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Research Article

On the Fiber Preserving Transformations for the Fifth-Order Ordinary Differential Equations

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This paper is devoted to the study of the linearization problem of fifth-order ordinary differential equations by means of fiber preserving transformations. The necessary and sufficient conditions for linearization are obtained. The procedure for obtaining the linearizing transformations is provided in explicit form. Examples demonstrating the procedure of using the linearization theorems are presented.

1. Introduction

1.1. The Research Problem and Its Significance. In mathematics, a nonlinear equation is an equation which is not linear; that is, an equation which does not satisfy the superposition principle, or whose output is not directly proportional to its input. Less technically, a nonlinear equation is any problem where the variables to be solved for cannot be written as a linear combination of independent components.

Nonlinear problems are of interest to engineers, physicists, and mathematicians because most physical systems are inherently nonlinear in nature. Nonlinear equations are difficult to solve and give rise to interesting phenomena. While solving problems related to nonlinear ordinary differential equations, it is often expedient to simplify equations by a suitable change of variables. One of the fundamental methods to solve this relies upon the transformation of a given equation to another equation of standard form. The transformation may be to an equation of equal order or of greater or lesser order. In particular, the possibility that a given equation could be linearized, that is, transformed to a linear equation, was a most attractive proposition due to the special properties of linear differential equations. The reduction of an ordinary differential equation to a linear ordinary differential equation besides simplification allows us to construct an exact solution of the original equation.

One type of the classification problem is the equivalence problem. Two equations of differential equations are said to be equivalent if there exists an invertible transformation which transforms any solution of one equation to a solution of the other equation and vice versa. The linearization problem is a particular case of the equivalence problem, where one of the equations is a linear equation. It is one of the essential parts in the study of nonlinear equations.

The main difficulty in solving the linearization problem comes from the large number of complicated calculations. Because of this difficulty, no one attempts to solve this problem for nonlinear equations are higher than fourth. However if we can solve the linearization problem of fifth-order ordinary differential equations, then we should set a new process to solve the problems in Physics or Engineering.

1.2. Historical Review. The linearization, that is, mapping a nonlinear differential equation into a linear differential equation, is an important tool in the theory of differential equations. The problem of linearization of ordinary differential equations attracted attention of mathematicians such as Lie and Cartan. The first linearization problem for ordinary differential equations was solved by Lie [1, 2]. He found the general form of all ordinary differential equations of second-order that can be reduced to a linear equation by changing the independent and dependent variables. He showed that any linearizable second-order equation should be at most cubic in the first-order derivative and provided a linearization test in terms of its coefficients. The linearization criterion is written through relative invariants of the equivalence group. Liouville [3] and Tresse [4] treated the equivalence problem for second-order ordinary differential equations in terms of relative invariants of the equivalence group of point transformations. There are other approaches for solving the linearization problem of a second-order ordinary differential equation. For example, one was developed by Cartan [5]. The idea of his approach was to associate with every differential equation a uniquely defined geometric structure of a certain form.

In 1993, Bocharov et al. [6] considered the linearization problem of third-order with respect to point transformations. Grebot [7] studied the linearization of third-order ordinary differential equations by means of a restricted class of point transformations, namely, $t = \varphi(x)$ and $u = \psi(x, y)$. However, the problem was not completely solved. Complete criteria for linearization by means of point transformations were obtained by Ibragimov and Meleshko [8].

In 2008, Ibragimov et al. [9] solved the linearization problem for fourth-order ordinary differential equations by using point transformation.

Nowadays, the linearization problem of fifth-order ordinary differential equations via point transformations still is an unsolved one.

1.3. The Mapping of a Function by a Point Transformation

Definition 1. A transformation,

$$t = \varphi(x, y), \quad u = \psi(x, y), \tag{1}$$

where φ and ψ are sufficiently smooth functions, is called a *point transformation*. If $\varphi_y = 0$, a transformation (1) is called a fiber preserving transformation.

Let us explain how a point transformation maps one function into another.

Assume that $y_0(x)$ is a given function, the transformed function $u_0(t)$ is defined by the following two steps. On the first step one has to solve with respect to x the equation

$$t = \varphi(x, y_0(x)). \tag{2}$$

Using the Inverse Function Theorem we find that $x = \alpha(t)$ is a solution of this equation. The transformed function is determined by the formula

$$u_0(t) = \psi(\alpha(t), y_0(\alpha(t))). \tag{3}$$

Conversely, if one has the function $u_0(t)$, then for finding the function $y_0(x)$ one has to solve the ordinary differential equation

$$u_0(\varphi(x, y_0(x))) = \psi(x, y_0(x)). \tag{4}$$

2. Necessary Conditions of Linearization

We begin with investigating the necessary conditions for linearization. Recall that according to the Laguerre theorem a linear fifth-order ordinary differential equation has the form

$$u^{(5)} + \alpha(t) u'' + \beta(t) u' + \gamma(t) u = 0. \tag{5}$$

Here we consider the fifth-order ordinary differential equations

$$y^{(5)} = f(x, y, y'', y''', y^{(4)}), \tag{6}$$

which can be transformed to the linear equation (5) with $\alpha = \beta = \gamma = 0$ under the fiber preserving transformation

$$t = \varphi(x), \quad u = \psi(x, y). \tag{7}$$

So we arrive at the following theorem.

Theorem 2. *Any fifth-order ordinary differential equation* (6) *obtained from a linear equation* (5) *with* $\alpha = \beta = \gamma = 0$ *by a fiber preserving transformation* (7) *has to be the form*

$$\begin{aligned} y^{(5)} &+ (A_1 y' + A_0) y^{(4)} \\ &+ (B_3 y'' + B_2 y'^2 + B_1 y' + B_0) y''' \\ &+ (C_1 y' + C_0) y''^{(2)} \\ &+ (D_3 y'^3 + D_2 y'^2 + D_1 y' + D_0) y'' \\ &+ E_5 y'^5 + E_4 y'^4 + E_3 y'^3 + E_2 y'^2 \\ &+ E_1 y' + E_0 = 0, \end{aligned} \tag{8}$$

where

$$A_1 = \frac{5\psi_{yy}}{\psi_y}, \tag{9}$$

$$A_0 = \frac{-5(2\varphi_{xx}\psi_y - \varphi_x\psi_{xy})}{(\varphi_x\psi_y)}, \tag{10}$$

$$B_3 = \frac{10\psi_{yy}}{\psi_y}, \tag{11}$$

$$B_2 = \frac{10\psi_{yyy}}{\psi_y}, \tag{12}$$

$$B_1 = \frac{-20(2\varphi_{xx}\psi_{yy} - \varphi_x\psi_{xyy})}{(\varphi_x\psi_y)}, \tag{13}$$

$$B_0 = -5\left(2\varphi_{xxx}\varphi_x\psi_y - 9\varphi_{xx}^2\psi_y + 8\varphi_{xx}\varphi_x\psi_{xy} - 2\varphi_x^2\psi_{xxy}\right)\left(\varphi_x^2\psi_y\right)^{-1}, \tag{14}$$

$$C_1 = \frac{15\psi_{yyy}}{\psi_y}, \tag{15}$$

$$C_0 = \frac{-15\left(2\varphi_{xx}\psi_{yy} - \varphi_x\psi_{xyy}\right)}{\left(\varphi_x\psi_y\right)}, \tag{16}$$

$$D_3 = \frac{10\psi_{yyyy}}{\psi_y}, \tag{17}$$

$$D_2 = \frac{-30\left(2\varphi_{xx}\psi_{yyy} - \varphi_x\psi_{xyyy}\right)}{\left(\varphi_x\psi_y\right)}, \tag{18}$$

$$D_1 = (15\left(9\varphi_{xx}^2\psi_{yy} - 8\varphi_{xx}\varphi_x\psi_{xyy} + 2\varphi_x^2\psi_{xxyy} - 2\varphi_{xxx}\varphi_x\psi_{yy}\right))\left(\varphi_x^2\psi_y\right)^{-1}, \tag{19}$$

$$\begin{aligned} D_0 = -5&\left(\varphi_{xxxx}\varphi_x^2\psi_y - 12\varphi_{xxx}\varphi_{xx}\varphi_x\psi_y + 6\varphi_{xxx}\varphi_x^2\psi_{xy} + 21\varphi_{xx}^3\psi_y - 27\varphi_{xx}^2\varphi_x\psi_{xy} + 12\varphi_{xx}\varphi_x^2\psi_{xxy} - 2\varphi_x^3\psi_{xxxy}\right) \\ &\times\left(\varphi_x^3\psi_y\right)^{-1}, \end{aligned} \tag{20}$$

$$E_5 = \frac{\psi_{yyyyy}}{\psi_y}, \tag{21}$$

$$E_4 = \frac{-5\left(2\varphi_{xx}\psi_{yyyy} - \varphi_x\psi_{xyyyy}\right)}{\left(\varphi_x\psi_y\right)}, \tag{22}$$

$$E_3 = \left(-5\left(\left(2\varphi_{xxx}\varphi_x - 9\varphi_{xx}^2\right)\psi_{yyy} + 2\left(4\varphi_{xx}\psi_{xyyy} - \varphi_x\psi_{xxyyy}\right)\varphi_x\right)\right.\times\left(\varphi_x^2\psi_y\right)^{-1}, \tag{23}$$

$$\begin{aligned} E_2 = 5&\left(\left(6\left(2\varphi_{xx}\psi_{yy} - \varphi_x\psi_{xyy}\right)\varphi_{xxx} - \varphi_{xxxx}\varphi_x\psi_{yy}\right)\varphi_x - \left(21\varphi_{xx}^3\psi_{yy} - 27\varphi_{xx}^2\varphi_x\psi_{xyy} + 12\varphi_{xx}\varphi_x^2\psi_{xxyy} - 2\varphi_x^3\psi_{xxxyy}\right)\right) \\ &\times\left(\varphi_x^3\psi_y\right)^{-1}, \end{aligned} \tag{24}$$

$$\begin{aligned} E_1 = &\left(5\left(21\varphi_{xx}^4\psi_y - 42\varphi_{xx}^3\varphi_x\psi_{xy} + 27\varphi_{xx}^2\varphi_x^2\psi_{xxy} - 8\varphi_{xx}\varphi_x^3\psi_{xxxy} + \varphi_x^4\psi_{xxxxy} + 2\varphi_{xxx}^2\varphi_x^2\psi_y\right)\right. \\ &- \psi_{xxxxx}\varphi_x^3\psi_y + 5\left(3\varphi_{xx}\psi_y - 2\varphi_x\psi_{xy}\right)\varphi_{xxxx}\varphi_x^2 - 15\left(7\varphi_{xx}^2\psi_y - 8\varphi_{xx}\varphi_x\psi_{xy} + 2\varphi_x^2\psi_{xxy}\right) \\ &\left.\times\varphi_{xxx}\varphi_x\right)\left(\varphi_x^4\psi_y\right)^{-1}, \end{aligned} \tag{25}$$

$$\begin{aligned} E_0 = -&\left(\psi_{xxxxx}\varphi_x^3\psi_x - 15\varphi_{xxxx}\varphi_{xx}\varphi_x^2\psi_x + 5\varphi_{xxxx}\varphi_x^3\psi_{xx} - 10\varphi_{xxx}^2\varphi_x^2\psi_x + 105\varphi_{xxx}\varphi_{xx}^2\varphi_x\psi_x - 60\varphi_{xxx}\varphi_{xx}\varphi_x^2\psi_{xx}\right. \\ &+ 10\varphi_{xxx}\varphi_x^3\psi_{xxx} - 105\varphi_{xx}^4\psi_x + 105\varphi_{xx}^3\varphi_x\psi_{xx} \\ &\left.-45\varphi_{xx}^2\varphi_x^2\psi_{xxx} + 10\varphi_{xx}\varphi_x^3\psi_{xxxx} - \varphi_x^4\psi_{xxxxx}\right) \\ &\times\left(\varphi_x^4\psi_y\right)^{-1}. \end{aligned} \tag{26}$$

Proof. Applying a fiber preserving transformation (7), one obtains the following transformation of derivatives:

$$u'(t) = \frac{D_x\psi}{D_x\varphi} = \frac{\psi_x + y'\psi_y}{\varphi_x} = P\left(x, y, y'\right), \tag{27}$$

$$\begin{aligned} u''(t) &= \frac{D_xP}{D_x\varphi} = \frac{P_x + y'P_y + y''P_{y'}}{\varphi_x} \\ &= \frac{1}{\varphi_x^3}\left[\left(\varphi_x\psi_y\right)y'' + \left(\varphi_x\psi_{yy}\right)y'^2 + \left(-\varphi_{xx}\psi_y + 2\varphi_x\psi_{xy}\right)y' - \varphi_{xx}\psi_x + \varphi_x\psi_{xx}\right] \\ &= Q\left(x, y, y', y''\right), \end{aligned} \tag{28}$$

$$\begin{aligned} u'''(t) &= \frac{D_xQ}{D_x\varphi} = \frac{Q_x + y'Q_y + y''Q_{y'} + y'''Q_{y''}}{\varphi_x} \\ &= \frac{1}{\varphi_x^5}\left[\left(\varphi_x^2\psi_y\right)y''' + \left(3\varphi_x^2\psi_{yy}\right)y'y'' + 3\varphi_x\left(-\varphi_{xx}\psi_y + \varphi_x\psi_{xy}\right)y'' + \cdots\right] \\ &= R\left(x, y, y', y'', y'''\right), \end{aligned} \tag{29}$$

$$\begin{aligned} u^{(4)}(t) &= \frac{D_xR}{D_x\varphi} \\ &= \frac{R_x + y'R_y + y''R_{y'} + y'''R_{y''} + y^{(4)}R_{y'''}}{\varphi_x} \\ &= \frac{1}{\varphi_x^7}\left[\left(\varphi_x^3\psi_y\right)y^{(4)} + \left(4\varphi_x^3\psi_{yy}\right)y'y''' + 2\varphi_x^2\left(-3\varphi_{xx}\psi_y + 2\varphi_x\psi_{xy}\right)y''' + \cdots\right] \\ &= S\left(x, y, y', y'', y''', y^{(4)}\right), \end{aligned} \tag{30}$$

$$\begin{aligned} u^{(5)}(t) &= \frac{D_xS}{D_x\varphi} \\ &= \frac{S_x + y'S_y + y''S_{y'} + y'''S_{y''} + y^{(4)}S_{y'''} + y^{(5)}S_{y^{(4)}}}{\varphi_x} \\ &= \frac{1}{\varphi_x^9}\left[\left(\varphi_x^4\psi_y\right)y^{(5)} + \left(5\varphi_x^4\psi_{yy}\right)y'y^{(4)} + 5\varphi_x^3\left(-2\varphi_{xx}\psi_y + \varphi_x\psi_{xy}\right)y^{(4)} + \cdots\right], \end{aligned} \tag{31}$$

where $D_x = (\partial/\partial x) + y'(\partial/\partial y) + y''(\partial/\partial y') + y'''(\partial/\partial y'') + y^{(4)}(\partial/\partial y''') + y^{(5)}(\partial/\partial y^{(4)}) + \cdots$ is a total derivative. Substituting $u^{(5)}(t)$ into the linear equation (5), we have

$$\begin{aligned} y^{(5)} &+ \left(\left(\frac{5\psi_{yy}}{\psi_y} \right) y' - 5\left(2\varphi_{xx}\psi_y - \varphi_x\psi_{xy}\right) \right) y^{(4)} \\ &+ \left(\left(\frac{10\psi_{yy}}{\psi_y} \right) y'' + \left(\frac{10\psi_{yyy}}{\psi_y} \right) y'^2 \right. \\ &\left. - \left(\frac{20\left(2\varphi_{xx}\psi_{yy} - \varphi_x\psi_{xyy}\right)}{\left(\varphi_x\psi_y\right)} \right) y' + \cdots \right) y''' \\ &+ \left(\left(\frac{15\psi_{yyy}}{\psi_y} \right) y' + \cdots \right) y''^2 + \cdots = 0. \end{aligned} \tag{32}$$

Denoting A_i, B_i, C_i, D_i, and E_i as (10)–(21), we obtain the necessary form (8). These prove the theorem. □

3. Formulation of the Linearization Theorem

We have shown in the previous section that every linearizable fifth-order ordinary differential equation belongs to the class of (8). In this section, we formulate the main theorems containing necessary and sufficient conditions for linearization as well as the methods for constructing the linearizing transformations.

Theorem 3. *Sufficient conditions for* (8) *to be linearizable via a fiber preserving transformation are as follows:*

$$A_{0y} = A_{1x}, \tag{33}$$

$$B_3 = 2A_1, \tag{34}$$

$$A_{1y} = \frac{-\left(2A_1^2 - 5B_2\right)}{10}, \tag{35}$$

$$A_{1x} = \frac{-\left(4A_0A_1 - 5B_1\right)}{20}, \tag{36}$$

$$B_2 = \frac{2C_1}{3}, \tag{37}$$

$$B_1 = \frac{4C_0}{3}, \tag{38}$$

$$C_{1y} = \frac{-\left(2A_1C_1 - 15D_3\right)}{10}, \tag{39}$$

$$C_{0y} = \frac{-\left(2A_1C_0 - 5D_2\right)}{10}, \tag{40}$$

$$D_{3y} = \frac{-\left(A_1D_3 - 50E_5\right)}{5}, \tag{41}$$

$$D_{2y} = \frac{-\left(A_1D_2 - 30E_4\right)}{5}, \tag{42}$$

$$B_{0y} = \frac{-2\left(15A_{0x}A_1 - 25C_{0x} + 3A_0^2A_1 - 5A_0C_0\right)}{75}, \tag{43}$$

$$C_{0x} = \frac{\left(30A_{0x}A_1 + 6A_0^2A_1 - 10A_0C_0 - 15A_1B_0 + 25D_1\right)}{50}, \tag{44}$$

$$A_{0xx} = \frac{-\left(60A_{0x}A_0 - 75B_{0x} + 8A_0^3 - 30A_0B_0 + 50D_0\right)}{50}, \tag{45}$$

$$D_{1y} = \frac{-\left(A_1D_1 - 15E_3\right)}{5}, \tag{46}$$

$$\begin{aligned} D_{0y} = \big(&60A_{0x}A_0A_1 - 100A_{0x}C_0 - 75B_{0x}A_1 \\ &+ 125D_{1x} + 12A_0^3A_1 - 20A_0^2C_0 - 45A_0A_1B_0 \\ &+ 25A_0D_1 - 75A_1D_0 + 50B_0C_0 + 375E_2\big)(750)^{-1}, \end{aligned} \tag{47}$$

$$\begin{aligned} B_{0xx} = \Big(2\big(&175A_{0x}^2 + 70A_{0x}A_0^2 - 325A_{0x}B_0 \\ &- 75B_{0x}A_0 + 500D_{0x} + 7A_0^4 \\ &- 65A_0^2B_0 + 100A_0D_0 + 100B_0^2 - 625E_1\big)\Big) \\ &\times (375)^{-1}. \end{aligned} \tag{48}$$

Proof. For obtaining sufficient conditions, one has to solve the compatibility problem. Considering the representations of the coefficients A_i, B_i, C_i, D_i, and E_i through the unknown functions φ and ψ. We first rewrite the expressions (9) and (10) for A_1 and A_0 in the following form:

$$\psi_{yy} = \frac{\psi_y A_1}{5}, \tag{49}$$

$$\varphi_{xx} = \frac{\left(5\psi_{xy} - \psi_y A_0\right)\varphi_x}{\left(10\psi_y\right)}. \tag{50}$$

Differentiating (50) with respect to y, one obtains the condition (33). Substituting the expressions of ψ_{yy} and φ_{xx} into (11), (12), (13), (15), (16), (17), (18), (21), and (22) one gets conditions (34)–(42), respectively. From (14) we have

$$\begin{aligned} \psi_{xxy} = -\big(&20A_{0x}\psi_y^2 - 125\psi_{xy}^2 + 10\psi_{xy}\psi_y A_0 \\ &+ 7\psi_y^2A_0^2 - 20\psi_y^2B_0\big)\left(100\psi_y\right)^{-1}. \end{aligned} \tag{51}$$

Comparing the mixed derivative $(\psi_{xxy})_y = (\psi_{yy})_{xx}$ one obtains the condition (43). Equations (19), (20), (23), (24), and (25) provide the conditions (44)–(48), respectively.

Consider the form of $\psi_{yy} : \psi_{yy} = (\psi_y A_1/5)$ one can solve that

$$\psi_y = \omega_1(x, y)\,\psi_1(x), \tag{52}$$

where $\omega_1(x, y) = e^{\int (A_1/5)dy}$ and $\psi_1(x) = e^{K_1(x)}$. Since $\psi_y \neq 0$ then ψ_1 and ω_1 cannot be zero. From $\omega_1 = e^{\int (A_1/5)dy}$, we found the relation

$$A_1 = 5\frac{\omega_{1y}}{\omega_1}. \tag{53}$$

Relations $(A_1)_x = A_{1x}$ and $(A_1)_y = A_{1y}$ provide the conditions

$$\omega_{1xy} = \frac{\left(15\omega_{1x}\omega_{1y} - 3A_0\omega_{1y}\omega_1 + C_0\omega_1^2\right)}{(15\omega_1)}, \qquad \omega_{1yy} = \frac{C_1\omega_1}{15}, \tag{54}$$

respectively, and the relation (52) satisfied the condition $(\psi_y)_y = \psi_{yy}$. cComposing the relation $(\psi_y)_{xx} = \psi_{xxy}$, one has the equation

$$\begin{aligned}\psi_{1xx} = {} & \left(-20A_{0x}\omega_1^2\psi_1^2 - 100\omega_{1xx}\omega_1\psi_1^2 + 125\omega_{1x}^2\psi_1^2\right.\\ & + 50\omega_{1x}\psi_{1x}\omega_1\psi_1 - 10\omega_{1x}A_0\omega_1\psi_1^2\\ & + 125\psi_{1x}^2\omega_1^2 - 10\psi_{1x}A_0\omega_1^2\psi_1 - 7A_0^2\omega_1^2\psi_1^2\\ & \left.+ 20B_0\omega_1^2\psi_1^2\right)\left(100\omega_1^2\psi_1\right)^{-1}.\end{aligned} \tag{55}$$

Consider $\psi_y = \omega_1(x, y)\psi_1(x)$; one can solve that

$$\psi = \psi_1(x)\,\omega_2(x, y) + \psi_2(x), \tag{56}$$

where $\omega_2(x, y) = \int \omega_1(x, y)dy$. Because of $\int \omega_1(x, y)dy = \omega_2(x, y)$ then

$$\omega_1(x, y) = \omega_{2y}. \tag{57}$$

Since $\omega_1 \neq 0$ then $\omega_{2y} \neq 0$. Substituting ω_1 into ω_{1xy} and ω_{1yy} we obtain the additional conditions

$$\omega_{2xyy} = \frac{\left(\left(-3\omega_{2yy}A_0 + \omega_{2y}C_0\right)\omega_{2y} + 15\omega_{2xy}\omega_{2yy}\right)}{\left(15\omega_{2y}\right)}, \qquad \omega_{2yyy} = \frac{\omega_{2y}C_1}{15}, \tag{58}$$

respectively, and these satisfied the relations $(\psi)_y = \psi_y$, $(\psi)_{yy} = \psi_{yy}$, and $(\psi)_{xxy} = \psi_{xxy}$. From (26), setting $\mu_1(x, y)$, $\mu_2(x, y)$, and $\mu_3(x, y)$ as (A.1), (see Appendix) then we obtain

$$\begin{aligned}\psi_{2xxxxx} = {} & \left(-140625\psi_{1x}^4\psi_{2x}\omega_{2y}^5 + 1125000\psi_{1x}^3\psi_{2xx}\omega_{2y}^5\psi_1\right.\\ & + 112500\psi_{1x}^3\psi_{2x}\omega_{2y}^4\psi_1\left(-5\omega_{2xy} + \omega_{2y}A_0\right)\\ & - 2250000\psi_{1x}^2\psi_{2xxx}\omega_{2y}^5\psi_1^2\\ & + 675000\psi_{1x}^2\psi_{2xx}\omega_{2y}^4\psi_1^2\left(5\omega_{2xy} - \omega_{2y}A_0\right)\\ & + 11250\psi_{1x}^2\psi_{2x}\omega_{2y}^3\psi_1^2\\ & \times\left(-40A_{0x}\omega_{2y}^2 - 75\omega_{2xy}^2 + 30\omega_{2xy}\omega_{2y}A_0\right.\\ & \qquad \left.- 11\omega_{2y}^2A_0^2 + 20\omega_{2y}^2B_0\right)\\ & + 1500000\psi_{1x}\psi_{2xxxx}\omega_{2y}^5\psi_1^3\\ & + 900000\psi_{1x}\psi_{2xxx}\omega_{2y}^4\psi_1^3\left(-5\omega_{2xy} + \omega_{2y}A_0\right)\\ & + 75000\psi_{1x}\psi_{2xx}\omega_{2y}^3\psi_1^3\\ & \times\left(16A_{0x}\omega_{2y}^2 + 45\omega_{2xy}^2 - 18\omega_{2xy}\omega_{2y}A_0\right.\\ & \qquad \left.+ 5\omega_{2y}^2A_0^2 - 8\omega_{2y}^2B_0\right) + 1500\psi_{1x}\psi_{2x}\omega_{2y}^2\psi_1^3\\ & \times\left(200A_{0x}\omega_{2xy}\omega_{2y}^2 - 40A_{0x}\omega_{2y}^3A_0\right.\\ & \quad + 375\omega_{2xy}^3 - 225\omega_{2xy}^2\omega_{2y}A_0 + 85\omega_{2xy}\omega_{2y}^2A_0^2\\ & \quad - 100\omega_{2xy}\omega_{2y}^2B_0 - 11\omega_{2y}^3A_0^3 + 20\omega_{2y}^3A_0B_0\\ & \quad \left.- 2\mu_3\right) + 300000\psi_{2xxxx}\omega_{2y}^4\psi_1^4\\ & \quad \times\left(5\omega_{2xy} - \omega_{2y}A_0\right)\\ & \quad + 30000\psi_{2xxx}\omega_{2y}^3\psi_1^4\\ & \quad \times\left(-20A_{0x}\omega_{2y}^2 - 75\omega_{2xy}^2 + 30\omega_{2xy}\omega_{2y}A_0\right.\\ & \qquad \left.-7\,\omega_{2y}^2A_0^2 + 10\omega_{2y}^2B_0\right)\\ & \quad + 3000\psi_{2xx}\omega_{2y}^2\mu_3\psi_1^4\\ & \quad \left.+ 25\psi_{2x}\omega_{2y}\mu_2\psi_1^4 + 16\mu_1\psi_1^5\right)\\ & \times\left(300000\omega_{2y}^5\psi_1^4\right)^{-1}.\end{aligned} \tag{59}$$

By the proof of Theorem 3, we arrive at the following Corollary. □

Corollary 4. *Provided that the sufficient conditions in Theorem 3 are satisfied, the transformation* (7) *of mapping equation* (8) *to a linear equation $u^{(5)}(t)$ is obtained by solving the compatible system of equations* (49), (50), (56), (55), *and* (59) *for the functions $\varphi(x)$ and $\psi(x, y)$.*

4. Examples

Example 1. Consider the nonlinear ordinary differential equation

$$\begin{aligned}& x^2yy^{(5)} + 5\left(x^2y' + 2xy\right)y^{(4)}\\ & \quad + 10\left(x^2y'' + 4xy' + 2y\right)y''' + 30xy''^2\\ & \quad + 60y'y'' = 0.\end{aligned} \tag{60}$$

It is an equation of the form (8) with the coefficients

$$\begin{gathered}A_1 = \frac{5}{y}, \quad A_0 = \frac{10}{x}, \quad B_3 = \frac{10}{y},\\ B_2 = 0, \quad B_1 = \frac{40}{xy}, \quad B_0 = \frac{20}{x^2},\\ C_1 = 0, \quad C_0 = \frac{30}{xy}, \quad D_3 = 0,\\ D_2 = 0, \quad D_1 = \frac{60}{x^2y}, \quad D_0 = 0,\\ E_5 = 0, \quad E_4 = 0, \quad E_3 = 0,\\ E_2 = 0, \quad E_1 = 0, \quad E_0 = 0,\\ \omega_1 = y, \quad \omega_2 = \frac{y^2}{2}, \quad \mu_1 = 0,\\ \mu_2 = \frac{-90000y^4}{x^4}, \quad \mu_3 = \frac{-3000y^3}{x^3}.\end{gathered} \tag{61}$$

Applying Theorem 3 for checking the linearity, the coefficients in (61) obey all the conditions in Theorem 3, so that one concludes that equation (60) is linearizable. Applying Corollary 4, the linearizing transformation is found by solving the following equations:

$$\varphi_{xx} = \frac{\left(\varphi_x \psi_{1x} x - 2\psi_1\right)}{\left(2\psi_1 x\right)}, \tag{62}$$

$$\psi_{1xx} = \frac{\left(5\psi_{1x}^2 x^2 - 4\psi_{1x}\psi_1 x - 4\psi_1^2\right)}{\left(4\psi_1 x^2\right)}, \tag{63}$$

$$\begin{aligned}\psi_{2xxxxx} = \Big(5\Big(&-3\psi_{1x}^4\psi_{2x}x^4 + 24\psi_{1x}^3\psi_{2xx}\psi_1 x^4 \\ &+ 24\psi_{1x}^3\psi_{2x}\psi_1 x^3 - 48\psi_{1x}^2\psi_{2xxx}\psi_1^2 x^4 \\ &- 144\psi_{1x}^2\psi_{2xx}\psi_1^2 x^3 - 72\psi_{1x}^2\psi_{2x}\psi_1^2 x^2 \\ &+ 32\psi_{1x}\psi_{2xxxx}\psi_1^3 x^4 + 192\psi_{1x}\psi_{2xxx}\psi_1^3 x^3 \\ &+ 288\psi_{1x}\psi_{2xx}\psi_1^3 x^2 + 96\psi_{1x}\psi_{2x}\psi_1^3 x \\ &- 64\psi_{2xxxx}\psi_1^4 x^3 - 192\psi_{2xxx}\psi_1^4 x^2 \\ &-192\psi_{2xx}\psi_1^4 x - 48\psi_{2x}\psi_1^4\Big)\Big) \\ \times\left(32\psi_1^4 x^4\right)^{-1}&\end{aligned} \tag{64}$$

$$\psi = \frac{\left(\psi_1 y^2 + 2\psi_2\right)}{2}. \tag{65}$$

Since $\varphi_y = 0$, then one can take the simplest solution

$$\varphi = x. \tag{66}$$

Thus, (62) becomes $\psi_{1x}x - 2\psi_1 = 0$; the solution for this equation is

$$\psi_1 = Cx^2. \tag{67}$$

Choosing $C = 2$, we have

$$\psi_1 = 2x^2. \tag{68}$$

This solution satisfied (63). Equation (64) becomes

$$\psi_{2xxxxx} = 0. \tag{69}$$

Choosing the particular solution

$$\psi_2 = 0. \tag{70}$$

Hence (65) is in the form

$$\psi = x^2 y^2. \tag{71}$$

So one obtains the linearizing transformation

$$t = x, \qquad u = x^2 y^2. \tag{72}$$

Thus, the nonlinear equation (60) can be mapped by transformation of (72) into the linear equation $u^{(5)}(t) = 0$. Next, we will find the solution of (60). Since

$$u^{(5)}(t) = 0, \tag{73}$$

then we get the general solution

$$u(t) = C_0\frac{t^4}{24} + C_1\frac{t^3}{6} + C_2\frac{t^2}{2} + C_3 t + C_4, \tag{74}$$

where C_0, C_1, C_2, C_3, and C_4 are arbitrary constants. Substituting (72) into (74) we get

$$x^2y^2 = C_0\frac{x^4}{24} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4. \tag{75}$$

Example 2. Consider the nonlinear ordinary differential equation

$$\begin{aligned}&16y^4y^{(5)} - 40\left(y^3y' + 4y^4\right)y^{(4)} \\ &\quad - 40\left(2y^3y'' - 3y^2y'^2 - 8y^3y' - 14y^4\right)y''' \\ &\quad + 60\left(3y'y^2 + 4y^3\right)y''^2 \\ &\quad - 20\left(15yy'^3 + 36y^2y'^2 + 42y^3y' + 40y^4\right)y'' \\ &\quad \times 105y'^5 + 300yy'^4 + 420y^2y'^3 \\ &\quad + 400y^3y'^2 + 384y^4y' = 0.\end{aligned} \tag{76}$$

It is an equation of the form equation (8) with the coefficients

$$\begin{gathered} A_1 = \frac{-5}{2y}, \qquad A_0 = -10, \qquad B_3 = \frac{-5}{y}, \\ B_2 = \frac{15}{2y^2}, \qquad B_1 = \frac{20}{y}, \qquad B_0 = 35, \\ C_1 = \frac{45}{4y^2}, \qquad C_0 = \frac{15}{y}, \qquad D_3 = \frac{-75}{4y^3}, \\ D_2 = \frac{-45}{y^2}, \qquad D_1 = \frac{-105}{2y}, \qquad D_0 = -50, \\ E_5 = \frac{105}{16y^4}, \qquad E_4 = \frac{75}{4y^3}, \qquad E_3 = \frac{105}{4y^2}, \\ E_2 = \frac{25}{y}, \qquad E_1 = 24, \qquad E_0 = 0, \\ \omega_1 = \frac{1}{\sqrt{y}}, \qquad \omega_2 = 2\sqrt{y}, \\ \mu_1 = 0, \qquad \mu_2 = \frac{-288000}{y^2}, \qquad \mu_3 = \frac{5000}{y\sqrt{y}}. \end{gathered} \tag{77}$$

Applying Theorem 3 for checking the linearity, the coefficients in (77) obey all conditions in Theorem 3, so that one concludes that (76) is linearizable. Applying Corollary 4, the linearizing transformation is found by solving the following equations:

$$\varphi_{xx} = \frac{\varphi_x(\psi_{1x} + 2\psi_1)}{(2\psi_1)}, \quad (78)$$

$$\psi_{1xx} = \psi_{1x}(5\psi_{1x} + 4\psi_1), \quad (79)$$

$$\begin{aligned}\psi_{2xxxxxx} = \big(&-15\psi_{1x}^4\psi_{2x} + 120\psi_{1x}^3\psi_{2xx}\psi_1\\ &- 120\psi_{1x}^3\psi_{2x}\psi_1 - 240\psi_{1x}^2\psi_{2xxx}\psi_1^2\\ &+ 720\psi_{1x}^2\psi_{2xx}\psi_1^2 - 480\psi_{1x}^2\psi_{2x}\psi_1^2\\ &+ 160\psi_{1x}\psi_{2xxxx}\psi_1^3 - 960\psi_{1x}\psi_{2xxx}\psi_1^3\\ &+ 1760\psi_{1x}\psi_{2xx}\psi_1^3 - 960\psi_{1x}\psi_{2x}\psi_1^3\\ &+ 320\psi_{2xxxx}\psi_1^4 - 1120\psi_{2xxx}\psi_1^4\\ &+1600\psi_{2xx}\psi_1^4 - 768\psi_{2x}\psi_1^4\big)\\ \times\big(32\psi_1^4\big)^{-1},&\end{aligned} \quad (80)$$

$$\psi = \frac{(\psi_1 y^2 + 2\psi_2)}{2}. \quad (81)$$

From (78),

$$\frac{\varphi_{xx}}{\varphi_x} = \frac{\psi_{1x}}{2\psi_1} + 1. \quad (82)$$

Taking the particular solution $\varphi = e^x$, then the solution of (82) is

$$\psi_1 = C. \quad (83)$$

Choosing $C = (1/2)$, we have

$$\psi_1 = \frac{1}{2}. \quad (84)$$

This solution satisfied (79). Equation (80) becomes

$$\psi_{2xxxxxx} = 10\psi_{2xxxx} - 35\psi_{2xxx} + 50\psi_{2xx} - 24\psi_{2x}. \quad (85)$$

Choosing the particular solution

$$\psi_2 = 0. \quad (86)$$

Hence (81) is in the form

$$\psi = \sqrt{y}. \quad (87)$$

So one obtains the linearizing transformation

$$t = e^x, \qquad u = \sqrt{y}. \quad (88)$$

Thus, the nonlinear equation (76) can be mapped by transformation of (88) into the linear equation $u^{(5)}(t) = 0$. Next, we will find the solution of (76). Since

$$u^{(5)}(t) = 0, \quad (89)$$

then we get the general solution

$$u(t) = C_0\frac{t^4}{24} + C_1\frac{t^3}{6} + C_2\frac{t^2}{2} + C_3 t + C_4, \quad (90)$$

where C_0, C_1, C_2, C_3, and C_4 are arbitrary constants. Substituting (88) into (90) we get

$$\sqrt{y} = C_0\frac{e^{4x}}{24} + C_1\frac{e^{3x}}{6} + C_2\frac{e^{2x}}{2} + C_3 e^x + C_4. \quad (91)$$

Appendix

Equations in Section 3

Consider

$$\begin{aligned}\mu_1 = {}& 700A_{0x}^2\omega_{2y}^5A_0\omega_2 - 1750A_{0x}B_{0x}\omega_{2y}^5\omega_2\\ &+ 280A_{0x}\omega_{2y}^5A_0^3\omega_2 - 1050A_{0x}\omega_{2y}^5A_0B_0\omega_2\\ &+ 2750A_{0x}\omega_{2y}^5D_0\omega_2 - 350B_{0x}\omega_{2y}^5A_0^2\omega_2\\ &+ 875B_{0x}\omega_{2y}^5B_0\omega_2 + 1250D_{0xx}\omega_{2y}^5\omega_2\\ &+ 500D_{0x}\omega_{2y}^5A_0\omega_2 - 6250E_{1x}\omega_{2y}^5\omega_2\\ &+ 2250000\omega_{2xy}^5\omega_2 - 2250000\omega_{2xy}^4\omega_{2x}\omega_{2y}\\ &- 450000\omega_{2xy}^4\omega_{2y}A_0\omega_2 - 4500000\omega_{2xy}^3\omega_{2xxy}\omega_{2y}\omega_2\\ &+ 1125000\omega_{2xy}^3\omega_{2xx}\omega_{2y}^2\\ &+ 450000\omega_{2xy}^3\omega_{2x}\omega_{2y}^2A_0 + 112500\omega_{2xy}^3\omega_{2y}^2B_0\omega_2\\ &+ 1125000\omega_{2xy}^2\omega_{2xxxy}\omega_{2y}^2\omega_2\\ &- 375000\omega_{2xy}^2\omega_{2xxx}\omega_{2y}^3\\ &+ 3375000\omega_{2xy}^2\omega_{2xxy}\omega_{2x}\omega_{2y}^2\\ &+ 675000\omega_{2xy}^2\omega_{2xxy}\omega_{2y}^2A_0\omega_2\\ &- 225000\omega_{2xy}^2\omega_{2xx}\omega_{2y}^3A_0\\ &- 112500\omega_{2xy}^2\omega_{2x}\omega_{2y}^3B_0\\ &- 37500\omega_{2xy}^2\omega_{2y}^3D_0\omega_2\\ &- 187500\omega_{2xy}\omega_{2xxxxy}\omega_{2y}^3\omega_2\\ &+ 93750\omega_{2xy}\omega_{2xxxx}\omega_{2y}^4\\ &- 750000\omega_{2xy}\omega_{2xxxy}\omega_{2x}\omega_{2y}^3\\ &- 150000\omega_{2xy}\omega_{2xxxy}\omega_{2y}^3A_0\omega_2\\ &+ 75000\omega_{2xy}\omega_{2xxx}\omega_{2y}^4A_0\\ &+ 1687500\omega_{2xy}\omega_{2xxy}^2\omega_{2y}^2\omega_2\\ &- 1125000\omega_{2xy}\omega_{2xxy}\omega_{2xx}\omega_{2y}^3\\ &- 450000\omega_{2xy}\omega_{2xxy}\omega_{2x}\omega_{2y}^3A_0\\ &- 112500\omega_{2xy}\omega_{2xxy}\omega_{2y}^3B_0\omega_2\\ &+ 56250\omega_{2xy}\omega_{2xx}\omega_{2y}^4B_0\end{aligned}$$

$$
\begin{aligned}
&+ 37500\omega_{2xy}\omega_{2x}\omega_{2y}^4 D_0 \\
&+ 18750\omega_{2xy}\omega_{2y}^4 E_1\omega_2 \\
&+ 18750\omega_{2xxxxxy}\omega_{2y}^4\omega_2 \\
&- 18750\omega_{2xxxxx}\omega_{2y}^5 \\
&+ 93750\omega_{2xxxxy}\omega_{2x}\omega_{2y}^4 \\
&+ 18750\omega_{2xxxxy}\omega_{2y}^4 A_0\omega_2 \\
&- 18750\omega_{2xxxx}\omega_{2y}^5 A_0 \\
&- 375000\omega_{2xxxy}\omega_{2xxy}\omega_{2y}^3\omega_2 \\
&+ 187500\omega_{2xxxy}\omega_{2xx}\omega_{2y}^4 \\
&+ 75000\omega_{2xxxy}\omega_{2x}\omega_{2y}^4 A_0 \\
&+ 18750\omega_{2xxxy}\omega_{2y}^4 B_0\omega_2 \\
&+ 187500\omega_{2xxx}\omega_{2xxy}\omega_{2y}^4 \\
&- 18750\omega_{2xxx}\omega_{2y}^5 B_0 \\
&- 562500\omega_{2xxy}^2\omega_{2x}\omega_{2y}^3 \\
&- 112500\omega_{2xxy}^2\omega_{2y}^3 A_0\omega_2 \\
&+ 112500\omega_{2xxy}\omega_{2xx}\omega_{2y}^4 A_0 \\
&+ 56250\omega_{2xxy}\omega_{2x}\omega_{2y}^4 B_0 \\
&+ 18750\omega_{2xxy}\omega_{2y}^4 D_0\omega_2 \\
&- 18750\omega_{2xx}\omega_{2y}^5 D_0 \\
&- 18750\omega_{2x}\omega_{2y}^5 E_1 + 18750\omega_{2y}^6 E_0 \\
&+ 28\omega_{2y}^5 A_0^5\omega_2 - 210\omega_{2y}^5 A_0^3 B_0\omega_2 \\
&+ 550\omega_{2y}^5 A_0^2 D_0\omega_2 + 350\omega_{2y}^5 A_0 B_0^2\omega_2 \\
&- 1250\omega_{2y}^5 A_0 E_1\omega_2 - 1250\omega_{2y}^5 B_0 D_0\omega_2, \\
\mu_2 = &-3200A_{0x}^2\omega_{2y}^4 - 18000A_{0x}\omega_{2xy}^2\omega_{2y}^2 \\
&- 2400A_{0x}\omega_{2xy}\omega_{2y}^3 A_0 - 2000A_{0x}\omega_{2y}^4 A_0^2 \\
&+ 5600A_{0x}\omega_{2y}^4 B_0 + 12000B_{0x}\omega_{2xy}\omega_{2y}^3 \\
&- 4000D_{0x}\omega_{2y}^4 - 5625\omega_{2xy}^4 \\
&+ 4500\omega_{2xy}^3\omega_{2y}A_0 - 4950\omega_{2xy}^2\omega_{2y}^2 A_0^2 \\
&+ 9000\omega_{2xy}^2\omega_{2y}^2 B_0 - 300\omega_{2xy}\omega_{2y}^3 A_0^3 \\
&+ 3600\omega_{2xy}\omega_{2y}^3 A_0 B_0 - 12000\omega_{2xy}\omega_{2y}^3 D_0 \\
&- 281\omega_{2y}^4 A_0^4 + 1480\omega_{2y}^4 A_0^2 B_0 \\
&- 800\omega_{2y}^4 A_0 D_0 - 2000\omega_{2y}^4 B_0^2 + 8000\omega_{2y}^4 E_1, \\
\mu_3 = &\ 400A_{0x}\omega_{2xy}\omega_{2y}^2 - 100B_{0x}\omega_{2y}^3 \\
&+ 375\omega_{2xy}^3 - 225\omega_{2xy}^2\omega_{2y}A_0 \\
&+ 125\omega_{2xy}\omega_{2y}^2 A_0^2 - 200\omega_{2xy}\omega_{2y}^2 B_0 \\
&- 3\omega_{2y}^3 A_0^3 - 20\omega_{2y}^3 A_0 B_0 + 100\omega_{2y}^3 D_0.
\end{aligned}
\tag{A.1}
$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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