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Research Article

Least-Squares Solutions of the Matrix Equations $AXB + CYD = H$ and $AXB + CXD = H$ for Symmetric Arrowhead Matrices and Associated Approximation Problems

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The least-squares solutions of the matrix equations $AXB + CYD = H$ and $AXB + CXD = H$ for symmetric arrowhead matrices are discussed. By using the Kronecker product and stretching function of matrices, the explicit representations of the general solution are given. Also, it is shown that the best approximation solution is unique and an explicit expression of the solution is derived.

1. Introduction

An $n \times n$ matrix A is called an arrowhead matrix if it has the following form:

$$A = \begin{bmatrix} a_1 & b_1 & b_2 & \cdots & b_{n-1} \\ c_1 & a_2 & 0 & \cdots & 0 \\ c_2 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & 0 & 0 & \cdots & a_n \end{bmatrix}. \quad (1)$$

If $b_i = c_i$, $i = 1, \ldots, n-1$, then A is said to be a symmetric arrowhead matrix. We denote all real-valued symmetric $n \times n$ arrowhead matrices by $\mathbf{SAR}^{n\times n}$. Such matrices arise in the description of radiationless transitions in isolated molecules [1], oscillators vibrationally coupled with a Fermi liquid [2] and quantum optics [3], and so forth. Numerically efficient algorithms for computing eigenvalues and eigenvectors of arrowhead matrices were discussed in [4–8]. The inverse problem of constructing the symmetric arrowhead matrix from spectral data has been investigated by Xu [9], Peng et al. [10], and Borges et al. [11]. In this paper, we will further consider the least-squares solutions of the matrix equations for symmetric arrowhead matrices and associated approximation problems, which can be described as follows.

Problem 1. Given $A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{n\times p}$, $C \in \mathbf{R}^{m\times q}$, $D \in \mathbf{R}^{q\times p}$, and $H \in \mathbf{R}^{m\times p}$, find nontrivial real-valued symmetric arrowhead matrices $X \in \mathbf{R}^{n\times n}$ and $Y \in \mathbf{R}^{q\times q}$ such that

$$\|AXB + CYD - H\| = \min. \quad (2)$$

Problem 2. Given real-valued symmetric arrowhead matrices $\widetilde{X} \in \mathbf{R}^{n\times n}$, $\widetilde{Y} \in \mathbf{R}^{q\times q}$, find $(\widehat{X}, \widehat{Y}) \in \mathcal{S}_1$ such that

$$\left\|\widehat{X} - \widetilde{X}\right\|^2 + \left\|\widehat{Y} - \widetilde{Y}\right\|^2 = \min_{(X,Y)\in\mathcal{S}_1} \left(\left\|X - \widetilde{X}\right\|^2 + \left\|Y - \widetilde{Y}\right\|^2\right), \quad (3)$$

where $\mathcal{S}_1$ is the solution set of Problem 1.

Problem 3. Given $A, C \in \mathbf{R}^{m\times n}$, $B, D \in \mathbf{R}^{n\times p}$, and $H \in \mathbf{R}^{m\times p}$, find nontrivial real-valued symmetric arrowhead matrix X such that

$$\|AXB + CXD - H\| = \min. \quad (4)$$

Problem 4. Given a real-valued symmetric arrowhead matrix $\widetilde{X} \in \mathbf{R}^{n\times n}$, find $\widehat{X} \in \mathcal{S}_3$ such that

$$\left\|\widehat{X} - \widetilde{X}\right\| = \min_{X\in\mathcal{S}_3} \left\|X - \widetilde{X}\right\|, \tag{5}$$

where $\mathcal{S}_3$ is the solution set of Problem 3.

Recently, Li et al. [12] considered the least-squares solutions of the matrix equation $AXB + CYD = E$ for symmetric arrowhead matrices. By using Moore-Penrose inverses and the Kronecker product, the minimum-norm and least-squares solution to the matrix equation for symmetric arrowhead matrices was provided. However, we can easily see that the method used in [12] involves complicated computations for Moore-Penrose generalized inverses of partitioned matrices, and the expression of the minimum-norm and least-squares solution was not explicit. Compared with the approach proposed in [12], the method in this paper is more concise and easy to perform.

The paper is organized as follows. In Section 2, using the Kronecker product and stretching function $\operatorname{vec}(\cdot)$ of matrices, we give an explicit representation of the solution set $\mathcal{S}_1$ of Problem 1. Furthermore, we show that there exists a unique solution in Problem 2 and present the expression of the unique solution $(\widehat{X}, \widehat{Y})$ of Problem 2. In Section 3, we provide an explicit representation of the solution set $\mathcal{S}_3$ of Problem 3 and present the expression of the unique solution $\widehat{X}$ of Problem 4. In Section 4, a numerical algorithm to acquire the optimal approximation solution for Problem 2 under the Frobenius norm sense is described and a numerical example is provided. Some concluding remarks are given in Section 5.

Throughout this paper, we denote the real $m \times n$ matrix space by $\mathbf{R}^{m\times n}$ and the transpose and the Moore-Penrose generalized inverse of a real matrix A by $A^\top$ and A^+, respectively. I_n represents the identity matrix of size n. For $A, B \in \mathbf{R}^{m\times n}$, an inner product in $\mathbf{R}^{m\times n}$ is defined by $(A, B) = \operatorname{trace}(B^\top A)$; then $\mathbf{R}^{m\times n}$ is a Hilbert space. The matrix norm $\|\cdot\|$ induced by the inner product is the Frobenius norm. Given two matrices $A = [a_{ij}] \in \mathbf{R}^{m\times n}$ and $B = [b_{ij}] \in \mathbf{R}^{p\times q}$, the Kronecker product of A and B is defined by $A \otimes B = [a_{ij}B] \in \mathbf{R}^{mp\times nq}$. Also, for an $m \times n$ matrix $A = [a_1, a_2, \dots, a_n]$, where a_i, $i = 1, \dots, n$, is the ith column vector of A, the stretching function $\operatorname{vec}(A)$ is defined by $\operatorname{vec}(A) = [a_1^\top, a_2^\top, \dots, a_n^\top]^\top$.

2. The Solutions of Problems 1 and 2

To begin with, we introduce two lemmas.

Lemma 5 (see [13]). *If $L \in \mathbf{R}^{m\times q}$, $b \in \mathbf{R}^m$, then the general solution of $\|Ly - b\| = \min$ can be expressed as $y = L^+b + (I_q - L^+L)z$, where $z \in \mathbf{R}^q$ is an arbitrary vector.*

Lemma 6 (see [14]). *Let $D \in \mathbf{R}^{m\times n}$, $H \in \mathbf{R}^{n\times l}$, $J \in \mathbf{R}^{l\times s}$. Then*

$$\operatorname{vec}(DHJ) = \left(J^\top \otimes D\right)\operatorname{vec}(H). \tag{6}$$

Let $d_1 = 2n - 1$ and $d_2 = 2q - 1$. It is easily seen that $\dim(\mathbf{SAR}^{n\times n}) = d_1$ and $\dim(\mathbf{SAR}^{q\times q}) = d_2$. Define

$$Z_{ij} = \begin{cases} \dfrac{\sqrt{2}}{2}\left(e_i^{(n)}\left(e_j^{(n)}\right)^\top + e_j^{(n)}\left(e_i^{(n)}\right)^\top\right), & i = 1;\ j = 2, \dots, n, \\ e_i^{(n)}\left(e_i^{(n)}\right)^\top, & i = j = 1, \dots, n, \end{cases} \tag{7}$$

$$W_{kl} = \begin{cases} \dfrac{\sqrt{2}}{2}\left(e_k^{(q)}\left(e_l^{(q)}\right)^\top + e_l^{(p)}\left(e_k^{(q)}\right)^\top\right), & k = 1;\ l = 2, \dots, q, \\ e_k^{(q)}\left(e_k^{(q)}\right)^\top, & k = l = 1, \dots, q, \end{cases} \tag{8}$$

where $e_i^{(n)}$ is the ith column vector of the identity matrix I_n. It is easy to verify that $\{Z_{ij}\}$ and $\{W_{kl}\}$ form orthonormal bases of the subspaces $\mathbf{SAR}^{n\times n}$ and $\mathbf{SAR}^{q\times q}$, respectively. That is,

$$\left(Z_{ij}, Z_{kl}\right) = \begin{cases} 0, & i \neq k \text{ or } j \neq l, \\ 1, & i = k,\ j = l, \end{cases}$$
$$\left(W_{ij}, W_{kl}\right) = \begin{cases} 0, & i \neq k \text{ or } j \neq l, \\ 1, & i = k,\ j = l. \end{cases} \tag{9}$$

Now, if $X \in \mathbf{SAR}^{n\times n}$ and $Y \in \mathbf{SAR}^{q\times q}$, then X and Y can be expressed as

$$X = \sum_{i,j}\alpha_{ij}Z_{ij}, \qquad Y = \sum_{k,l}\beta_{kl}W_{kl}, \tag{10}$$

where the real numbers α_{ij}, $i = 1$, $j = 2, \dots, n$; $i = j = 1, \dots, n$, and β_{kl}, $k = 1$, $l = 2, \dots, q$; $k = l = 1, \dots, q$, are yet to be determined.

It follows from (10) that the relation of (2) can be equivalently written as

$$\left\|\sum_{i,j}\alpha_{ij}AZ_{ij}B + \sum_{k,l}\beta_{kl}CW_{kl}D - H\right\| = \min. \tag{11}$$

When setting

$$\alpha = \left[\alpha_{11}, \dots, \alpha_{n,n}, \alpha_{12}, \dots, \alpha_{1,n}\right]^\top,$$
$$\beta = \left[\beta_{11}, \dots, \beta_{q,q}, \beta_{12}, \dots, \beta_{1,q}\right]^\top, \tag{12}$$

$$G = \left[\operatorname{vec}\left(Z_{11}\right), \dots, \operatorname{vec}\left(Z_{n,n}\right), \operatorname{vec}\left(Z_{12}\right), \dots, \operatorname{vec}\left(Z_{1,n}\right)\right] \in \mathbf{R}^{n^2\times d_1} \tag{13}$$

$$L = \left[\operatorname{vec}\left(W_{11}\right), \dots, \operatorname{vec}\left(W_{q,q}\right), \operatorname{vec}\left(W_{12}\right), \dots, \operatorname{vec}\left(W_{1,q}\right)\right] \in \mathbf{R}^{q^2\times d_2}, \tag{14}$$

$$M = \left(B^\top \otimes A\right)G, \qquad N = \left(D^\top \otimes C\right)L, \qquad h = \operatorname{Vec}(H). \tag{15}$$

By Lemma 6, we see that the relation of (11) is equivalent to

$$\|M\alpha + N\beta - h\| = \min. \quad (16)$$

We note that

$$\begin{aligned} &\|M\alpha + N\beta - h\|^2 \\ &\quad = \|M[\alpha + M^{+}(N\beta - h)] + E_M(N\beta - h)\|^2 \\ &\quad = \|M[\alpha + M^{+}(N\beta - h)]\|^2 + \|E_M(N\beta - h)\|^2 \\ &\quad = \|M[\alpha + M^{+}(N\beta - h)]\|^2 \\ &\qquad + \Big\|E_M N\left[\beta - (E_M N)^{+} E_M h\right] \\ &\qquad\quad - (I_{mp} - E_M N(E_M N)^{+}) E_M h\Big\|^2 \\ &\quad = \|M[\alpha + M^{+}(N\beta - h)]\|^2 + \|E_M N[\beta - (E_M N)^{+} E_M h]\|^2 \\ &\qquad + \|(I_{mp} - E_M N(E_M N)^{+}) E_M h\|^2, \end{aligned} \quad (17)$$

where $E_M = I_{mp} - MM^{+}$. It follows from Lemma 5 and (17) that $\|M\alpha + N\beta - h\| = \min$ if and only if

$$\alpha = -M^{+}N\beta + M^{+}h + F_M v, \quad (18)$$

$$\beta = (E_M N)^{+} E_M h + Wu, \quad (19)$$

where $F_M = I_{d_1} - M^{+}M$, $W = I_{d_2} - (E_M N)^{+} E_M N$, and $u \in \mathbf{R}^{d_2}$, $v \in \mathbf{R}^{d_1}$ are arbitrary vectors.

Substituting (19) into (18), we obtain

$$\alpha = \widetilde{\alpha} - M^{+}NWu + F_M v, \quad (20)$$

where $\widetilde{\alpha} = M^{+}h - M^{+}N(E_M N)^{+} E_M h$.

In summary of the above discussion, we have proved the following result.

Theorem 7. *Suppose that* $A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{n\times p}$, $C \in \mathbf{R}^{m\times q}$, $D \in \mathbf{R}^{q\times p}$, *and* $H \in \mathbf{R}^{m\times p}$. *Let* $\{Z_{ij}\}, \{W_{kl}\}, G, L, M, N, h$ *be given as in* (7), (8), (13), (14), *and* (15), *respectively. Write* $d_1 = 2n - 1$, $d_2 = 2q - 1$, $E_M = I_{mp} - MM^{+}$, $F_M = I_{d_1} - M^{+}M$, $W = I_{d_2} - (E_M N)^{+} E_M N$, *and* $\widetilde{\alpha} = M^{+}h - M^{+}N(E_M N)^{+} E_M h$. *Then the solution set* $\mathcal{S}_1$ *of Problem 1 can be expressed as*

$$\begin{aligned} \mathcal{S}_1 = \Big\{ &(X, Y) \in \mathbf{SAR}^{n\times n} \times \mathbf{SAR}^{q\times q} \mid \\ &X = K_1 (\alpha \otimes I_n),\ Y = K_2 (\beta \otimes I_q) \Big\}, \end{aligned} \quad (21)$$

where

$$K_1 = [Z_{11}, \ldots, Z_{n,n}, Z_{12}, \ldots, Z_{1,n}] \in \mathbf{R}^{n\times nd_1}, \quad (22)$$

$$K_2 = [W_{11}, \ldots, W_{q,q}, W_{12}, \ldots, W_{1,q}] \in \mathbf{R}^{q\times qd_2}, \quad (23)$$

α, β *are, respectively, given by* (20) *and* (19) *with* $u \in \mathbf{R}^{d_2}$, $v \in \mathbf{R}^{d_1}$ *being arbitrary vectors.*

From (17), we can easily obtain the following corollary.

Corollary 8. *Under the same assumptions as in Theorem 7, the matrix equation*

$$AXB + CYD = H \quad (24)$$

has a solution if and only if

$$E_M N(E_M N)^{+} E_M N = E_M h. \quad (25)$$

In this case, the solution set $\mathcal{S}_1$ *of* (24) *is given by* (21).

It follows from Theorem 7 that the solution set $\mathcal{S}_1$ is always nonempty. It is easy to verify that $\mathcal{S}_1$ is a closed convex subset of $\mathbf{SAR}^{n\times n} \times \mathbf{SAR}^{q\times q}$. From the best approximation theorem [15], we know there exists a unique solution $(\widehat{X}, \widehat{Y})$ in $\mathcal{S}_1$ such that (3) holds.

We now focus our attention on seeking the unique solution $(\widehat{X}, \widehat{Y})$ in $\mathcal{S}_1$. For the real-valued symmetric arrowhead matrices $\widetilde{X}$ and $\widetilde{Y}$, it is easily seen that $\widetilde{X}, \widetilde{Y}$ can be expressed as the linear combinations of the orthonormal bases $\{Z_{ij}\}$ and $\{W_{ij}\}$; that is,

$$\widetilde{X} = \sum_{i,j} \gamma_{ij} Z_{ij}, \qquad \widetilde{Y} = \sum_{k,l} \delta_{kl} W_{kl}, \quad (26)$$

where γ_{ij}, $i = 1$, $j = 2, \ldots, n$; $i = j = 1, \ldots, n$, and δ_{kl}, $k = 1$, $l = 2, \ldots, q$; $k = l = 1, \ldots, q$, are uniquely determined by the elements of $\widetilde{X}$ and $\widetilde{Y}$. Let

$$\begin{aligned} \gamma &= [\gamma_{11}, \ldots, \gamma_{n,n}, \gamma_{12}, \ldots, \gamma_{1,n}]^{\top}, \\ \delta &= [\delta_{11}, \ldots, \delta_{q,q}, \delta_{12}, \ldots, \delta_{1,q}]^{\top}. \end{aligned} \quad (27)$$

Then, for any pair of matrices $(X, Y) \in \mathcal{S}_1$ in (21), by the relations of (9) and (26), we see that

$$\begin{aligned} f &= \|X - \widetilde{X}\|^2 + \|Y - \widetilde{Y}\|^2 \\ &= \Big\|\sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Z_{ij}\Big\|^2 + \Big\|\sum_{k,l} (\beta_{kl} - \delta_{kl}) W_{kl}\Big\|^2 \\ &= \left(\sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Z_{ij}, \sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Z_{ij}\right) \\ &\quad + \left(\sum_{k,l} (\beta_{kl} - \delta_{kl}) W_{kl}, \sum_{k,l} (\beta_{kl} - \delta_{kl}) W_{kl}\right) \\ &= \sum_{i,j} (\alpha_{ij} - \gamma_{ij}) \left(Z_{ij}, \sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Z_{ij}\right) \\ &\quad + \sum_{k,l} (\beta_{kl} - \delta_{kl}) \left(W_{kl}, \sum_{k,l} (\beta_{kl} - \delta_{kl}) W_{kl}\right) \\ &= \sum_{i,j} (\alpha_{ij} - \gamma_{ij})^2 + \sum_{k,l} (\beta_{kl} - \delta_{kl})^2 \\ &= \|\alpha - \gamma\|^2 + \|\beta - \delta\|^2. \end{aligned} \quad (28)$$

Substituting (19) and (20) into the function of f, we have

$$\begin{aligned} f &= \|\tilde{\alpha} - M^{+}NWu + F_M v - \gamma\|^2 \\ &\quad + \left\|(E_M N)^{+} E_M h + Wu - \delta\right\|^2 \\ &= u^\top WN^\top (MM^\top)^{+} NWu + 2(\gamma - \tilde{\alpha})^\top M^{+} NWu \\ &\quad - 2(\gamma - \tilde{\alpha})^\top F_M v + v^\top F_M v + (\gamma - \tilde{\alpha})^\top (\gamma - \tilde{\alpha}) + u^\top Wu \\ &\quad - 2\left(\delta - (E_M N)^{+} E_M h\right)^\top Wu \\ &\quad + \left(\delta - (E_M N)^{+} E_M h\right)^\top \left(\delta - (E_M N)^{+} E_M h\right). \end{aligned} \tag{29}$$

Therefore,

$$\begin{aligned} \frac{\partial f}{\partial u} &= 2WN^\top (MM^\top)^{+} NWu + 2WN^\top (M^{+})^\top (\gamma - \tilde{\alpha}) \\ &\quad + 2Wu - 2W\left(\delta - (E_M N)^{+} E_M h\right), \end{aligned} \tag{30}$$

$$\frac{\partial f}{\partial v} = 2F_M v - 2F_M (\gamma - \tilde{\alpha}).$$

Clearly, $\|X - \widetilde{X}\|^2 + \|Y - \widetilde{Y}\|^2 = \min$ if and only if

$$\frac{\partial f}{\partial u} = 0, \qquad \frac{\partial f}{\partial v} = 0 \tag{31}$$

which yields

$$\begin{aligned} Wu &= \left(I_{d_2} + WN^\top (MM^\top)^{+} NW\right)^{-1} \\ &\quad \times W\left(\delta - (E_M N)^{+} E_M h - N^\top (M^{+})^\top (\gamma - \tilde{\alpha})\right), \\ F_M v &= F_M (\gamma - \tilde{\alpha}). \end{aligned} \tag{32}$$

Upon substituting (32) into (19) and (20), we obtain

$$\begin{aligned} \hat{\alpha} &= -M^{+}NW\left(I_{d_2} + WN^\top (MM^\top)^{+} NW\right)^{-1} \\ &\quad \times W\left(\delta - (E_M N)^{+} E_M h - N^\top (M^{+})^\top (\gamma - \tilde{\alpha})\right) \\ &\quad + \tilde{\alpha} + F_M \gamma, \end{aligned} \tag{33}$$

$$\begin{aligned} \hat{\beta} &= (E_M N)^{+} E_M h + \left(I_{d_2} + WN^\top (MM^\top)^{+} NW\right)^{-1} \\ &\quad \times W\left(\delta - (E_M N)^{+} E_M h - N^\top (M^{+})^\top (\gamma - \tilde{\alpha})\right). \end{aligned} \tag{34}$$

By now, we have proved the following result.

Theorem 9. *Let the real-valued symmetric arrowhead matrices $\widetilde{X}$ and $\widetilde{Y}$ be given. Then Problem 2 has a unique solution and the unique solution of Problem 2 can be expressed as*

$$\begin{aligned} \widehat{X} &= K_1 (\hat{\alpha} \otimes I_n), \\ \widehat{Y} &= K_2 (\hat{\beta} \otimes I_q), \end{aligned} \tag{35}$$

where $\hat{\alpha}, \hat{\beta}$ are given by (33) *and* (34), *respectively.*

3. The Solutions of Problems 3 and 4

It follows from (10) that the minimization problem of (4) can be equivalently written as

$$\left\|\sum_{i,j} \alpha_{ij} A Z_{ij} B + \sum_{i,j} \alpha_{ij} C Z_{ij} D - H\right\| = \min. \tag{36}$$

Using Lemma 6, we see that the relation of (36) is equivalent to

$$\|M\alpha + Q\alpha - h\| = \min, \tag{37}$$

where $Q = (D^\top \otimes C)G$. It follows from Lemma 5 that the general solution of $\|M\alpha + Q\alpha - h\| = \min$ with respect to α can be expressed as

$$\alpha = U^{+}h + \left(I_{d_1} - U^{+}U\right) z, \tag{38}$$

where $U = M + Q$ and $z \in \mathbf{R}^{d_1}$ is an arbitrary vector.

To summarize, we have obtained the following result.

Theorem 10. *Suppose that $A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{n\times p}$, $C \in \mathbf{R}^{m\times q}$, $D \in \mathbf{R}^{q\times p}$, and $H \in \mathbf{R}^{m\times p}$. Let $\{Z_{ij}\}, G, M, h$ be given as in* (7), (13), *and* (15), *respectively. Write $Q = (D^\top \otimes C)G$, $d_1 = 2n - 1$, and $U = M + Q$. Then the solution set $\mathcal{S}_3$ of Problem 3 can be expressed as*

$$\mathcal{S}_3 = \left\{X \in \mathbf{SAR}^{n\times n} \mid X = K_1 (\alpha \otimes I_n)\right\}, \tag{39}$$

where K_1 and α are given by (22) *and* (38) *with $z \in \mathbf{R}^{d_1}$ being arbitrary vectors.*

Similarly, for the real-valued symmetric arrowhead matrix $\widetilde{X}$, it is easily seen that $\widetilde{X}$ can be expressed as the linear combination of the orthonormal basis $\{Z_{ij}\}$; that is, $\widetilde{X} = \sum_{i,j} \gamma_{ij} Z_{ij}$, where γ_{ij}, $i = 1$, $j = 2, \ldots, n$; $i = j = 1, \ldots, n$, are uniquely determined by the elements of $\widetilde{X}$. Then, for any matrix $X \in \mathcal{S}_3$ in (39), by the relation of (9), we have

$$\begin{aligned} \left\|X - \widetilde{X}\right\|^2 &= \left\|\sum_{i,j} (\alpha_{ij} - \gamma_{ij}) Z_{ij}\right\|^2 \\ &= \left(\sum_{i,j} \left(\alpha_{ij} - \delta_{ij}\right) Z_{ij}, \sum_{i,j} \left(\alpha_{ij} - \gamma_{ij}\right) Z_{ij}\right) \\ &= \sum_{i,j} \left(\alpha_{ij} - \gamma_{ij}\right) \left(Z_{ij}, \sum_{i,j} \left(\alpha_{ij} - \gamma_{ij}\right) Z_{ij}\right) \\ &= \sum_{i,j} \left(\alpha_{ij} - \gamma_{ij}\right)^2 = \|\alpha - \gamma\|^2 = \left\|Jz - (\gamma - U^{+}h)\right\|^2, \end{aligned} \tag{40}$$

where $J = I_{d_1} - U^{+}U$, $\gamma = [\gamma_{11}, \ldots, \gamma_{n,n}, \gamma_{12}, \ldots, \gamma_{1,n}]^\top$.

In order to solve Problem 4, we need the following lemma [16].

Lemma 11. *Suppose that $P \in \mathbf{R}^{q\times m}$, $\Delta \in \mathbf{R}^{q\times q}$, and $\Gamma \in \mathbf{R}^{m\times m}$ where $\Delta^2 = \Delta = \Delta^\top$ and $\Gamma^2 = \Gamma = \Gamma^\top$. Then*

$$\|P - \Delta D\Gamma\| = \min_{E\in\mathbf{R}^{q\times m}} \|P - \Delta E\Gamma\| \tag{41}$$

if and only if $\Delta(P - D)\Gamma = 0$, in which case, $\|P - \Delta D\Gamma\| = \|P - \Delta P\Gamma\|$.

It follows from Lemma 11 and $J^2 = J = J^\top$ that

$$\left\|X - \widetilde{X}\right\| = \left\|Jz - (\gamma - U^+h)\right\| = \min \tag{42}$$

if and only if $J(\gamma - U^+h - z) = 0$; that is,

$$Jz = J(\gamma - U^+h). \tag{43}$$

Substituting (43) into (38), we obtain

$$\widehat{\alpha} = U^+h + J(\gamma - U^+h). \tag{44}$$

By now, we have proved the following result.

Theorem 12. *Let the real-valued symmetric arrowhead matrix $\widetilde{X}$ be given. Then Problem 4 has a unique solution and the unique solution of Problem 4 can be expressed as*

$$\widehat{X} = K_1(\widehat{\alpha}\otimes I_n), \tag{45}$$

where $J = I_{d_1} - U^+U$, $\gamma = [\gamma_{11},\ldots,\gamma_{n,n},\gamma_{12},\ldots,\gamma_{1,n}]^\top$, and $\widehat{\alpha}$ is given by (44).

4. A Numerical Example

Based on Theorems 7 and 9 we can state the following algorithm.

Algorithm 13 (an algorithm for solving the optimal approximation solution of Problem 2). Consider the following.

(1) Input $A, B, C, D, H, \widetilde{X}, \widetilde{Y}$.

(2) Form the orthonormal bases $\{Z_{ij}\}$ and $\{W_{kl}\}$ by (7) and (8), respectively.

(3) Compute G, L, M, N, h according to (13), (14) and (15), respectively.

(4) Compute $E_M = I_{mp} - MM^+$, $F_M = I_{d_1} - M^+M$, $W = I_{d_2} - (E_MN)^+E_MN$, $\widetilde{\alpha} = M^+h - M^+N(E_MN)^+E_Mh$.

(5) Form the vectors γ, δ by (26), (27).

(6) Compute K_1, K_2 by (22) and (23), respectively.

(7) Compute $\widehat{\alpha}, \widehat{\beta}$ by (33) and (34), respectively.

(8) Compute the unique optimal approximation solution $(\widehat{X}, \widehat{Y})$ of (2) by (35).

Example 14. Given

$$A = \begin{bmatrix} 9.9733 & 3.8706 & 2.0328 & 6.1573 & 3.1647 & 0.0822 & 9.9417 & 5.1706 \\ 1.0809 & 0.7027 & 0.6888 & 5.9745 & 1.9242 & 5.8168 & 3.8545 & 1.5801 \\ 4.5676 & 6.8105 & 8.0014 & 6.8524 & 2.3317 & 5.5635 & 7.8625 & 7.4438 \\ 1.3305 & 8.5615 & 4.2306 & 9.7382 & 2.1984 & 6.8938 & 8.6961 & 4.8908 \\ 7.0384 & 0.971 & 8.9232 & 2.8033 & 0.7944 & 7.6095 & 0.852 & 6.4964 \\ 3.0627 & 3.9186 & 9.9191 & 1.7397 & 1.2387 & 8.2429 & 4.8371 & 0.4176 \\ 0.6432 & 3.7595 & 4.0088 & 9.2191 & 5.3758 & 5.2769 & 8.4455 & 3.3482 \\ 1.7377 & 5.1552 & 3.4069 & 2.6024 & 6.8886 & 3.4151 & 4.6584 & 4.3336 \\ 1.925 & 8.9427 & 3.1674 & 3.7506 & 7.2012 & 0.6829 & 5.9924 & 0.9859 \\ 6.7035 & 6.4182 & 3.6429 & 6.4474 & 9.388 & 2.6404 & 2.0815 & 6.8092 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.2712 & 0.8096 & 1.0469 & 2.7236 \\ 3.2085 & 3.0149 & 3.2898 & 4.2318 \\ 0.5248 & 1.8002 & 2.6111 & 6.9898 \\ 3.6956 & 7.1617 & 3.884 & 8.1743 \\ 3.8388 & 7.9026 & 8.9737 & 8.2547 \\ 3.3588 & 3.3133 & 7.6678 & 4.0326 \\ 8.9069 & 7.8162 & 2.4429 & 6.516 \\ 2.3814 & 7.7666 & 4.5465 & 1.5243 \end{bmatrix}, \qquad C = \begin{bmatrix} 3.3991 & 4.2993 & 5.1435 & 1.0278 & 8.3341 & 7.1445 \\ 0.1929 & 0.4898 & 0.4551 & 3.8017 & 4.7598 & 2.015 \\ 7.6829 & 9.0488 & 9.9136 & 7.5245 & 3.4013 & 1.0817 \\ 4.5788 & 9.7823 & 3.0961 & 1.4917 & 5.4221 & 7.3803 \\ 0.4356 & 6.5778 & 2.5647 & 6.6088 & 5.615 & 7.6202 \\ 0.9956 & 7.2696 & 6.101 & 2.3694 & 9.2522 & 2.7898 \\ 9.5692 & 4.8863 & 4.4698 & 7.2761 & 2.7439 & 8.4522 \\ 6.088 & 6.8578 & 4.1844 & 2.5853 & 2.9564 & 6.2439 \\ 2.4675 & 3.6294 & 4.1184 & 4.6951 & 6.1255 & 6.2167 \\ 9.2952 & 4.01 & 7.1406 & 8.5751 & 8.9064 & 9.5489 \end{bmatrix},$$

$$D=\begin{bmatrix}2.9541 & 4.8815 & 6.0597 & 0.6076\\ 6.3182 & 9.4265 & 8.1108 & 8.4724\\ 5.3129 & 3.4653 & 4.6176 & 4.5411\\ 8.7117 & 1.9969 & 0.9372 & 2.4818\\ 5.8446 & 0.981 & 1.027 & 9.8858\\ 6.0263 & 6.5212 & 8.7454 & 5.6432\end{bmatrix},\quad H=\begin{bmatrix}2.2053 & 6.8314 & 1.5734 & 7.5244\\ 8.7751 & 8.7539 & 9.8088 & 6.155\\ 7.2756 & 5.0769 & 2.9875 & 2.5417\\ 8.5308 & 4.6789 & 1.546 & 3.5041\\ 1.7768 & 2.4484 & 9.3574 & 6.1301\\ 5.403 & 5.6603 & 6.7386 & 3.3772\\ 4.7218 & 7.6072 & 4.2791 & 6.3599\\ 7.3239 & 3.4517 & 8.3281 & 5.166\\ 9.806 & 4.4755 & 1.7132 & 5.1954\\ 6.6682 & 5.1579 & 4.4064 & 6.2645\end{bmatrix},$$

$$\widetilde{X}=\begin{bmatrix}-4 & 1 & -7 & 10 & -6 & 13 & -2 & 11\\ 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0\\ -7 & 0 & 6 & 0 & 0 & 0 & 0 & 0\\ 10 & 0 & 0 & -2 & 0 & 0 & 0 & 0\\ -6 & 0 & 0 & 0 & -4 & 0 & 0 & 0\\ 13 & 0 & 0 & 0 & 0 & 18 & 0 & 0\\ -2 & 0 & 0 & 0 & 0 & 0 & -7 & 0\\ 11 & 0 & 0 & 0 & 0 & 0 & 0 & -24\end{bmatrix},\quad \widetilde{Y}=\begin{bmatrix}-7 & 2 & 19 & 9 & 3 & -15\\ 2 & -13 & 0 & 0 & 0 & 0\\ 19 & 0 & 8 & 0 & 0 & 0\\ 9 & 0 & 0 & -6 & 0 & 0\\ 3 & 0 & 0 & 0 & -3 & 0\\ -15 & 0 & 0 & 0 & 0 & 28\end{bmatrix}. \tag{46}$$

According to Algorithm 13, we can figure out

$$\widehat{X}=\begin{bmatrix}1.6112 & -0.34773 & -0.19013 & -0.3608 & 0.0018101 & 0.043733 & 0.12845 & 0.24356\\ -0.34773 & 0.087563 & 0 & 0 & 0 & 0 & 0 & 0\\ -0.19013 & 0 & 0.020941 & 0 & 0 & 0 & 0 & 0\\ -0.3608 & 0 & 0 & 0.070032 & 0 & 0 & 0 & 0\\ 0.0018101 & 0 & 0 & 0 & 0.10853 & 0 & 0 & 0\\ 0.043733 & 0 & 0 & 0 & 0 & 0.1462 & 0 & 0\\ 0.12845 & 0 & 0 & 0 & 0 & 0 & 0.048087 & 0\\ 0.24356 & 0 & 0 & 0 & 0 & 0 & 0 & -0.055789\end{bmatrix},$$
$$\widehat{Y}=\begin{bmatrix}0.032787 & 0.030243 & -0.064815 & 0.038152 & 0.00093986 & -0.072695\\ 0.030243 & -0.030004 & 0 & 0 & 0 & 0\\ -0.064815 & 0 & -0.0048278 & 0 & 0 & 0\\ 0.038152 & 0 & 0 & 0.025828 & 0 & 0\\ 0.00093986 & 0 & 0 & 0 & 0.03291 & 0\\ -0.072695 & 0 & 0 & 0 & 0 & -0.0038235\end{bmatrix}. \tag{47}$$

5. Concluding Remarks

The symmetric arrowhead matrix arises in many important practical applications. In this paper, the least-squares solutions of the matrix equations $AXB + CYD = H$ and $AXB + CXD = H$ for symmetric arrowhead matrices are provided by using the Kronecker product and stretching function of matrices. The explicit representations of the general solution are given. The best approximation solution to the given matrices is derived. A simple recipe for constructing the optimal approximation solution of Problem 2 is described, which can serve as the basis for numerical computation. The approach is demonstrated by a numerical example and reasonable results are produced.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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