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Research Article

Symmetry and Solution of Neutron Transport Equations in Nonhomogeneous Media

Ivan Tsyfra[1] and Tomasz Czyżycki[2]

[1] *AGH University of Science and Technology, Faculty of Applied Mathematics, 30 Mickiewicza Avenue, 30-059 Krakow, Poland*
[2] *Institute of Mathematics, University of Białystok, Akademicka 2, 15-267 Białystok, Poland*

Correspondence should be addressed to Ivan Tsyfra; tsyfra@agh.edu.pl

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We propose the group-theoretical approach which enables one to generate solutions of equations of mathematical physics in nonhomogeneous media from solutions of the same problem in a homogeneous medium. The efficiency of this method is illustrated with examples of thermal neutron diffusion problems. Such problems appear in neutron physics and nuclear geophysics. The method is also applicable to nonstationary and nonintegrable in quadratures differential equations.

1. Introduction

The group-theoretical analysis is known to be used for the construction of exact solutions of a number of linear and nonlinear equations of mathematical physics [1, 2]. One of the most efficient methods for the obtaining of explicit solutions is the method of group reduction [1–11]. Finite transformations of the invariance group of differential equations can also be applied to generate new solutions (both exact and approximate). Moreover, based on the notion of the equivalence group of transformations [2], we broaden possibilities of applications of the group-theoretical methods by extending the class of admissible operators. These operators are not the symmetry operators in the rigorous classical sense. In the present paper we show that the group analysis can be applied to the construction of solutions for equations of mathematical physics with varying coefficients characterizing properties of the media. Using the equivalence group we show the connections between nonhomogeneity of the medium and the group transformations. The nonhomogeneity manifests itself by dependency of the coefficients in the equation on the variable x. We show that from the solution for a homogeneous medium one can generate the solution for a nonhomogeneous medium and even obtain the results for the equations, in which the coefficients depend on time (nonstationary media), and for media with anisotropy. This approach enables one to obtain the class of not only the integrable, but also the nonintegrable equations in some sense. Regardless of integrability or nonintegrability of a given equation it is important whether or not the coefficients of this equation lie on one orbit of the action of the equivalence group, as we will show in Section 3. The following property takes place: if a potential lies on an orbit of a nonintegrable potential, then it is also nonintegrable, whilst if it lies on an orbit of an integrable potential, then it is also integrable. If the system of Lie equations is integrable and its solution is expressed by elementary functions, then by means of this group of transformations one can generate new nonintegrable potentials.

2. Application of Finite Group of Transformations to the Neutron Diffusion Problem

The group-theoretical methods are proved to be very effective for construction of exact solutions for many important equations in mathematical physics. In particular they enable one to reduce partial differential equation to ordinary differential equation by using the classical and conditional symmetry of initial equation. Integrating this reduced equation one can obtain exact solutions in explicit form, such as one-and many-soliton solutions. Moreover, the existence of Lie-Bäcklund symmetry enables one to construct conservation laws for initial equation [3–11]. In this section we use another property of an invariance group, namely, the generation of the new solutions by action of the group of finite transformations on the known one.

While studying the problems of the theory (and interpretation) of geophysical fields, it is necessary to solve boundary-value problems for equations of the form

$$u_{xx} + u_{yy} = a(x, y)\,\Phi(u, u_t), \tag{1}$$

where a is a parameter characterizing the nonhomogeneity of the medium and Φ is a smooth function. In mathematical formulation these problems are reduced to solve boundary-value problems for these equations.

Assume that $a(x, y, t)$ is a function parameter contained in (1) along with $u(x, y, t)$. Then in the extended space (x, y, t, u, a) (1) admits a sufficiently wide group of transformations of the form

$$\begin{gathered} x' = V(x, y), \qquad y' = W(x, y), \\ t' = t, \qquad u' = u, \qquad a' = \frac{a}{V_x^2 + W_x^2}, \end{gathered} \tag{2}$$

where $f(z) = V(x, y) + iW(x, y)$ is an arbitrary analytic function. These groups of transformations were used in [12] for investigation of inverse problems of geophysics. In this paper we use the group of transformations to solve the direct problem, namely, the problem of stationary diffusion of neutrons in a nonhomogeneous medium for the linear and point source in two- and three-dimensional space.

As an example illustrating the efficiency of the given approach, consider the following problem of stationary diffusion equation:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} - \frac{\Phi}{L^2} = 0. \tag{3}$$

Here L is the diffusion length of thermal neutrons, which is piecewise continuous function depending on the distance $r = \sqrt{x^2 + y^2}$ from point (x, y) to the origin of coordinates in two-dimensional Euclidean space. As a geometrical model for an investigated heterogeneous medium we take three-zone cylindrical system with parameters L and D (coefficients of diffusion) given by relations

$$\begin{gathered} D = D_1 = \text{const}, \quad L = L_1 = \text{const}, \quad 0 < \sqrt{x^2 + y^2} < r_1, \\ D = D_2 = \text{const}, \quad L = \frac{L_2\left(x^2 + y^2\right)}{r_2^2}, \\ L_2 = \text{const}, \quad r_1 < \sqrt{x^2 + y^2} < r_2, \\ D = D_3 = \text{const}, \quad L = L_3 = \text{const}, \quad \sqrt{x^2 + y^2} > r_2. \end{gathered} \tag{4}$$

Thus, the coefficient D is a piecewise constant function. This setting of the problem can be used for investigation of diffusion of particles (e.g., thermal neutrons) in a heterogeneous three-zone system that simulates a real system of "borehole layer" [13, 14]. Gradient variation of diffusion length L in the second zone is caused by penetration of the borehole fluid in the layer with absorption parameters different from the same parameters of fluid.

The solution of the problem in question in the first and third zones coincides with the corresponding solution for the infinite homogeneous medium with L = const. The solution in the middle zone is obtained from such solution [14] by the action of a finite group of transformations.

It is known that the sought solution must satisfy the continuity conditions for the flow of neutrons as well as for the normal component of a neutrons current across interfaces. It means that the functions Φ, $D\cdot(\partial\Phi/\partial r)$ must be continuous at points r_m, $m = 1, 2$. These properties together with the condition at infinity

$$\Phi(r) \longrightarrow 0 \quad \text{if } r \longrightarrow \infty \tag{5}$$

and the condition of the neutrons balance

$$Q = 2\pi \int_0^\infty r \frac{D}{L^2} \Phi\, dr, \tag{6}$$

where Q is a number of neutrons, emitted by 1 cm length of linear source per 1 second, are used to determine the constants A_i, B_i, on which the solution of the boundary-value problem for different source models of thermal neutrons depends.

The formulas for the coefficients A_i, B_i seem to be rather complicated and therefore we give system of algebraic equations instead of them. From this (linear) system of equations one can easily calculate coefficients A_i, B_i in each particular case. Finally the solution of this problem can be presented in the following form:

(i) for a linear source of heat neutrons located on the symmetry axis of the system

$$\Phi = A_1 I_0\left(\frac{r}{L_1}\right) + B_1 K_0\left(\frac{r}{L_1}\right), \quad 0 < r < r_1, \tag{7}$$

$$\Phi = A_2 I_0\left(\frac{r_2^2}{L_2 r}\right) + B_2 K_0\left(\frac{r_2^2}{L_2 r}\right), \quad r_1 < r < r_2, \tag{8}$$

$$\Phi = B_3 K_0\left(\frac{r}{L_3}\right), \quad r_2 < r < \infty, \tag{9}$$

$$\frac{Q}{2\pi} = A_1 D_1 \frac{r_1}{L_1} I_1\left(\frac{r_1}{L_1}\right) + B_1 D_1\left[1 - \frac{r_1}{L_1} K_1\left(\frac{r_1}{L_1}\right)\right] + A_2 D_2\left[\frac{r_2^2}{L_2 r_1} I_1\left(\frac{r_2^2}{L_2 r_1}\right) - \frac{r_2}{L_2} I_1\left(\frac{r_2}{L_2}\right)\right] + B_2 D_2\left[\frac{r_2}{L_2} K_1\left(\frac{r_2}{L_2}\right) - \frac{r_2^2}{L_2 r_1} K_1\left(\frac{r_2^2}{L_2 r_1}\right)\right] + B_3 D_3 \frac{r_2}{L_3} K_1\left(\frac{r_2}{L_3}\right), \tag{10}$$

$$A_1 I_0\left(\frac{r_1}{L_1}\right) + B_1 K_0\left(\frac{r_1}{L_1}\right) = A_2 I_0\left(\frac{r_2^2}{L_2 r_1}\right) + B_2 K_0\left(\frac{r_2^2}{L_2 r_1}\right), \tag{11}$$

$$A_1 I_1\left(\frac{r_1}{L_1}\right) - B_1 K_1\left(\frac{r_1}{L_1}\right) = \frac{D_2 L_1}{D_1 L_2}\left(\frac{r_2}{r_2}\right)^2 \left\{-A_2 I_1\left(\frac{r_2^2}{L_2 r_1}\right) + B_2 K_1\left(\frac{r_2^2}{L_2 r_1}\right)\right\}, \tag{12}$$

$$A_2 I_0\left(\frac{r_2}{L_2}\right) + B_2 K_0\left(\frac{r_2}{L_2}\right) = B_3 K_0\left(\frac{r_2}{L_3}\right), \tag{13}$$

$$A_2 I_1\left(\frac{r_2}{L_2}\right) - B_2 K_1\left(\frac{r_2}{L_2}\right) = \frac{D_3 L_2}{D_2 L_3} B_3 K_1\left(\frac{r_2}{L_3}\right); \tag{14}$$

(iiA) for a thin cylinder layer emitting heat neutrons

$$\Phi = A_1 I_0\left(\frac{r}{L_1}\right), \quad 0 < r < r^*, \tag{15}$$

$$\Phi = A_1^* I_0\left(\frac{r}{L_1}\right) + B_1^* K_0\left(\frac{r}{L_1}\right), \quad r^* < r < r_1, \tag{16}$$

$$\Phi = A_2 I_0\left(\frac{r_2^2}{L_2 r}\right) + B_2 K_0\left(\frac{r_2^2}{L_2 r}\right), \quad r_1 < r < r_2, \tag{17}$$

$$\Phi = B_3 K_0\left(\frac{r}{L_3}\right), \quad r_2 < r < \infty, \tag{18}$$

$$A_1 I_0\left(\frac{r^*}{L_1}\right) = A_1^* I_0\left(\frac{r^*}{L_1}\right) + B_1^* K_0\left(\frac{r^*}{L_1}\right), \tag{19}$$

$$(A_1 - A_1^*) I_1\left(\frac{r^*}{L_1}\right) + B_1^* K_1\left(\frac{r^*}{L_1}\right) = \frac{Q L_1}{2\pi D_1 r^*}, \tag{20}$$

$$A_1^* I_0\left(\frac{r_1}{L_1}\right) + B_1^* K_0\left(\frac{r_1}{L_1}\right) = A_2 I_0\left(\frac{r_2^2}{L_2 r_1}\right) + B_2 K_0\left(\frac{r_2^2}{L_2 r_1}\right), \tag{21}$$

$$A_1^* I_1\left(\frac{r_1}{L_1}\right) - B_1^* K_1\left(\frac{r_1}{L_1}\right) = \frac{D_2 L_1}{D_1 L_2}\left(\frac{r_2}{r_1}\right)^2 \left\{-A_2 I_1\left(\frac{r_2^2}{L_2 r_1}\right) + B_2 K_1\left(\frac{r_2^2}{L_2 r_1}\right)\right\}, \tag{22}$$

$$A_2 I_0\left(\frac{r_2}{L_2}\right) + B_2 K_0\left(\frac{r_2}{L_2}\right) = B_3 K_0\left(\frac{r_2}{L_3}\right), \tag{23}$$

$$A_2 I_1\left(\frac{r_2}{L_2}\right) - B_2 K_1\left(\frac{r_2}{L_2}\right) = B_3 \frac{D_3 L_2}{D_2 L_3} K_1\left(\frac{r_2}{L_3}\right) \tag{24}$$

in case when a cylinder source is located in the inner homogeneous zone $0 \le r < r_1$;

(iiB) for a thin cylinder layer emitting heat neutrons

$$\Phi = A_1 I_0\left(\frac{r}{L_1}\right), \quad 0 < r < r_1, \tag{25}$$

$$\Phi = A_2 I_0\left(\frac{r_2^2}{L_2 r}\right) + B_2 K_0\left(\frac{r_2^2}{L_2 r}\right), \quad r_1 < r < r^*, \tag{26}$$

$$\Phi = A_2^* I_0\left(\frac{r_2^2}{L_2 r}\right) + B_2^* K_0\left(\frac{r_2^2}{L_2 r}\right), \quad r^* < r < r_2, \tag{27}$$

$$\Phi = B_3 K_0\left(\frac{r}{L_3}\right), \quad r_2 < r < \infty, \tag{28}$$

$$A_1 I_0\left(\frac{r_1}{L_1}\right) = A_2 I_0\left(\frac{r_2^2}{L_2 r_1}\right) + B_2 K_0\left(\frac{r_2^2}{L_2 r_1}\right), \tag{29}$$

$$A_1 \frac{D_1 L_2}{D_2 L_1}\left(\frac{r_1}{r_2}\right)^2 I_1\left(\frac{r_1}{r_2}\right) = -A_2 I_1\left(\frac{r_2^2}{L_2 r_1}\right) + B_2 K_1\left(\frac{r_2^2}{L_2 r_1}\right), \tag{30}$$

$$(A_2 - A_2^*) I_0\left(\frac{r_2^2}{L_2 r^*}\right) + (B_2 - B_2^*) K_0\left(\frac{r_2^2}{L_2 r^*}\right) = 0, \tag{31}$$

$$(A_2 - A_2^*) I_1\left(\frac{r_2^2}{L_2 r^*}\right) - (B_2 - B_2^*) K_1\left(\frac{r_2^2}{L_2 r^*}\right) = -\frac{Q L_2 r^*}{2\pi D_2 r_2^2}, \tag{32}$$

$$A_2^* I_0\left(\frac{r_2}{L_2}\right) + B_2^* K_0\left(\frac{r_2}{L_2}\right) = B_3 K_0\left(\frac{r_2}{L_3}\right), \tag{33}$$

$$A_2^* I_1\left(\frac{r_2}{L_2}\right) - B_2^* K_1\left(\frac{r_2}{L_2}\right) = \frac{D_3 L_2}{D_2 L_3} B_3 K_1\left(\frac{r_2}{L_3}\right) \tag{34}$$

in case when the source is located in the middle zone $r_1 < r < r_2$; that is, the gradient is nonhomogeneous in the radial direction;

(iiC) for a thin cylinder layer emitting heat neutrons

$$\Phi = A_1 I_0\left(\frac{r}{L_1}\right), \quad 0 < r < r_1, \tag{35}$$

$$\Phi = A_2 I_0\left(\frac{r_2^2}{L_2 r}\right) + B_2 K_0\left(\frac{r_2^2}{L_2 r}\right), \quad r_1 < r < r_2, \tag{36}$$

$$\Phi = A_3 I_0\left(\frac{r}{L_3}\right) + B_3 K_0\left(\frac{r}{L_3}\right), \quad r_2 < r < r^*, \tag{37}$$

$$\Phi = B_3^* K_0\left(\frac{r}{L_3}\right), \quad r^* < r < \infty, \tag{38}$$

$$A_1 I_0\left(\frac{r_1}{L_1}\right) = A_2 I_0\left(\frac{r_2^2}{L_2 r_1}\right) + B_2 K_0\left(\frac{r_2^2}{L_2 r_1}\right), \tag{39}$$

$$A_1 \frac{D_1 L_2}{D_2 L_1}\left(\frac{r_1}{r_2}\right)^2 I_1\left(\frac{r_1}{L_1}\right) = -A_2 I_1\left(\frac{r_2^2}{L_2 r_1}\right) + B_2 K_1\left(\frac{r_2^2}{L_2 r_1}\right), \tag{40}$$

$$A_2 I_0\left(\frac{r_2}{L_2}\right) + B_2 K_0\left(\frac{r_2}{L_2}\right) = A_3 I_0\left(\frac{r_2}{L_2}\right) + B_3 K_0\left(\frac{r_2}{L_3}\right), \tag{41}$$

$$A_2 I_1\left(\frac{r_2}{L_2}\right) - B_2 K_1\left(\frac{r_2}{L_2}\right) = -\frac{D_3 L_2}{D_2 L_3}\left[A_3 I_1\left(\frac{r_2}{L_2}\right) - B_3 K_1\left(\frac{r_2}{L_3}\right)\right], \tag{42}$$

$$A_3 I_0\left(\frac{r^*}{L_3}\right) + \left(B_3 - B_3^*\right) K_0\left(\frac{r^*}{L_3}\right) = 0, \tag{43}$$

$$A_3 I_1\left(\frac{r^*}{L_3}\right) - \left(B_3 - B_3^*\right) K_1\left(\frac{r^*}{L_3}\right) = \frac{Q L_3}{2\pi D_3 r^*} \tag{44}$$

in case when a cylinder source is located in the exterior homogeneous zone $r_1 < r < \infty$.

Here r^* is a radius of a cylinder layer emitting thermal neutrons; I_0 and K_0 are modified Bessel functions of the first and second kinds. Solutions (8), (17), (26), (27), and (36) are obtained from the corresponding solutions for the homogeneous medium with $L = L_2$ by using the transformations

$$x' = \frac{r_2^2\, x}{x^2 + y^2}, \qquad y' = \frac{r_2^2\, y}{x^2 + y^2}. \tag{45}$$

Further we consider the case when D is not constant but a smooth function on r. By using the polar coordinates and assuming $\Phi = \Phi(r)$, $D = D(r)$, $\Sigma = \Sigma(r)$, one can write the stationary neutron diffusion equation

$$-\operatorname{div}\left(D(x,y)\cdot\nabla\Phi(x,y)\right) + \Sigma(x,y)\cdot\Phi(x,y) = 0, \tag{46}$$

where $D(x, y)$ is the coefficient of thermal neutron diffusion and $\Sigma(x, y)$ is the macroscopic cross-section of neutron absorption in the following form:

$$\Phi_{rr} + \left(\frac{1}{r} + \frac{D_r}{D}\right)\cdot\Phi_r + \frac{\Sigma}{D}\cdot\Phi = 0. \tag{47}$$

After the substitution

$$\Phi(r) = \frac{W(r)}{\sqrt{r\cdot D(r)}}, \tag{48}$$

we obtain the equation

$$W'' + W\cdot\left[-\frac{\Sigma}{D} + \frac{1}{4r^2} - \frac{D_{rr}}{2D} + \frac{D_r^2}{4D^2} - \frac{D_r}{2rD}\right] = 0 \tag{49}$$

which can be written in the form

$$W''(r) + V(r)\,W(r) = 0, \tag{50}$$

where the potential $V(r)$ is given by formula

$$V(r) = \frac{1}{4r^2} - \frac{\Sigma}{r} - \frac{D_{rr}}{2D} + \frac{D_r^2}{4D^2} - \frac{D_r}{2rD}. \tag{51}$$

We have constructed the generator of one parameter Lie group of equivalence transformations for (50):

$$Y = \tau(r)\,\partial_r + \frac{1}{2}\tau'(r)\,W\partial_W - \left(\frac{1}{2}\tau'''(r) + 2\tau'(r)\,V\right)\partial_V, \tag{52}$$

where $\tau(r)$ is an arbitrary smooth function. Hence we obtain the finite group transformations:

$$\begin{aligned}
\tilde{r} &= \tilde{\tau}^{-1}\left(a + \tilde{\tau}(r)\right), \\
\widetilde{W} &= W\cdot\sqrt{\frac{\tau(\tilde{r})}{\tau(r)}}, \\
\widetilde{V} &= V\cdot\frac{\tau^2(r)}{\tau^2(\tilde{r})} - \frac{1}{4\tau^2(\tilde{r})} \cdot\left[2\tau''(\tilde{r})\cdot\tau(\tilde{r}) - 2\tau''(r)\cdot\tau(r) - \tau'^2(\tilde{r}) + \tau'^2(r)\right],
\end{aligned} \tag{53}$$

where $\tilde{\tau}(s) = \int((1/\tau(s))ds)$.

Now we can generate new solution W_n for new potential V_n from a given solution. Let $W = \varphi(r)$, $V = V_1(r)$, where φ, V_1 are given smooth functions. Then we obtain

$$\begin{aligned}
W_n &= \varphi(\tilde{r})\cdot\sqrt{\frac{\tau(r)}{\tau(\tilde{r})}}, \\
V_n &= V_1(\tilde{r})\cdot\frac{\tau^2(\tilde{r})}{\tau^2(r)} + \frac{1}{4\tau^2(r)} \cdot\left[2\tau''(\tilde{r})\cdot\tau(\tilde{r}) - 2\tau''(r)\cdot\tau(r) - \tau'^2(\tilde{r}) + \tau'^2(r)\right].
\end{aligned} \tag{54}$$

If $D = D_2 = \text{const}$, $\Sigma = \Sigma_2 = \text{const}$, then the solution of (47) is given by

$$\Phi = C_2 I_0\left(\frac{r}{L_2}\right) + E_2 K_0\left(\frac{r}{L_2}\right), \tag{55}$$

where $C_2 = \text{const}$, $E_2 = \text{const}$, and $L_2 = \sqrt{D_2/\Sigma_2}$.

Thus $W = \sqrt{D_2 r}\cdot\Phi(r)$ and $V = V_1 = (1/4r^2) - (\Sigma_2/D_2)$. By using (54), one can construct new solution W_n for new potential V_n and therefore

$$\Phi_n(r) = \frac{W_n(r)}{\sqrt{r\cdot D(r)}}. \tag{56}$$

So, the function Φ_n (56) is the solution of (47) provided that $D(r)$ and $\Sigma(r)$ satisfy the following equation:

$$V_n = -\frac{\Sigma(r)}{D(r)} + \frac{1}{4r^2} - \frac{D_{rr}}{2D} + \frac{D_r^2}{4D^2} - \frac{D_r}{2rD}. \tag{57}$$

Thus, by virtue of the group method, we can construct the solution of neutron diffusion equation in nonhomogeneous medium where both coefficients D and Σ are not constant but variable functions on r. Obviously, such a model extends to more adequately describe the properties of the nonhomogeneous medium. Note that two functions $D(r)$, $\Sigma(r)$ have to satisfy one differential equation. Hence, there is the class of nonhomogeneous media for which the function (56) is the solution of diffusion equation (47). The media are described by coefficients D, Σ satisfying (57).

By analogy with the previous example, making use of the conformal transformations, we can construct solutions of diffusion equations in the three-dimensional space (for a point source) for a nonhomogeneous medium

$$\frac{\partial^2\Phi}{\partial x_1^2} + \frac{\partial^2\Phi}{\partial x_2^2} + \frac{\partial^2\Phi}{\partial x_3^2} - \frac{\Phi}{L^2} = 0, \tag{58}$$

where

$$\begin{gathered} D = D_1 = \text{const.},\quad L = L_1 = \text{const.},\quad 0 < r \le r_1, \\ D = D_2 = \text{const.},\quad L = L_2\cdot\frac{r^2}{r_2^2}, \\ L_2 = \text{const.},\quad r_1 < r \le r_2, \\ D = D_3 = \text{const.},\quad L = L_3 = \text{const.},\quad r_2 < r < \infty, \\ r = \sqrt{x_1^2 + x_2^2 + x_3^2}. \end{gathered} \tag{59}$$

If we consider the parameter L as a dependent variable, then (58) is invariant with respect to the continuous Lie group of conformal transformations. We choose the following transformations, forming a discrete group with two elements:

$$r' = \frac{r_2^2}{r},\qquad \Phi' = \frac{r}{r_2}\Phi,\qquad L' = \frac{r_2^2 L}{r^2}. \tag{60}$$

Applying transformations (60) to the solution of (58) in a homogenous medium for $L = L_2$, we construct the solution of (58), (59) for the point source in the following form:

$$\begin{gathered} \Phi = \frac{A_1}{r}\exp\left(-\frac{r}{L_1}\right) + \frac{B_1}{r}\exp\left(\frac{r}{L_1}\right), \\ 0 < r \le r_1, \\ \Phi = A_2\exp\left(-\frac{r_2^2}{L_2 r}\right) + B_2\exp\left(\frac{r_2^2}{L_2 r}\right), \\ r_1 < r \le r_2, \\ \Phi = \frac{A_3}{r}\exp\left(-\frac{r}{L_3}\right),\quad r_2 < r \le \infty, \\ \frac{Q}{4\pi} = D_1\left[B_1\left(\frac{r_1}{L_1} - 1\right)\exp\left(\frac{r_1}{L_1}\right) + B_1 - A_1\left(\frac{r_1}{L_1} + 1\right)\exp\left(-\frac{r_1}{L_1}\right) + A_1\right] \\ + \frac{D_2 r_2^2}{L_2}\left[A_2\left(\exp\left(-\frac{r_2}{L_2}\right) - \exp\left(-\frac{r_2^2}{L_2 r_1}\right)\right) - B_2\left(\exp\left(\frac{r_2}{L_2}\right) - \exp\left(\frac{r_2^2}{L_2 r_1}\right)\right)\right] \\ + A_3 D_3\left(\frac{r_2}{L_3} + 1\right)\exp\left(-\frac{r_2}{L_3}\right), \\ \frac{A_1}{r_1}\exp\left(-\frac{r_1}{L_1}\right) + \frac{B_1}{r_1}\left(\frac{r_1}{L_1}\right) = A_2\exp\left(-\frac{r_2^2}{L_2 r_1}\right) + B_2\exp\left(\frac{r_2^2}{L_2 r_1}\right), \\ -\frac{A_1}{r_1}\left(\frac{1}{L_1} + \frac{1}{r_1}\right)\exp\left(-\frac{r_1}{L_1}\right) + \frac{B_1}{r_1}\left(\frac{1}{L_1} - \frac{1}{r_1}\right)\exp\left(\frac{r_1}{L_1}\right) \\ = \frac{A_2}{L_2}\left(\frac{r_2}{r_1}\right)^2\exp\left(-\frac{r_2^2}{L_2 r_1}\right) - \frac{B_2}{L_2}\left(\frac{r_2}{r_1}\right)^2\exp\left(\frac{r_2^2}{L_2 r_1}\right), \\ A_2\exp\left(-\frac{r_2}{L_2}\right) + B_2\exp\left(\frac{r_2}{L_2}\right) = \frac{A_3}{r_2}\exp\left(-\frac{r_2}{L_3}\right), \\ \frac{A_2}{L_2}\exp\left(-\frac{r_2}{L_2}\right) - \frac{B_2}{L_2}\exp\left(\frac{r_2}{L_2}\right) = -\frac{A_3}{r_2}\left(\frac{1}{L_3} + \frac{1}{r_2}\right)\exp\left(-\frac{r_2}{L_3}\right). \end{gathered} \tag{61}$$

For the medium where $D \neq \text{const}$ the stationary thermal neutron diffusion equation has the form

$$-\operatorname{div}\left(D\left(x_1, x_2, x_3\right) \nabla \Phi\left(x_1, x_2, x_3\right)\right) + \Sigma\left(x_1, x_2, x_3\right) \Phi\left(x_1, x_2, x_3\right) = 0. \tag{62}$$

By using the spherical coordinates $x = r\cos\varphi\sin\theta$, $y = r\sin\varphi\sin\theta$, $z = r\cos\theta$, one can write it in the following form:

$$\Phi_{rr} + \left(\frac{2}{r} + \frac{D_r}{D}\right) \cdot \Phi_r + \frac{\Sigma}{D} \cdot \Phi = 0. \tag{63}$$

Then, after substitution: $\Phi(r) = W(r)/(r \cdot \sqrt{D(r)})$, we obtain the equation in the form

$$W'' + VW = 0, \tag{64}$$

where

$$V(r) = -\frac{\Sigma}{D} - \frac{D_{rr}}{2D} + \frac{3D_r^2}{4D^2} - \frac{D_r}{rD}. \tag{65}$$

We can construct the solution of (63) in nonhomogeneous media from the solution in homogeneous medium when $\Sigma(r)$ and $D(r)$ are constants, by using the group transformations (53). If $D = D_2 = \text{const}$, $\Sigma = \Sigma_2 = \text{const}$, then the solution of (63) is given by

$$\Phi = \frac{M_2}{r}\exp\left(-\frac{r}{L_2}\right) + \frac{N_2}{r}\exp\left(\frac{r}{L_2}\right), \tag{66}$$

where $M_2 = \text{const}$, $N_2 = \text{const}$, and $L_2 = \sqrt{D_2/\Sigma_2}$.

Thus we have

$$W = r\sqrt{D_2}\Phi(r), \quad V = V_1 = -\frac{\Sigma_2}{D_2} \tag{67}$$

and solution for nonhomogeneous medium

$$W_n = \tilde{r}\sqrt{D_2}\Phi(\tilde{r}) \cdot \sqrt{\frac{\tau(r)}{\tau(\tilde{r})}},$$

$$V_n = -\frac{\Sigma_2}{D_2} \cdot \frac{\tau^2(\tilde{r})}{\tau^2(r)} + \frac{1}{4\tau^2(r)} \cdot \left[2\tau''(\tilde{r}) \cdot \tau(\tilde{r}) - 2\tau''(r) \cdot \tau(r) - \tau'^2(\tilde{r}) + \tau'^2(r)\right]. \tag{68}$$

Then we can obtain

$$\Phi_n(r) = \frac{W_n(r)}{r\sqrt{D(r)}}. \tag{69}$$

Therefore the function Φ_n (69) is the solution of (63) provided that $D(r)$ and $\Sigma(r)$ satisfy the following differential equation:

$$V_n = -\frac{\Sigma(r)}{D(r)} - \frac{D_{rr}}{2D} + \frac{3D_r^2}{4D^2} - \frac{D_r}{rD}. \tag{70}$$

Thus, by using the group method, we can construct the solution of neutron diffusion equation for point source in nonhomogeneous medium.

3. Further Applications of the Method

In the framework of this approach one can also study nonstationary problems. It turns out that the group of transformations (2) is sufficiently wide to solve the Cauchy problem for (1). Using the invariance property of (1) with respect to transformations (2), we can easily verify that the following assertion is true.

Theorem 1. *Assume that $u = f(x, y, t)$ is a solution of the Cauchy problem*

$$u|_{t=0} = \varphi(x, y), \quad u_t|_{t=0} = \psi(x, y) \tag{71}$$

for (1) *with $a = a_0 =$* const *. Then $u = f(x', y', t')$ is a solution to the Cauchy problem*

$$u|_{t=0} = \varphi(x', y'), \quad u_t|_{t=0} = \psi(x', y') \tag{72}$$

for (1) *with $a = a_0(V_x^2 + W_x^2)$.*

Thus, if the medium has a nonhomogeneity defined by the relation

$$a = a_0\left(V_x^2 + W_x^2\right), \tag{73}$$

then the solution of the Cauchy problem for (1) can be obtained from the solution of the same problem for the homogeneous medium ($a = a_0$), by transformations (2).

One can obviously map a known solution of the Schrödinger equation with a given potential to the solution of the equation for a changed (in general time-dependent) potential as it has been done above for the neutrons diffusion equation. The group equivalence transformations are also used in this case.

Consider the Schrödinger equation

$$i\frac{\partial\psi}{\partial t} + \Delta\psi + W(t, \vec{x}, |\psi|)\psi = 0 \tag{74}$$

in the n-dimensional space. Symmetry properties of this equation were investigated in [15, 16]. For arbitrary $W(t, \vec{x}, |\psi|)$, this equation admits only the group of identity transformations:

$$\vec{x}' = \vec{x}, \qquad t' = t, \qquad \psi' = \psi. \tag{75}$$

If we treat W as a new independent variable, then we obtain quite a broad invariance group of (74). This approach is equivalent to the construction of the group of equivalence transformations [16, 17].

Theorem 2 (see [16]). *Equation* (74) *is invariant under the action of the elements of the infinite-dimensional Lie algebra with basis operators:*

$$J_{ab} = x_a \frac{\partial}{\partial x_b} - x_b \frac{\partial}{\partial x_a},$$

$$Q_a = U_a \frac{\partial}{\partial x_a} + \frac{i}{2}\dot{U}_a x_a \left(\psi \frac{\partial}{\partial \psi} - \psi^* \frac{\partial}{\partial \psi^*}\right) + \frac{1}{2}\ddot{U}_a x_a \frac{\partial}{\partial W},$$

$$Q_A = 2A\frac{\partial}{\partial t} + \dot{A}x_c \frac{\partial}{\partial x_c} + \frac{i}{4}\ddot{A}x_c x_c \left(\psi \frac{\partial}{\partial \psi} - \psi^* \frac{\partial}{\partial \psi^*}\right) - \frac{n\dot{A}}{2}\left(\psi \frac{\partial}{\partial \psi} + \psi^* \frac{\partial}{\partial \psi^*}\right) + \left(\frac{1}{4}\dddot{A}\ x_c x_c - 2W \cdot \dot{A}\right)\frac{\partial}{\partial W},$$

$$Q_B = iB\left(\psi \frac{\partial}{\partial \psi} - \psi^* \frac{\partial}{\partial \psi^*}\right) + \dot{B}\frac{\partial}{\partial W},$$

$$Z_1 = \psi \frac{\partial}{\partial \psi}, \qquad Z_2 = \psi^* \frac{\partial}{\partial \psi^*}, \tag{76}$$

where $U_a(t)$, $A(t)$, and $B(t)$ are arbitrary smooth functions of t and we mean by the repeated index c the summation from 1 to n. The upper dot stands for the derivative with respect to time.

Theorem 2 has been proved by using the Lie infinitesimal criterion of invariance in [16]. Note that the invariance algebra (76) includes subalgebras such as the Galilean algebra and the projective algebra.

By using the Lie equations we obtain the following finite group transformations generated by operators from (76).

For operators Q_a we get

$$t' = t,$$
$$x'_a = U_a(t)\beta_a + x_a,$$
$$x'_b = x_b, \quad b \neq a,$$
$$\psi' = \psi \exp\left(\frac{i}{4}\dot{U}_a U_a \beta_a^2 + \frac{i}{2}\dot{U}_a x_a \beta_a\right),$$
$$\psi^{*\prime} = \psi^* \exp\left(-\frac{i}{4}\dot{U}_a U_a \beta_a^2 - \frac{i}{2}\dot{U}_a x_a \beta_a\right),$$
$$W' = W + \frac{1}{2}\ddot{U}_a x_a \beta_a + \frac{1}{4}\ddot{U}_a U_a \beta_a^2, \tag{77}$$

where β_a are group parameters and $U_a(t)$ are arbitrary smooth functions.

Remark 3. If $U_a(t) = t$ we obtain from Q_a the Galilean operators

$$G_a = t\partial_{x_a} + \frac{i}{2}x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right). \tag{78}$$

Remark 4. Let us apply these transformations to the Schrödinger equation with cubic nonlinearity in the two-dimensional case:

$$i\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + 2\left(\psi^*\psi\right)\psi = 0. \tag{79}$$

This equation is known to be integrated by the inverse problem method.

Let $\psi = \varphi(t, x)$ be a solution to (79). Then using (77), we obtain the solution of the equation

$$i\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + W\psi = 0, \tag{80}$$

where

$$W = 2\left(\psi^*\psi\right) - \frac{1}{2}\ddot{U}x\beta - \frac{1}{4}\ddot{U}U\beta^2, \tag{81}$$

in the following form:

$$\psi_{\text{new}} = \exp\left\{-\frac{i}{2}\dot{U}x\beta - \frac{i}{4}\dot{U}U\beta^2\right\}\varphi\left(t, x + U(t)\beta\right). \tag{82}$$

Thus any solution of the integrable (79) is transformed into the solution of (80).

For operators Q_A one gets

$$t' = \widetilde{A}^{-1}\left(2a + \widetilde{A}(t)\right), \tag{83}$$

where $\widetilde{A}(s) = \int ds/A(s)$, $\widetilde{A}^{-1}$ denotes the inverse function to $\widetilde{A}$, and a is the group parameter. Consider

$$x'_c = x_c \cdot \sqrt{\frac{A(t')}{A(t)}},$$
$$\psi' = \psi \cdot \left(\frac{A(t)}{A(t')}\right)^{n/4} \cdot \exp\left(\frac{i\vec{x}^2\left(\dot{A}(t') - \dot{A}(t)\right)}{8A(t)}\right),$$
$$\psi^{*\prime} = \psi^* \cdot \left(\frac{A(t)}{A(t')}\right)^{n/4} \cdot \exp\left(\frac{-i\vec{x}^2\left(\dot{A}(t') - \dot{A}(t)\right)}{8A(t)}\right),$$
$$W' = \frac{\vec{x}^2}{8A(t)\cdot A(t')} \cdot \int_t^{t'} \dddot{A}(s)\cdot A(s)\,ds + \frac{A(t)}{A(t')}\cdot W. \tag{84}$$

Remark 5. It can be easily verified that transformations (84) preserve the integral

$$\int_{\mathbb{R}^n} \psi^*\psi \, d\vec{x}. \tag{85}$$

Indeed

$$\int_{\mathbb{R}^n} \psi^{*\prime}\psi' d\vec{x}' = \int_{\mathbb{R}^n} \left(\frac{A(t)}{A(t')}\right)^{n/2} \psi^*\psi \cdot \left(\frac{A(t')}{A(t)}\right)^{n/2} d\vec{x} = \int_{\mathbb{R}^n} \psi^*\psi \, d\vec{x}. \tag{86}$$

We use a well-known property of symmetry transformations, which map the solution of studied differential equation to another solution of this equation but not necessarily the physical solution to physical one. Formulas (84) give the example of the transformations which preserve the physical normalization condition

$$\int_{\mathbb{R}^n} \psi\psi^* d\vec{x} = 1 \tag{87}$$

for wave function ψ.

Moreover, one can generate the solution of the nonstationary Schrödinger equation applying transformations (84) to the solution of the stationary Schrödinger equation. More exactly, if $\psi(x)$ satisfies the stationary equation

$$-\Delta\psi - V(x)\psi = E_n\psi, \tag{88}$$

then

$$\Psi(t,x) = e^{-iE_n t'} \cdot \psi\left(\vec{x} \cdot \sqrt{\frac{A(t')}{A(t)}}\right) \cdot \left(\frac{A(t')}{A(t)}\right)^{n/4} \cdot \exp\left(\frac{-i\vec{x}^2 \left(\dot{A}(t') - \dot{A}(t)\right)}{8A(t)}\right) \tag{89}$$

satisfies the nonstationary Schrödinger equation with time-dependent potential:

$$\begin{aligned} W(t,x) = {} & -\frac{\vec{x}^2}{8A^2(t)} \cdot \int_t^{t'} \ddot{A}(s) \cdot A(s)\, ds \\ & + \frac{A(t')}{A(t)} \cdot V\left(x \cdot \sqrt{\frac{A(t')}{A(t)}}\right), \end{aligned} \tag{90}$$

where $t' = \widetilde{A}^{-1}(2a + \widetilde{A}(t))$ as above.

Note that the obtained solution $\Psi(t,x)$ satisfies condition (87) provided that $\psi(t,x)$ satisfies it. This procedure can also be effectively applied to exactly solvable quantum mechanical models. Hence we can obtain solutions of the Schrödinger equation with time-dependent potentials, generated from a number of exactly solvable potentials, such as trigonometric and hyperbolic Pöschel-Teller, Eckart potentials and others. Recall that the spectral problem for the stationary Schrödinger equation in one-dimensional case is reduced to the pure algebraic problem of finding eigenvalues of a constant matrix for an exactly solvable model.

It is obvious that one can compose equivalence transformations (84) and the Moutard-type transformations for the two-dimensional stationary Schrödinger equation:

$$\psi_{x_1x_1} + \psi_{x_2x_2} = V(x_1, x_2)\psi, \quad \psi = \psi(x_1, x_2) \tag{91}$$

in the form

$$\begin{aligned} V_1 &= -V_0 + \frac{2\left(\omega_{x_1}^2 + \omega_{x_2}^2\right)}{\omega^2}, \\ (\omega\vartheta)_{x_1} &= -\omega^2\left(\frac{\varphi}{\omega}\right)_{x_2}, \\ (\omega\vartheta)_{x_2} &= -\omega^2\left(\frac{\varphi}{\omega}\right)_{x_1}, \end{aligned} \tag{92}$$

where $\psi = \omega(x_1, x_2)$, $\psi = \varphi(x_1, x_2)$ are two solutions of (91) with potential $V = V_0(x_1, x_2)$ and $\psi = \vartheta(x_1, x_2)$ is the solution of this equation with changed potential:

$$V = -V_0 + \frac{2\left(\omega_{x_1}^2 + \omega_{x_2}^2\right)}{\omega^2}. \tag{93}$$

Let us consider an application of the method discussed above to the new class of problems, namely, nonintegrable differential equations. We have proved the following theorem.

Theorem 6. *If the second-order equation*

$$y'' + V(x)\, y = 0 \tag{94}$$

is nonintegrable in quadratures, then equation

$$y'' + V_{new}(x)\, y = 0, \tag{95}$$

where

$$V_{new} = V\left(\frac{x}{1-ax}\right) \cdot (1-ax)^{-4}, \tag{96}$$

is also nonintegrable in quadratures.

Proof. We use the equivalence transformations of (94). The generator of equivalence group is in the form

$$\begin{aligned} Y = {} & \tau(x)\,\partial_x + \left(\frac{1}{2}\tau'(x) + C\right) y\partial_y \\ & - \left(\frac{1}{2}\tau'''(x) + 2\tau'(x)\, V\right)\partial_V, \end{aligned} \tag{97}$$

where $\tau(x)$ is a smooth function, $C = \text{const}$.

Applying the group transformations for $\tau(x) = x^2$, we construct the needed substitution.

Let $g(x)$ be a solution of (95). Consider the following function: $h(x) = (1 + ax)g(x/(1 + ax))$. Hence

$$\begin{aligned} h'(x) &= ag\left(\frac{x}{1+ax}\right) + g'\left(\frac{x}{1+ax}\right)\frac{1}{1+ax}, \\ h''(x) &= g''\left(\frac{x}{1+ax}\right)\frac{1}{(1+ax)^3}. \end{aligned} \tag{98}$$

Note that

$$\begin{aligned} & h''(x) + V(x)h(x) \\ & = \frac{1}{(1+ax)^3} g''\left(\frac{x}{1+ax}\right) + V(x)(1+ax) g\left(\frac{x}{1+ax}\right) \\ & = \frac{1}{(1+ax)^3}\left[g''\left(\frac{x}{1+ax}\right) + V(x)(1+ax)^4 g\left(\frac{x}{1+ax}\right)\right] \\ & = \frac{1}{(1+ax)^3}\left[g''\left(\frac{x}{1+ax}\right) + \frac{V(x)}{1+ax}\right. \\ & \qquad \left. \cdot \frac{1}{(1-a(x/(1+ax)))^5} g\left(\frac{x}{1+ax}\right)\right] = 0. \end{aligned} \quad (99)$$

Hence h satisfies (94). If (95) were integrable in quadratures, (94) would be integrable too, which would lead to the contradiction with the assumption of the theorem.

Analogically, taking $\tau = e^x$ and using the fact that the Airy equation is nonintegrable in quadratures, one can show that the equation

$$y'' + \left[-\frac{1}{4} + e^{2x}\left(\ln(e^x - a) + \frac{1}{4}\right)\frac{1}{(e^x - a)^2}\right] \cdot y = 0 \quad (100)$$

is nonintegrable in quadratures too.

It seems to us that the method used in proof of theorem is more simple in comparison with a technique developed in [18].

By applying Theorem 6 to neutron diffusion problems we conclude that if the potentials V_n in (57) or (70) have the form given by Theorem 6, then the corresponding neutron diffusion equations (47) and (63) are not integrable in quadratures. □

4. Conclusions

In Section 2 we have shown that the finite group of transformations helps us to construct the solution of the neutron diffusion equation in two- and three-dimensional cases for the nonhomogeneous medium from the known solution for the homogeneous medium. We considered linear point and thin cylinder layer emitting heat neutrons. It turns out that the method is also applicable to the nonstationary and integrable and nonintegrable differential equations as was shown in Section 3. We can generate a family of ordinary differential equations nonintegrable (integrable) in quadratures by using finite group transformations. It should be noted that one can successfully apply the method discussed above to the Maxwell equations in nonhomogeneous medium.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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