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Research Article

Positive Solutions for a Nonhomogeneous Kirchhoff Equation with the Asymptotical Nonlinearity in R^3

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We study the following nonhomogeneous Kirchhoff equation: $-(a+b\int_{R^3}|\nabla u|^2dx)\Delta u+u=k(x)f(u)+h(x)$, $x\in R^3$, $u\in H^1(R^3)$, $u>0$, $x\in R^3$, where f is asymptotically linear with respect to t at infinity. Under appropriate assumptions on k, f, and h, existence of two positive solutions is proved by using the Ekeland's variational principle and the Mountain Pass Theorem in critical point theory.

1. Introduction and Main Results

In this paper, we consider the following nonhomogeneous Kirchhoff equation:

$$-\left(a+b\int_{R^3}|\nabla u|^2dx\right)\Delta u+u=k(x)f(u)+h(x),\quad x\in R^3,$$
$$u\in H^1\left(R^3\right),\quad u>0,\ x\in R^3, \tag{1}$$

where constants $a,b>0$, and functions k,f and h satisfy the following conditions: k is a positive bounded condition, $f\in C(R,R^+)$, $f(t)\equiv 0$ if $t<0$ and $h\in L^2(R^3)$, $h\geq 0$. Note that, with $a=1$, $b=0$, and R^3 replaced by R^N, problem (1) reduces to

$$-\Delta u+u=k(x)f(u)+h(x)\quad \text{in } R^N, \tag{2}$$

which can be looked at as a generalization of the well known Schrödinger equation.

When Ω is a smooth bounded domain in R^N, the problem

$$-\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=g(x,u),\quad x\in\Omega,$$
$$u=0,\quad x\in\partial\Omega, \tag{3}$$

is related to the stationary analogue of the Kirchhoff equation which was proposed by Kirchhoff in 1883 (see [1]) as a generalization of the well known d'Alembert's wave equation

$$\rho\frac{\partial^2u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2u}{\partial x^2}=g(x,u) \tag{4}$$

for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. Moreover, Kirchhoff's type problems also model several physical systems and biological systems and there are many interesting results for problem (3) which can be found in [2–8] and the references therein.

Some interesting studies for Kirchhoff-type problem (3) in a bounded domain Ω of R^N by variational methods can be found in [2, 9–22]. Very recently, some authors had studied the Kirchhoff equation on the whole space R^N and obtained the existence of multiple solutions (see [23–31]). In the same spirit of [24–26, 28–31], we study a nonhomogeneous Kirchhoff equation (1) on the whole space R^3. Especially, inspired by the paper [32, 33], we consider the asymptotically linear nonlinearity at infinity of problem (1). For the nonhomogeneous Kirchhoff problem, Chen and Li in

[23] study it under the condition of superlinear nonlinearity at infinity. In [33], Wang and Zhou study the existence of two positive solutions for a nonhomogeneous elliptic equation ((1) with $a = 1$ and $b = 0$). In [32], Sun et al. study the existence of a ground state solution for some nonautonomous Schrödinger-Poisson systems involving the asymptotically linear nonlinearity at infinity without the nonhomogeneous term. But we will study the existence of two positive solutions for Kirchhoff-type problem (1) with $a, b > 0$, the asymptotically linear nonlinearity at infinity and the nonhomogeneous term. So, we can not obtain the existence of a ground state solution for Kirchhoff-type problem (1) and the compactness result as in [32] because of the nonhomogeneous term, and we cannot easily obtain the compactness result as in [33] due to the nonlocal term (or $b \neq 0$). To our best knowledge, little has been done for nonhomogeneous Kirchhoff problems with respect to the asymptotically linear nonlinearity at infinity.

Before stating our main results, we give some notations. For any $1 \leq q \leq +\infty$, we denote by $\|\cdot\|_q$ the usual norm of the Lebesgue space $L^q(R^3)$. Define the function space

$$H^1(R^3) := \{u \in L^2(R^3) : \nabla u \in L^2(R^3)\} \tag{5}$$

with the product and equivalent norm

$$(u, v) = \int_{R^3} (a\nabla u \cdot \nabla v + uv)\, dx, \qquad \|u\| := \left(\int_{R^3} (a|\nabla u|^2 + |u|^2)\, dx\right)^{1/2}. \tag{6}$$

Define the function space

$$D^{1,2}(R^3) := \{u \in L^6(R^3) : \nabla u \in L^2(R^3)\} \tag{7}$$

with the standard product and norm

$$(u, v) = \int_{R^3} \nabla u \cdot \nabla v dx, \qquad \|u\|_D := \left(\int_{R^3} |\nabla u|^2 dx\right)^{1/2}. \tag{8}$$

Recall that the Sobolev's inequality with the best constant is

$$\|v\|_6 \leq S\|v\|. \tag{9}$$

Moreover, problem (1) has a variational structure. Indeed the corresponding action functional $I : H^1(R^3) \to R$ of (1) is defined by

$$I(u) = \frac{a}{2}\int_{R^3} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{R^3} |\nabla u|^2 dx\right)^2 + \frac{1}{2}\int_{R^3} |u|^2 dx - \int_{R^3} k(x) F(u)\, dx - \int_{R^3} h(x) u\, dx. \tag{10}$$

By Lemma 2.1 in [24] or Lemma 1 in [25], the functional I is $C^1(H^1(R^3), R)$ with the derivative given by

$$\langle I'(u), u\rangle = \left(a + b\int_{R^3} |\nabla u|^2 dx\right)\int_{R^3} \nabla u \cdot \nabla v\, dx + \int_{R^3} uv\, dx - \int_{R^3} k(x) f(u) u\, dx - \int_{R^3} h(x) v\, dx. \tag{11}$$

Hence, if $u \in H^1(R^3)$ is a nonzero critical point of I, then it is also a nonnegative solution of (1). In fact, by $f(t) \equiv 0$ if $t < 0$ and $h \geq 0$, we have $\langle I'(u), u^-\rangle = -(a + b\|u\|^2)\int_{R^3} |\nabla u^-|^2 dx - \int_{R^3} (u^-)^2 dx - \int_{R^3} k(x) f(u) u^- dx - \int_{R^3} h(x) u^- dx = 0$, where $u^- = \max\{-u, 0\}$. This yields that $u^- = 0$; then $u = u^+ - u^- = u^+ \geq 0$, where $u^+ = \max\{u, 0\}$. By the maximum principle, the nonzero critical point of I is the positive solution for problem (1).

Here is the main result of this paper.

Theorem 1. *Suppose that $h \in L^2(R^3)$, $h \geq 0$, and the following conditions hold.*

(f1) *$f \in C(R, R^+)$, $f(0) = 0$, and $f(t) \equiv 0$ for $t < 0$.*

(f2) $\lim_{t\to 0}(f(t)/t) = 0$.

(f3) $\lim_{t\to +\infty}(f(t)/t) = l < +\infty$.

(k1) *$k(x)$ is a positive continuous function and there exists $R_0 > 0$ such that*

$$\sup\left\{\frac{f(t)}{t} : t > 0\right\} < \inf\left\{\frac{1}{k(x)} : |x| \geq R_0\right\}. \tag{12}$$

(k2) *Let*

$$l_0 > \mu^* := \inf\left\{\int_{R^3} (a|\nabla u|^2 + u^2)\, dx : u \in H^1(R^3), \int_{R^3} k(x) F(u)\, dx \geq \frac{b}{2} l^2\right\} \tag{13}$$

hold, where $l_0 = \min\{l, (b/2)l^2\}$, $F(t) = \int_0^t f(s) ds$.

Then problem (1) has at least two positive solutions $u_0, u \in H^1(R^3)$ satisfying $I(u_0) < 0$ and $I(u) > 0$ if $\|h\|_2 < m$ for some small $m > 0$.

Remark 2. It is not difficult to find some functions k, f satisfying conditions of Theorem 1. For example, for any $R_0 > 0$, let

$$f(t) = \begin{cases} \dfrac{R_0 t^2}{1 + t}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases} \tag{14}$$

Clearly, f satisfies (f1)–(f3) with $l = R_0$. Moreover, $F(t) = R_0((1/2)t^2 - t + \ln(1 + t))$ and $\sup\{f(t)/t : t > 0\} = R_0$. Taking a positive continuous function $k(x)$,

$$k(x) = \begin{cases} \dfrac{C_0}{1 + |x|}, & \text{if } |x| \leq \dfrac{R_0}{2}, \\ \dfrac{1}{1 + |R_0|}, & \text{if } |x| \geq R_0, \end{cases} \tag{15}$$

where $C_0 = 3M^3(1 + R_0/2)/4\pi(\ln 2 - 1/2)$ for some $M > 0$. Note that

$$\inf\left\{\frac{1}{k(x)} : |x| \geq R_0\right\} = 1 + R_0 > R_0 = \sup\left\{\frac{f(t)}{t} : t > 0\right\}; \tag{16}$$

then (k1) holds. To verify the condition (k2), we have to choose some special $R_0 > 0$. For any $R > 0$, taking $\psi \in C_0^\infty(R^3, [0,1])$ such that $\psi(x) = 1$ if $|x| \le R$, $\psi(x) = 0$ if $|x| \ge 2R$ and $|\nabla\psi(x)| \le C/\sqrt{a}R$ for all $x \in R^3$, where $C > 0$ is an arbitrary constant independent of x. Then, for any $R_0 > 2R$, we have

$$\begin{aligned}\int_{R^3} k(x) F(\psi)\, dx &\ge \int_{|x|\le R} k(x) F(\psi)\, dx \\ &\ge \frac{R_0 C_0}{1+R}\left(\ln 2 - \frac{1}{2}\right)\left|B_R(0)\right| \\ &\ge \frac{3M^3(1+R_0/2)}{4\pi(\ln 2 - 1/2)}\frac{4\pi R^3 R_0}{3(1+R_0/2)} \\ &\quad \times\left(\ln 2 - \frac{1}{2}\right) \\ &= M^3 R_0 R^3,\end{aligned} \tag{17}$$

$$\begin{aligned}\int_{R^3}\left(a|\nabla\psi|^2 + |\psi|^2\right) dx &\le \int_{|x|\le 2R}\frac{C^2}{R^2}dx + \int_{|x|\le 2R} dx \\ &\le \left(1+\frac{C^2}{R^2}\right)\frac{32\pi}{3}R^3 \\ &\le \frac{32\pi}{3}R\left(C^2+R^2\right).\end{aligned} \tag{18}$$

Taking $R_0 = l = 1$, $R = (1/M)R_0\sqrt[3]{b/2} = (1/M)\sqrt[3]{b/2}$, where M is large enough such that $2R < R_0/4$, $40\pi/3M^3 < 1$, and $(40\pi/3M^3)(b/2) < 1$. Let $C = (1/4M)\sqrt[3]{b/2}$. Then, we obtain that $\int_{R^3} k(x)F(\psi)dx \ge bR_0/2 = bl^2/2$. Moreover, in view of the definition of μ^* and (18), one has

$$\begin{aligned}\mu^* &\le \int_{R^3}\left(a|\nabla\psi|^2 + |\psi|^2\right) dx \\ &\le \frac{32\pi}{3}R\left(C^2+R^2\right) < \frac{40\pi}{3M^3}\frac{b}{2} < l_0.\end{aligned} \tag{19}$$

So, condition (k2) holds. In particular, the condition (k1) and above examples can also be found in [34] in which the asymptotically linear term $k(x)f(u)$ satisfying (k1) appeared first.

Remark 3. If $h \equiv 0$, we know that problem (1) has a positive ground state solution by using the method in [32] and a trivial solution ($u(x) \equiv 0$). If $h \not\equiv 0$, a trivial solution ($u(x) \equiv 0$) is replaced by the local minimum solution by Theorem 1. Note that the local minimal solution exists due to the homogeneous term which is looked at as a small perturbation because $\|h\|_2 < m$ for small m.

In order to obtain our results, we have to overcome various difficulties. Since the embedding of $H^1(R^3)$ into $L^p(R^3)$, $p \in [2,6]$, is not compact, condition (k1) and (k2) are crucial to obtain the boundedness of Cerami sequence. Furthermore, in order to recover the compactness, we establish a compactness result $\int_{|x|\ge R}(|\nabla u_n|^2 + |u_n|^2)dx \le \varepsilon$ which is similar to [32] but different from the one in [24–26, 28–31]. In fact, this difficulty can be avoided, when problems are considered, restricting I to the subspace of $H^1(R^3)$ consisting of radially symmetric functions [23, 24, 29] and constraint potential functions [25, 30], or when one is looking for semiclassical states [28], by using perturbation methods or a reduction to a finite dimension by the projections method. Third, it is not difficult to find that every (PS) sequence is bounded because a variant of Ambrosetti-Rabinowitz condition is satisfied (see [23, 25, 31]). However, for the asymptotically linear case, we have to find another method to verify the boundedness of (PS) sequence.

This paper is organized as follows. In Section 2, we manage to give proofs of Theorem 1. In the following discussion, we denote various positive constants as C or C_i ($i = 1, 2, 3, \ldots$) for convenience.

2. Proof of Main Result

In this section, we prove that problem (1) has a mountain pass type solution and a local minimum solution with $h \not\equiv 0$. For this purpose, we use a variant version of Mountain Pass Theorem [35], which allows us to find a so-called Cerami type (PS) sequence (Cerami sequence, in short). The properties of this kind of Cerami sequence sequences are very helpful in showing its boundedness in the asymptotical cases. The following lemmas will show that I has the so-called mountain pass geometry.

Lemma 4. *Suppose that $h \in L^2(R^3)$, $h \ge 0$, (f1)–(f3), and (k1) hold. Then there exist $\rho, \alpha, m > 0$ such that $I(u)|_{\|u\|=\rho} \ge \alpha > 0$ for $\|h\|_2 < m$.*

Proof. For any $\varepsilon > 0$, it follows from (f1)–(f3) that there exists $C_\varepsilon > 0$ such that

$$\left|f(t)\right| \le \varepsilon|t| + C_\varepsilon|t|^5 \quad \forall t \in R. \tag{20}$$

Therefore, we have

$$|F(t)| \le \frac{1}{2}\varepsilon|t|^2 + \frac{C_\varepsilon}{6}|t|^6 \quad \forall t \in R. \tag{21}$$

Furthermore, by (f1)–(f3) and (k1), there exists $C_1 > 0$ such that

$$k(x) \le C_1 \quad \forall x \in R^3. \tag{22}$$

According to (21), (22), and the Sobolev inequality, we deduce that

$$\begin{aligned}\left|\int_{R^3} k(x) F(u)\, dx\right| &\le \frac{\varepsilon C_1}{2}\int_{R^3}|u|^2 dx + \frac{C_1 C_\varepsilon}{6}\int_{R^3}|u|^6 dx \\ &\le \frac{\varepsilon C_1}{2}\|u\|^2 + C_2\|u\|^6,\end{aligned} \tag{23}$$

where $C_2 = C_1 C_\varepsilon S^6/6$. By $b > 0$, $h \in L^2(R^3)$, and the Hölder inequality, one has

$$\begin{aligned} I(u) &= \frac{a}{2}\int_{R^3}|\nabla u|^2 dx + \frac{b}{4}\left(\int_{R^3}|\nabla u|^2 dx\right)^2 + \frac{1}{2}\int_{R^3}|u|^2 dx \\ &\quad - \int_{R^3} k(x) F(u)\, dx - \int_{R^3} h(x)\, u\, dx \\ &\geq \frac{a}{2}\int_{R^3}|\nabla u|^2 dx + \frac{1}{2}\int_{R^3}|u|^2 dx - \frac{\varepsilon C_1}{2}\|u\|^2 \\ &\quad - C_2\|u\|^6 - \|h\|_2\|u\|_2 \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon C_1}{2}\|u\|^2 - C_2\|u\|^6 - \|h\|_2\,\|u\| \\ &\geq \|u\|\left(\frac{1-\varepsilon C_1}{2}\,\|u\| - C_2\|u\|^5 - \|h\|_2\right). \end{aligned} \tag{24}$$

Taking $\varepsilon = 1/2C_1$ and setting $g(t) = (1/4)t - C_2 t^5$ for $t \geq 0$, we see there exists $\rho = (1/20C_2)^{1/4}$ such that $\max_{t\geq 0} g(t) = g(\rho) := m > 0$. Then it follows from (24) that there exists $\alpha > 0$ such that $I(u)|_{\|u\|=\rho} \geq \alpha > 0$ for $\|h\|_2 < m$. Of course, ρ, m can be chosen small enough; we can obtain the same result: there exists $\alpha > 0$ such that $I(u)|_{\|u\|=\rho} \geq \alpha > 0$ for $\|h\|_2 < m$. □

Lemma 5. *Suppose that $h \in L^2(R^3)$, $h \geq 0$, (f1)–(f3), and (k1)-(k2) hold. Then there exists $v \in H^1(R^3)$ with $\|v\| > \rho$, ρ is given by Lemma 4, such that $I(v) < 0$.*

Proof. By (k2) and $h \geq 0$, in view of the definition of μ^* and $l_0 > \mu^*$ with $l_0 = \min\{l, (b/2)l^2\}$, there is a nonnegative function $v \in H^1(R^3)$ such that

$$\int_{R^3} k(x) F(v)\, dx \geq \frac{b}{2} l^2, \qquad \int_{R^3} h(x)\, v\, dx \geq 0, \tag{25}$$

and $\mu^* \leq \|v\|^2 < l_0$. Then, we have

$$\begin{aligned} I(v) &= \frac{a}{2}\int_{R^3}|\nabla v|^2 dx + \frac{b}{4}\left(\int_{R^3}|\nabla v|^2 dx\right)^2 + \frac{1}{2}\int_{R^3}|v|^2 dx \\ &\quad - \int_{R^3} k(x) F(v)\, dx - \int_{R^3} h(x)\, v dx \\ &\leq \frac{1}{2}\|v\|^2 + \frac{b}{4}\|v\|^4 - \frac{b}{2} l^2 \\ &\leq \frac{1}{2}\|v\|^2 - \frac{b}{4} l^2 \\ &< 0. \end{aligned} \tag{26}$$

Choosing $\rho > 0$ small enough in Lemma 4 such that $\|v\| > \rho$, then this Lemma is proved.

From Lemmas 4 and 5 and Mountain Pass Lemma in [35], there is a Cerami sequence $\{u_n\} \subset H^1(R^3)$ such that

$$\begin{gathered} \left\|I'(u_n)\right\|_{H^{-1}}(1 + \|u_n\|) \longrightarrow 0, \\ I(u_n) \longrightarrow c \quad \text{as } n \longrightarrow \infty, \end{gathered} \tag{27}$$

where H^{-1} denotes the dual space of $H^1(R^3)$. In the following Lemmas 6 and 7, we shall prove that I satisfies the Cerami condition, that is; the Cerami sequence $\{u_n\}$ has a convergence subsequence. □

Lemma 6. *Suppose that $h \in L^2(R^3)$, $h \geq 0$, (f1)–(f3), and (k1) hold. Then $\{u_n\}$ defined in (27) is bounded in $H^1(R^3)$.*

Proof. By contradiction, let $\|u_n\| \to \infty$. Define $w_n = u_n\|u_n\|^{-1}$. Clearly, $\{w_n\}$ is bounded in $H^1(R^3)$ and there is a $w \in H^1(R^3)$ such that, up to a sequence,

$$\begin{gathered} w_n \longrightarrow w \quad \text{weakly in } H^1(R^3), \\ w_n \longrightarrow w \quad \text{a.e. in } R^3, \\ w_n \longrightarrow w \quad \text{strongly in } L^2_{loc}(R^3) \end{gathered} \tag{28}$$

as $n \to \infty$.

Firstly, we claim that w is nontrivial; that is, $w \not\equiv 0$. Otherwise, if $w \equiv 0$, the Sobolev embedding implies that $w_n \to 0$ strongly in $L^2(B_{R_0})$; R_0 is given by (k1). By (f1)–(f3), there exists $C_3 > 0$ such that

$$\frac{f(t)}{t} \leq C_3 \quad \forall t \in R. \tag{29}$$

Then, for all $n \in N$, we have

$$0 \leq \int_{|x|<R_0} k(x)\frac{f(u_n)}{u_n} w_n^2 dx \leq C_3\|k\|_\infty \int_{|x|<R_0} w_n^2 dx \longrightarrow 0. \tag{30}$$

This yields

$$\lim_{n\to\infty}\int_{|x|<R_0} k(x)\frac{f(u_n)}{u_n} w_n^2 dx = 0. \tag{31}$$

Furthermore, by (k1), there exists a constant $\theta \in (0,1)$ such that

$$\sup\left\{\frac{f(t)}{t} : t > 0\right\} \leq \theta \inf\left\{\frac{1}{k(x)} : |x| \geq R_0\right\}. \tag{32}$$

Then, for all $n \in N$, we have

$$\int_{|x|\geq R_0} k(x)\frac{f(u_n)}{u_n} w_n^2 dx \leq \theta \int_{|x|\geq R_0} w_n^2 dx \leq \theta\|w\|^2 = \theta < 1. \tag{33}$$

Combining (31) and (33), we obtain

$$\limsup_{n\to\infty}\int_{R^3} k(x)\frac{f(u_n)}{u_n} w_n^2 dx < 1. \tag{34}$$

By (27), we get

$$\begin{aligned} 0 &\leq \left|\left\langle I'(u_n), u_n\right\rangle\right| \leq \left\|I'(u_n)\right\|_{H^{-1}}\|u_n\| \\ &\leq \left\|I'(u_n)\right\|_{H^{-1}}(1 + \|u_n\|) \longrightarrow 0 \end{aligned} \tag{35}$$

as $n \to \infty$. Together with $\|u_n\| \to \infty$ as $n \to \infty$, it follows that

$$\frac{\langle I'(u_n), u_n\rangle}{\|u_n\|^2} = o(1). \tag{36}$$

Together with $b > 0$, we have

$$\begin{aligned} o(1) &= \frac{1}{\|u_n\|^2}\left(a\int_{R^3}|\nabla u_n|^2dx + \int_{R^3}|u_n|^2dx\right. \\ &\qquad + b\left(\int_{R^3}|\nabla u_n|^2dx\right)^2 \\ &\qquad \left. - \int_{R^3}k(x)f(u_n)u_ndx - \int_{R^3}h(x)u_ndx\right) \\ &\geq \|w_n\|^2 - \int_{R^3}k(x)\frac{f(u_n)}{u_n}w_n^2dx \\ &\geq 1 - \int_{R^3}k(x)\frac{f(u_n)}{u_n}w_n^2dx, \end{aligned} \tag{37}$$

where, and in what follows, $o(1)$ denotes a quantity which goes to zero as $n \to \infty$. Therefore, we deduce that

$$\int_{R^3}k(x)\frac{f(u_n)}{u_n}w_n^2dx + o(1) \geq 1, \tag{38}$$

which contradicts (34). So, $w \not\equiv 0$.

Furthermore, because $\|u_n\| \to \infty$ as $n \to \infty$, it follows from (35) that

$$\frac{\langle I'(u_n), u_n\rangle}{\|u_n\|^4} = o(1); \tag{39}$$

that is,

$$o(1) = \frac{b\left(\int_{R^3}|\nabla u_n|^2dx\right)^2}{\|u_n\|^4} - \frac{1}{\|u_n\|^2}\int_{R^3}k(x)\frac{f(u_n)}{u_n}w_n^2dx. \tag{40}$$

Together with (22), (29), and $b > 0$, one has

$$\begin{aligned} &\frac{\left(\int_{R^3}|\nabla u_n|^2dx\right)^2}{\|u_n\|^4} \\ &= \frac{\left(\int_{R^3}\left(|\nabla u_n|^2 + |u_n|^2\right)dx - \int_{R^3}|u_n|^2dx\right)^2}{\|u_n\|^4} = o(1). \end{aligned} \tag{41}$$

This yields

$$\begin{aligned} &\frac{\left(\int_{R^3}\left(|\nabla u_n|^2 + |u_n|^2\right)dx\right)^2}{\|u_n\|^4} \\ &\quad - \frac{2\int_{R^3}\left(|\nabla u_n|^2 + |u_n|^2\right)dx\int_{R^3}|u_n|^2dx}{\|u_n\|^4} \\ &\quad + \frac{\left(\int_{R^3}|u_n|^2dx\right)^2}{\|u_n\|^4} = o(1). \end{aligned} \tag{42}$$

This means

$$\begin{aligned} 1 - 2\int_{R^3}|w_n|^2dx + \left(\int_{R^3}|w_n|^2dx\right)^2 &= \left(1 - \int_{R^3}|w_n|^2dx\right)^2 \\ &= o(1). \end{aligned} \tag{43}$$

Therefore, we have

$$\int_{R^3}|w_n|^2dx \longrightarrow 1 \quad \text{as } n \longrightarrow \infty. \tag{44}$$

By $\|w_n\| = 1$, we get $\int_{R^3}|\nabla w_n|^2dx \to 0$ as $n \to \infty$; thus $w_n \to 0$ strongly in $D^{1,2}(R^3)$; therefore, $w_n \rightharpoonup 0$ weakly in $D^{1,2}(R^3)$. Since $w_n \rightharpoonup w$ weakly in $H^1(R^3)$; we have $w_n \rightharpoonup w$ weakly in $D^{1,2}(R^3)$. By the uniqueness of the weak limitation, we have $w = 0$ which contradicts $w \neq 0$. Therefore, the Cerami sequence $\{u_n\}$ is bounded in $H^1(R^3)$. □

Lemma 7. *Suppose that $h \in L^2(R^3)$, $h \geq 0$, (f1)–(f3), and (k1) hold. Then for any $\varepsilon > 0$, there exist $R(\varepsilon) > R_0$ and $n(\varepsilon) > 0$ such that $\{u_n\}$ defined in (27) satisfies $\int_{|x|\geq R}(|\nabla u_n|^2 + |u_n|^2)dx \leq \varepsilon$ for $n > n(\varepsilon)$ and $R \geq R(\varepsilon)$.*

Proof. Let $\xi_R : R^3 \to [0,1]$ be a smooth function such that

$$\xi_R(x) = \begin{cases} 0, & 0 \leq |x| \leq \dfrac{R}{2}, \\ 1, & |x| \geq R. \end{cases} \tag{45}$$

Moreover, there exists a constant C_4 independent of R such that

$$|\nabla\xi_R(x)| \leq \frac{C_4}{R} \quad \forall x \in R^3. \tag{46}$$

Then, for all $n \in N$ and $R \geq R_0$, by (45), (46), and the Hölder inequality, we have

$$\begin{aligned}\int_{R^3} |\nabla (u_n\xi_R)|^2 dx &\leq \int_{R^3} |u_n|^2 |\nabla \xi_R|^2 dx + \int_{R^3} |\nabla u_n|^2 |\xi_R|^2 dx \\ &\quad + 2\int_{R^3} |u_n| |\xi_R| |\nabla u_n| |\nabla \xi_R| dx \\ &\leq \int_{R/2<|x|<R} |\nabla u_n|^2 dx + \int_{|x|>R} |\nabla u_n|^2 dx \\ &\quad + \frac{C_4^2}{R^2}\int_{R^3} |u_n|^2 dx \\ &\quad + 2\left(\int_{R^3} |\nabla u_n|^2 |\xi_R^2| dx\right)^{1/2} \\ &\quad \times \left(\int_{R^3} |u_n|^2 |\nabla \xi_R|^2 dx\right)^{1/2} \\ &\leq \int_{R/2<|x|<R} |\nabla u_n|^2 dx + \int_{|x|>R} |\nabla u_n|^2 dx \\ &\quad + \frac{C_4^2}{R^2}\int_{R^3} |u_n|^2 dx \\ &\quad + 2\left(\int_{R/2<|x|<R} |\nabla u_n|^2 dx \right.\\ &\qquad \left. + \int_{|x|>R} |\nabla u_n|^2 dx\right)^{1/2} \\ &\quad \times \left(\frac{C_4^2}{R^2}\int_{R^3} |u_n|^2 dx\right)^{1/2} \\ &\leq \left(2 + \frac{C_4^2}{R^2} + \frac{2\sqrt{2}C_4}{R}\right)\|u_n\|^2 \\ &\leq \left(2 + \frac{C_4^2}{R_0^2} + \frac{2\sqrt{2}C_4}{R_0}\right)\|u_n\|^2.\end{aligned} \tag{47}$$

This implies that

$$\|u_n\xi_R\| \leq C_5 \|u_n\| \tag{48}$$

for all $n \in N$ and $R \geq R_0$, where $C_5 = \max\{(3 + (C_4^2/R_0^2) + 2\sqrt{2}C_4/R_0)^{1/2}, 1\}$. From Lemma 6, we know that $\{u_n\}$ is bounded in $H^1(R^3)$. Together with (27), we obtain that $I'(u_n) \to 0$ in $H^{-1}(R^3)$. Moreover, by (48), for $\varepsilon > 0$, there exists $n(\varepsilon) > 0$ such that

$$\langle I'(u_n), \xi_R u_n\rangle \leq C_5 \|I'(u_n)\|_{H^{-1}(R^3)} \|u_n\| \leq \frac{\varepsilon}{4} \tag{49}$$

for $n > n(\varepsilon)$ and $R > R_0$. Note that

$$\begin{aligned}\langle I'(u_n), \xi_R u_n\rangle &= \left(a + b\int_{R^3} |\nabla u_n|^2 dx\right) \\ &\quad \times \int_{R^3} |\nabla u_n|^2 \xi_R dx + \int_{R^3} |u_n|^2 \xi_R dx \\ &\quad + \left(a + b\int_{R^3} |\nabla u_n|^2 dx\right) \\ &\quad \times \int_{R^3} u_n \nabla u_n \cdot \nabla \xi_R dx \\ &\quad - \int_{R^3} k(x) f(u_n) u_n \xi_R dx \\ &\quad - \int_{R^3} h(x) u_n \xi_R dx \leq \frac{\varepsilon}{4}.\end{aligned} \tag{50}$$

This yields

$$\begin{aligned}&\left(a + b\int_{R^3} |\nabla u_n|^2 dx\right)\int_{R^3} |\nabla u_n|^2 \xi_R dx + \int_{R^3} |u_n|^2 \xi_R dx \\ &\quad + \left(a + b\int_{R^3} |\nabla u_n|^2 dx\right)\int_{R^3} u_n \nabla u_n \cdot \nabla \xi_R dx \\ &\quad \leq \int_{R^3} k(x) f(u_n) u_n \xi_R dx + \int_{R^3} h(x) u_n \xi_R dx + \frac{\varepsilon}{4}.\end{aligned} \tag{51}$$

By (32), we have

$$k(x) f(u_n) u_n \leq \theta u_n^2 \quad \text{for } \theta \in (0, \min\{1, a\}), \ |x| \geq R_0. \tag{52}$$

This yields

$$\int_{R^3} k(x) f(u_n) u_n \xi_R dx \leq \theta \int_{R^3} u_n^2 \xi_R dx \tag{53}$$

for all $n \in N$ and $|x| \geq R_0$. For any $\varepsilon > 0$, there exists $R(\varepsilon) \geq R_0$ such that

$$\frac{1}{R^2} \leq \frac{4\varepsilon^2}{C_4^2} \quad \forall R > R(\varepsilon). \tag{54}$$

Because $h \in L^2(R^3)$, $h \geq (\not\equiv)0$, there exists $\overline{\rho} = \overline{\rho}(\varepsilon)$ such that

$$\|h\|_{2, R^3 \setminus B_\rho(0)} < \varepsilon, \quad \forall \rho \geq \overline{\rho}. \tag{55}$$

By the Hölder inequality, (45), (55), and the boundedness of $\{u_n\}$ in $H^1(R^3)$, we have

$$\begin{aligned}\int_{R^3} h(x) u_n \xi_R dx &\leq \|h(x)\xi_R\|_2 \|u_n\|_2 \\ &\leq \|h(x)\|_{2, |x|>R/2} \|u_n\|_2 \leq \frac{\varepsilon}{4} \\ &\qquad \forall R > R(\varepsilon).\end{aligned} \tag{56}$$

By the Young inequality, (46), and (54), for all $n \in N$ and $R > R(\varepsilon)$, we obtain

$$\begin{aligned}\int_{R^3} |u_n \nabla u_n \cdot \nabla \xi_R|\, dx &\le \int_{R^3} \sqrt{2\varepsilon}\, |\nabla u_n| \frac{1}{\sqrt{2\varepsilon}} |u_n|\, |\nabla \xi_R|\, dx \\ &\le \varepsilon \int_{R^3} |\nabla u_n|^2 dx \\ &\quad + \frac{1}{4\varepsilon} \int_{|x|\le R} |u_n|^2 \frac{C_4^2}{R^2} dx \\ &\le \varepsilon \int_{R^3} |\nabla u_n|^2 dx \\ &\quad + \varepsilon \int_{|x|\le R} |u_n|^2 dx \le \varepsilon \max\{a, 1\} \|u_n\|^2. \end{aligned} \tag{57}$$

Combining $b > 0$, (51), (53), (56), (57), and the boundedness of $\{u_n\}$ in $H^1(R^3)$, there exists $C_6 > 0$ such that

$$\begin{aligned}\min\{1-\theta, 1\} &\int_{R^3} \left(a|\nabla u_n|^2 + |u_n|^2\right) \xi_R dx \\ &\le \frac{\varepsilon}{2} + \varepsilon \max\{a, 1\} \|u_n\|^2 \left(a + b\int_{R^3} |\nabla u_n|^2 dx\right) \\ &\le C_6 \varepsilon \quad \forall R > R(\varepsilon). \end{aligned} \tag{58}$$

Note that C_6 is independent of ε. So, for any $\varepsilon > 0$, we can choose $R(\varepsilon) > R_0$ and $n(\varepsilon) > 0$ such that $\int_{|x|\ge R}(|\nabla u_n|^2 + |u_n|^2)dx \le \varepsilon$ holds. □

Lemma 8. *Suppose that $h \in L^2(R^3)$, $h \ge 0$, (f1)–(f3), and (k1)-(k2) hold. Then the sequence $\{u_n\}$ in* (27) *has a convergent subsequence. Moreover, I possesses a nonzero critical point u in $H^1(R^3)$ and $I(u) > 0$.*

Proof. By Lemma 6, the sequence $\{u_n\}$ in (27) is bounded in $H^1(R^3)$. We may assume that up to a subsequence $u_n \rightharpoonup u$ weakly in $H^1(R^3)$ for some $u \in H^1(R^3)$. Now, we shall show that $\| u_n \| \to \| u \|$ as $n \to \infty$.

By (11), we have

$$\begin{aligned}\langle I'(u_n), u_n\rangle &= \left(a + b\int_{R^3} |\nabla u_n|^2 dx\right) \int_{R^3} |\nabla u_n|^2 dx \\ &\quad + \int_{R^3} u_n^2 dx - \int_{R^3} k(x) f(u_n) u_n \\ &\quad - \int_{R^3} h(x) u_n dx, \\ \langle I'(u_n), u\rangle &= \left(a + b\int_{R^3} |\nabla u_n|^2 dx\right) \int_{R^3} \nabla u_n \cdot \nabla u\, dx \\ &\quad + \int_{R^3} u_n u dx - \int_{R^3} k(x) f(u_n) u \\ &\quad - \int_{R^3} h(x) u dx. \end{aligned} \tag{59}$$

By (59), $b > 0$, and the boundedness of $\{u_n\}$ in $H^1(R^3)$, we easily get

$$\begin{aligned}\langle I'(u_n), u_n - u\rangle &= \left(a + b\int_{R^3} |\nabla u_n|^2 dx\right) \int_{R^3} |\nabla u_n|^2 dx \\ &\quad + \int_{R^3} u_n^2 dx - \int_{R^3} k(x) f(u_n) u_n dx \\ &\quad - \int_{R^3} h(x) u_n dx \\ &\quad - \left(a + b\int_{R^3} |\nabla u_n|^2 dx\right) \int_{R^3} \nabla u \cdot \nabla u_n dx \\ &\quad - \int_{R^3} u u_n dx + \int_{R^3} k(x) f(u_n) u\, dx \\ &\quad + \int_{R^3} h(x) u\, dx \\ &= \left(a + b\int_{R^3} |\nabla u_n|^2 dx\right) \\ &\quad \times \int_{R^3} |\nabla (u_n - u)|^2 dx \\ &\quad + \int_{R^3} |u_n - u|^2 dx \\ &\quad + \left(a + b\int_{R^3} |\nabla u_n|^2 dx\right) \\ &\quad \times \int_{R^3} \nabla u \nabla (u_n - u)\, dx \\ &\quad + \int_{R^3} u (u_n - u)\, dx \\ &\quad - \int_{R^3} k(x) f(u_n)(u_n - u)\, dx \\ &\quad - \int_{R^3} h(x)(u_n - u)\, dx \\ &\ge a\int_{R^3} \nabla|(u_n - u)|^2 dx + \int_{R^3} |u_n - u|^2 dx \\ &\quad + \left(a + b\int_{R^3} |\nabla u_n|^2 dx\right) \\ &\quad \times \int_{R^3} \nabla u \nabla (u_n - u)\, dx \\ &\quad + \int_{R^3} u (u_n - u)\, dx \\ &\quad - \int_{R^3} k(x) f(u_n)(u_n - u)\, dx \\ &\quad - \int_{R^3} h(x)(u_n - u)\, dx \end{aligned}$$

$$\begin{aligned}&\geq \|u_n-u\|^2+\left(a+b\int_{R^3}|\nabla u_n|^2dx\right)\\&\quad\times\int_{R^3}\nabla u\nabla(u_n-u)\,dx\\&\quad+\int_{R^3}u(u_n-u)\,dx\\&\quad-\int_{R^3}k(x)f(u_n)(u_n-u)\,dx\\&\quad-\int_{R^3}h(x)(u_n-u)\,dx.\end{aligned}\tag{60}$$

One has

$$\begin{aligned}\|u_n-u\|^2\leq{}&\left\langle I'(u_n),u_n-u\right\rangle\\&-\left(a+b\int_{R^3}|\nabla u_n|^2dx\right)\int_{R^3}\nabla u\nabla(u_n-u)\,dx\\&-\int_{R^3}u(u_n-u)\,dx\\&+\int_{R^3}k(x)f(u_n)(u_n-u)\,dx\\&+\int_{R^3}h(x)(u_n-u)\,dx.\end{aligned}\tag{61}$$

It is clear that

$$\langle I'(u_n),u_n-u\rangle\longrightarrow 0,\quad\text{as } n\longrightarrow\infty.\tag{62}$$

Since $u_n\rightharpoonup u$ weakly in $H^1(R^3)$, we obtain

$$\begin{aligned}\int_{R^3}(\nabla u_n\cdot\nabla u+u_nu)\,dx={}&\int_{R^3}\left(|\nabla u|^2+|u|^2\right)dx\\&+o(1),\quad\text{as } n\longrightarrow\infty.\end{aligned}\tag{63}$$

By the continuity of imbedding $H^1(R^3)\hookrightarrow L^2(R^3)$, we have that $u_n\rightharpoonup u$ weakly in $L^2(R^3)$; that is,

$$\int_{R^3}u_nu\,dx=\int_{R^3}u^2dx+o(1),\quad\text{as } n\longrightarrow\infty.\tag{64}$$

By (63) and (64), we deduce

$$\int_{R^3}\nabla u_n\cdot\nabla u\,dx=\int_{R^3}|\nabla u|^2dx+o(1),\quad\text{as } n\longrightarrow\infty.\tag{65}$$

Combining the boundedness of $\{u_n\}$ in $H^1(R^3)$, (64), and (65), we obtain

$$\begin{aligned}&\left(a+b\int_{R^3}|\nabla u_n|^2dx\right)\int_{R^3}\nabla u\nabla(u_n-u)\,dx\\&\quad+\int_{R^3}u(u_n-u)\,dx=o(1),\quad\text{as } n\longrightarrow\infty.\end{aligned}\tag{66}$$

Moreover, by (32), Lemma 7, and $u_n\rightharpoonup u$ in $H^1(R^3)$, for any $\varepsilon>0$ and n large enough, one has

$$\begin{aligned}&\int_{|x|\geq R(\varepsilon)}k(x)f(u_n)u_ndx-\int_{|x|\geq R(\varepsilon)}k(x)f(u_n)u\,dx\\&\quad=\int_{|x|\geq R(\varepsilon)}k(x)f(u_n)(u_n-u)\,dx\\&\quad\leq\int_{|x|\geq R(\varepsilon)}|k(x)f(u_n)||u_n-u|\,dx\\&\quad\leq\left(\int_{|x|\geq R(\varepsilon)}\left|k^2(x)f^2(u_n)\right|dx\right)^{1/2}\\&\qquad\times\left(\int_{|x|\geq R(\varepsilon)}|u_n-u|^2dx\right)^{1/2}\\&\quad\leq\theta\left(\int_{|x|\geq R(\varepsilon)}\left|u_n^2\right|dx\right)^{1/2}\left(\int_{|x|\geq R(\varepsilon)}|u_n-u|^2dx\right)^{1/2}\\&\quad\leq\theta\left(\int_{|x|\geq R(\varepsilon)}\left(a|\nabla u_n|^2+|u_n|^2\right)dx\right)^{1/2}\\&\qquad\times\left(\int_{|x|\geq R(\varepsilon)}|u_n-u|^2dx\right)^{1/2}\leq\theta\varepsilon.\end{aligned}\tag{67}$$

This and the compactness of embedding $H^1(R^3)\hookrightarrow L^2_{\text{loc}}(R^3)$ imply that

$$\int_{R^3}k(x)f(u_n)u_ndx=\int_{R^3}k(x)f(u_n)u\,dx+o(1).\tag{68}$$

Since u_n is bounded in $H^1(R^3)$ and the continuity of the Sobolev embedding of $H^1(R^3)$ imbedding in $L^2(R^3)$, for any choice of $\varepsilon>0$ and $\rho>0$, the relation

$$\|u_n-u\|_{2,B_\rho(0)}<\varepsilon\tag{69}$$

holds for large n. By $h\in L^2(R^3)$, for any $\varepsilon>0$ there exists $\overline{\rho}=\overline{\rho}(\varepsilon)$ such that

$$\|h\|_{2,R^3\backslash B_\rho(0)}<\varepsilon,\quad\forall\rho\geq\overline{\rho}.\tag{70}$$

By (70) and (69), we have

$$\begin{aligned}&\int_{R^3}h(x)u_ndx-\int_{R^3}h(x)u\,dx\\&\quad\leq\int_{R^3\backslash B_\rho(0)}|h(x)(u_n-u)|\,dx\\&\qquad+\int_{B_\rho(0)}|h(x)(u_n-u)|\,dx\\&\quad\leq\|h(x)\|_{2,R^3\backslash B_\rho(0)}\|u_n-u\|_{2,R^3\backslash B_\rho(0)}\\&\qquad+\|h(x)\|_{2,B_\rho(0)}\|u_n-u\|_{2,B_\rho(0)}\\&\quad\leq\varepsilon\|u_n-u\|_{2,R^3\backslash B_\rho(0)}+\varepsilon\|h(x)\|_{2,B_\rho(0)}\\&\quad\leq C_7\varepsilon.\end{aligned}\tag{71}$$

This yields

$$\int_{R^3} h(x) u_n dx = \int_{R^3} h(x) u \, dx + o(1). \quad (72)$$

By (61), (62), (66), (68), and (72), we have

$$\|u_n - u\|^2 = o(1), \quad \text{as } n \longrightarrow \infty, \quad (73)$$

by $a > 0$. This yields that $\|u_n\| \to \|u\|$ as $n \to \infty$ and u is a nonzero critical point of I in $H^1(R^3)$ and $I(u) > 0$ by Mountain Pass Theorem in [35]. □

Now, we give local properties of the variational functional I, which is required by using Ekeland's variational principle.

Lemma 9. *Suppose that $h \in L^2(R^3)$, $h \geq (\not\equiv)0$, (f1)–(f3), and (k1) hold. If $\|h\|_2 < m$, then there exists $u_0 \in H^1(R^3)$ such that*

$$I(u_0) = \inf\left\{I(u) : u \in \overline{B}_\rho\right\} < 0,$$
$$\text{where } B_\rho = \left\{u \in H^1\left(R^3\right) : \|u\| < \rho\right\}, \quad (74)$$

m, ρ are given by Lemma 4 and u_0 is a positive solution of system (1).

Proof. Because $h \in L^2(R^3)$, $h \geq (\not\equiv)0$, we can choose a function $\varphi \in H^1(R^3)$ such that

$$\int_{R^3} h(x) \varphi \, dx > 0. \quad (75)$$

Together with (f1), (k1), and (75), for $t > 0$, we have

$$\begin{aligned} I(t\varphi) = {} & \frac{at^2}{2} \int_{R^3} |\nabla\varphi|^2 dx + \frac{bt^4}{4}\left(\int_{R^3} |\nabla\varphi|^2 dx\right)^2 \\ & + \frac{t^2}{2} \int_{R^3} |\varphi|^2 dx \\ & - \int_{R^3} k(x) F(t\varphi) dx - \int_{R^3} h(x) t\varphi \, dx \\ \leq {} & \frac{t^2}{2}\|\varphi\|^2 + \frac{bt^4}{4}\|\varphi\|^4 - t\int_{R^3} h(x)\varphi \, dx \\ \leq {} & 0 \end{aligned} \quad (76)$$

for $t > 0$ small enough. Thus there exists u small enough such that $I(u) < 0$. By Lemma 4, we deduce that

$$c_0 := \inf_{u \in \overline{B}_\rho} I(u) < 0 < \inf_{u \in \partial \overline{B}_\rho} I(u). \quad (77)$$

By applying Ekeland's variational principle [36, Theorem 4.1] in $\overline{B}_\rho$, there is a minimizing sequence $\{u_n\} \subset \overline{B}_\rho$ such that

$$\begin{aligned} &\text{(i)} \ \ c_0 \leq I(u_n) < c_0 + \frac{1}{n}, \\ &\text{(ii)} \ \ I(w) \geq I(u_n) - \frac{1}{n}\|w - u_n\| \quad \forall w \in \overline{B}_\rho. \end{aligned} \quad (78)$$

Then, by a standard procedure, we can show that $\{u_n\}$ is a bounded (PS) sequence of I. Lemmas 7 and 8 imply that there exists $u_0 \in H^1(R^3)$ such that $I'(u_0) = 0$ and $I(u_0) = c_0 < 0$. So this lemma is proved. □

Proof of Theorem 1.2. By Lemmas 4–8, we obtain the existence of a mountain pass solution u for problem (1) and $I(u) > 0$. By Lemma 9, we know that problem (1) has a local minimum solution u_0 and $I(u_0) < 0$. Thus, $u \neq u_0$ and u, u_0 are positive. Thus this theorem is proved. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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