{n}{k} K_{\nu+2k}(z) &= (-2)^n K_{\nu+n}(z) \end{aligned} \quad (39)$$

(regarding the last formula see, e.g., [15, Entry 4.2.4-2]), we have

$$\begin{aligned} \Theta_0 &= K_{\sigma+1}(p_3) + 8(K_{\sigma-1}(p_3) + K_{\sigma+3}(p_3)) \\ &= 17K_{\sigma+1}(p_3) + \frac{2\sigma}{p_3}(K_{\sigma+2}(p_3) - K_\sigma(p_3)) \\ &\quad + \frac{4}{p_3}K_{\sigma+2}(p_3) \\ &= \left(17 + \frac{4\sigma(\sigma+1)}{p_3^2} \right) K_{\sigma+1}(p_3) + \frac{4}{p_3}K_{\sigma+2}(p_3), \quad (40) \\ \Theta_1 &= \frac{2(\sigma+2)}{p_3}K_{\sigma+2}(p_3) - \frac{2\sigma}{p_3}K_\sigma(p_3) \\ &= \frac{4\sigma(\sigma+1)}{p_3^2}K_{\sigma+1}(p_3) + \frac{4}{p_3}K_{\sigma+2}(p_3), \\ \Theta_2 &= K_{\sigma+1}(p_3), \end{aligned}$$

and, therefore,

$$\begin{aligned}
 & F_3(f_{p_3, q_3}^{**}, f_{p_1, -q_3}) \\
 &= \frac{4\pi p_3^{p_1 - \sigma - 1}}{\Gamma(|q_3| + (1/2))} \\
 &\cdot \left(\frac{17\Gamma((1/2) + p_1)}{\Gamma(1 + p_1 - |q_3|)\Gamma(p_1 - \sigma)} \right. \\
 &\quad \times K_{\sigma+1}(p_3) \sum_{j=0}^{\infty} (-1)_j \Xi_j \\
 &\quad \left. + \frac{4\sigma(\sigma + 1)\Gamma((1/2) + p_1)}{p_3^2 \Gamma(1 + p_1 - |q_3|)\Gamma(p_1 - \sigma)} \right. \\
 &\quad \times K_{\sigma+1}(p_3) \sum_{j=0}^{\infty} \frac{(-2)_j}{(1)_j} \Xi_j \\
 &\quad \left. + \frac{\Gamma(p_1 - (3/2))}{p_3^4 \Gamma(p_1 - |q_3| - 3)\Gamma(p_1 - \sigma - 4)} \right. \\
 &\quad \times K_{\sigma+1}(p_3) \sum_{j=2}^{\infty} \Xi_j \\
 &\quad \left. + \frac{4\Gamma((1/2) + p_1)}{p_3 \Gamma(1 + p_1 - |q_3|)\Gamma(p_1 - \sigma)} \right. \\
 &\quad \left. \times K_{\sigma+2}(p_3) \sum_{j=0}^{\infty} \frac{(-2)_j}{(1)_j} \Xi_j \right). \tag{41}
 \end{aligned}$$

It means that

$$\begin{aligned}
 & F_3(f_{p_3, q_3}^{**}, f_{p_1, -q_3}) = \frac{4\pi p_3^{p_1 - \sigma - 1}}{\Gamma\left(|q_3| + \frac{1}{2}\right)} \\
 &\cdot \left(\frac{17\Gamma((1/2) + p_1)}{\Gamma(1 + p_1 - |q_3|)\Gamma(p_1 - \sigma)} K_{\sigma+1}(p_3) \right. \\
 &\cdot {}_5F_1 \left(\begin{matrix} \frac{|q_3| - p_1}{2}, \frac{1 + |q_3| - p_1}{2}, \frac{1 + \sigma - p_1}{2}, 1 - \frac{\sigma + p_1}{2}, -1 \\ \frac{1}{2} - p_1 \end{matrix} \middle| \frac{4}{p_3^2} \right) \\
 &+ \frac{4\sigma(\sigma + 1)\Gamma((1/2) + p_1)}{p_3^2 \Gamma(1 + p_1 - |q_3|)\Gamma(p_1 - \sigma)} K_{\sigma+1}(p_3) \\
 &\cdot {}_5F_2 \left(\begin{matrix} \frac{|q_3| - p_1}{2}, \frac{1 + |q_3| - p_1}{2}, \frac{1 + \sigma - p_1}{2}, 1 - \frac{\sigma + p_1}{2}, -2 \\ \frac{1}{2} - p_1, 1 \end{matrix} \middle| \frac{4}{p_3^2} \right) \\
 &+ \frac{\Gamma(p_1 - (3/2))}{p_3^4 \Gamma(p_1 - |q_3| - 3)\Gamma(p_1 - \sigma - 4)} K_{\sigma+1}(p_3) \\
 &\cdot {}_5F_2 \left(\begin{matrix} 2 + \frac{|q_3| - p_1}{2}, \frac{5 + |q_3| - p_1}{2}, \frac{5 + \sigma - p_1}{2}, 3 - \frac{\sigma + p_1}{2}, -1 \\ \frac{1}{2} - p_1, 2 \end{matrix} \middle| \frac{4}{p_3^2} \right) \\
 &+ \frac{4\Gamma((1/2) + p_1)}{p_3 \Gamma(1 + p_1 - |q_3|)\Gamma(p_1 - \sigma)} K_{\sigma+2}(p_3) \\
 &\cdot {}_5F_2 \left(\begin{matrix} \frac{|q_3| - p_1}{2}, \frac{1 + |q_3| - p_1}{2}, \frac{1 + \sigma - p_1}{2}, 1 - \frac{\sigma + p_1}{2}, -2 \\ \frac{1}{2} - p_1, 1 \end{matrix} \middle| \frac{4}{p_3^2} \right) \Bigg). \tag{42}
 \end{aligned}$$

It should be remarked in passing that the same reasoning can be applied to any values of $p_1 - |q_3|$.

4. Composition of Basis Transformations and a Representation of the Macdonald Function

Let $k_{(p_3, q_3), (p_2, q_2, \pm)}$ be the matrix element of the linear operator acting in \mathfrak{D}_σ and mapping the hyperbolic basis into the parabolic basis; that is,

$$f_{p_3, q_3}^{**}(x) = \sum_{q_2 \in \mathbb{Z}} \int_0^{+\infty} k_{(p_3, q_3), (p_2, q_2, \pm)} f_{p_2, q_2, \pm}^*(x) dp_2. \quad (43)$$

The linear operator $(d_{(p_3, q_3), (p_1, q_1)})$ can be represented as the composition of the linear operators $(c_{(p_2, q_2, \pm), (p_1, q_1)})$ and $(k_{(p_3, q_3), (p_2, q_2, \pm)})$; namely,

$$(d_{(p_3, q_3), (p_1, q_1)}) = (k_{(p_3, q_3), (p_2, q_2, \pm)}) (c_{(p_2, q_2, \pm), (p_1, q_1)}). \quad (44)$$

From this equality, we have, in particular,

$$\begin{aligned} d_{(p_3, q_3), (|q_3|, -q_3)} \\ = \int_0^{+\infty} k_{(p_3, q_3), (p_2, q_2, \pm)} c_{(p_2, q_2, \pm), (|q_3|, -q_3)} dp_2. \end{aligned} \quad (45)$$

Theorem 6. *The following formula holds true: for $\Re(\sigma) < -(1/2)$,*

$$\begin{aligned} K_{\sigma+1}(p_3) \\ = 2\pi p_3^\sigma \Gamma(-\sigma) \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1} [n!]^2} B(n+1, -2\sigma-1) \\ \times {}_2F_1(n+1, -\sigma; n-2\sigma; -1). \end{aligned} \quad (46)$$

Proof. Note that the restrictions of the functions $f_{p_2, q_2, pm}^* \in \mathfrak{D}_{-\sigma-2}$ and $f_{p_3, q_3}^{**} \in \mathfrak{D}_\sigma$ to γ_3 are, respectively,

$$\begin{aligned} f_{p_2, q_2, pm}^*|_{\gamma_2} \\ \equiv f_{p_2, q_2, pm}^*(\alpha_3, \beta_3) \\ = 2^{\sigma+2} (1 - \alpha_3^2)_\pm P_{-(1/2)+ip_2}^{-|q_2|} \left(\frac{1 + \alpha_3^2}{1 - \alpha_3^2} \right) \exp(\mathbf{i}|q_2|\beta_3), \\ f_{p_3, q_3}^{**}|_{\gamma_2} \equiv f_{p_3, q_3}^{**}(\alpha_3, \beta_3) = J_{|q_3|}(\alpha_3 p_3) \exp(\mathbf{i}|q_3|\beta_3). \end{aligned} \quad (47)$$

Setting $q_3 = 0$, we can rewrite (45) as

$$\begin{aligned} K_{\sigma+1}(p_3) \\ = 2^{-\sigma-1} \sqrt{\pi} p_3^\sigma \\ \cdot \int_0^{+\infty} p_2 \sinh(\pi p_2) \Gamma\left(\frac{1}{2} + \mathbf{i}p_2\right) \Gamma\left(\frac{1}{2} - \mathbf{i}p_2\right) \\ \times \Gamma\left(\frac{-\sigma + \mathbf{i}p_2}{2} - \frac{1}{4}\right) \Gamma\left(\frac{-\sigma - \mathbf{i}p_2}{2} - \frac{1}{4}\right) \\ \cdot \int_0^1 \alpha_3 (1 - \alpha_3^2)^{-\sigma-2} J_0(p_3 \alpha_3) P_{-(1/2)+ip_2}^0 \\ \times \left(\frac{1 + \alpha_3^2}{1 - \alpha_3^2} \right) d\alpha_3 dp_2. \end{aligned} \quad (48)$$

Changing the order of integration and using formula (see, e.g., [15, Entry 2.17.27-9])

$$\begin{aligned} \int_0^{+\infty} x \sinh(\pi n x) \Gamma\left[\frac{1}{2} - \mu + \mathbf{i}, \frac{1}{2} - \mu - \mathbf{i}, \nu + \mathbf{i}r x, \nu - \mathbf{i}r x\right] \\ \times P_{\mathbf{i}x-(1/2)}^\mu(c) dx = R_{n,r} \left(\Re(\mu) \leq \frac{1}{2}; \Re(\nu) \geq 0 \right), \\ R_{1, (1/2)} = 2^{(3/2)-2\nu} \pi^{3/2} \Gamma\left(2\nu - \mu + \frac{1}{2}\right) c^{\mu-2\nu-(1/2)} \\ \times (c^2 - 1)^{-\mu/2}, \end{aligned} \quad (49)$$

we find, in the case of $\mu = 0$, $r = (1/2)$, and $\nu = -(\sigma/2) - (1/4)$, that

$$\begin{aligned} K_{\sigma+1}(p_3) = 2\pi p_3^\sigma \Gamma(-\sigma) \\ \times \int_0^1 \alpha_3 (1 - \alpha_3^2)^{-2\sigma-2} (1 + \alpha_3^2)^\sigma J_0(p_3 \alpha_3) d\alpha_3. \end{aligned} \quad (50)$$

Using here the decomposition

$$J_\nu(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(\nu + j + 1)} \left(\frac{z}{2}\right)^{\nu+2j}, \quad (51)$$

we obtain

$$\begin{aligned} K_{\sigma+1}(p_3) \\ = 2\pi p_3^\sigma \Gamma(-\sigma) \\ \times \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} [n!]^2} \int_0^1 \alpha_3^{2n+1} (1 - \alpha_3^2)^{-2\sigma-2} (1 + \alpha_3^2)^\sigma d\alpha_3. \end{aligned} \quad (52)$$

To evaluate the right-sided integral, considering new variable $\alpha_3 := s$, we rewrite it as

$$\frac{1}{2} \int_0^1 s^n (1-s)^{-2\sigma-2} (1+s)^\sigma ds \quad (53)$$

and use a known formula (see, e.g., [16, page 315, Entry 3.197-8])

$$\int_0^a x^{a-1} (a-x)^{\omega-1} (x+z)^{-\rho} dx = a^{\alpha+\omega-1} z^{-\rho} B(\alpha, \omega) {}_2F_1(\alpha, \rho; \alpha + \omega; -a/z) \quad (54)$$

(|arg z| < π ; $a > 0$; $\Re(a) > 0$; $\Re(\omega) > 0$).

This completes the proof. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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