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Research Article

Strong Convergence of an Iterative Algorithm for Hierarchical Problems

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We introduce the triple hierarchical problem over the solution set of the variational inequality problem and the fixed point set of a nonexpansive mapping. The strong convergence of the algorithm is proved under some mild conditions. Our results extend those of Yao et al., Iiduka, Ceng et al., and other authors.

1. Introduction

Let C be a closed convex subset of a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We denote weak convergence and strong convergence by notations $\rightharpoonup$ and $\rightarrow$, respectively. Let A be a nonlinear mapping. The *Hartman-Stampacchia variational inequality* [1] is to find $x \in C$ such that $\langle Ax, y-x\rangle \geq 0, \forall y \in C$. The set of solutions is denoted by $\mathrm{VI}(C,A)$. $f: C \rightarrow C$ is said to be a *ρ-contraction* if there exists a constant $\rho \in [0,1)$ such that $\|f(x)-f(y)\| \leq \rho\|x-y\|, \forall x, y \in C$. A mapping $A: H \rightarrow H$ is said to be *monotone* if $\langle Ax-Ay, x-y\rangle \geq 0, \forall x, y \in H$. A mapping $A: H \rightarrow H$ is said to be *α- strongly monotone* if there exists a positive real number α such that $\langle Ax-Ay, x-y\rangle \geq \alpha\|x-y\|^2, \forall x, y \in H$. A mapping $A: H \rightarrow H$ is said to be *β-inverse-strongly monotone* if there exists a positive real number β such that $\langle Ax-Ay, x-y\rangle \geq \beta\|Ax-Ay\|^2, \forall x, y \in H$. A mapping $A: H \rightarrow H$ is said to be *L-Lipschitz continuous* if there exists a positive real number L such that $\|Ax-Ay\| \leq L\|x-y\|, \forall x, y \in H$. A linear bounded operator A is said to be *strongly positive* on H if there exists a constant $\overline{\gamma} > 0$ with the property $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \forall x \in H$. A mapping $T: C \rightarrow C$ is said to be *nonexpansive* if $\|Tx-Ty\| \leq \|x-y\|, \forall x, y \in C$.

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. If C is bounded closed convex and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty (see [2]).

We discuss the following variational inequality problem over the fixed point set of a nonexpansive mapping (see [3–16]), which is said to be the *hierarchical problem*. Let a monotone, continuous mapping $A: H \rightarrow H$ and a nonexpansive mapping $T: H \rightarrow H$. Find $x \in \mathrm{VI}(F(T), A) = \{x \in F(T) : \langle Ax, y-x\rangle \geq 0, \forall y \in F(T)\}$, where $F(T) \neq \emptyset$. This solution set is denoted by Ξ.

We introduce the following variational inequality problem over the solution set of variational inequality problem and the fixed point set of a nonexpansive mapping (see [17, 18]), which is said to be the *triple hierarchical problem*. Let an inverse-strongly monotone $A: H \rightarrow H$, a strongly monotone and Lipschitz continuous $B: H \rightarrow H$, and a nonexpansive mapping $T: H \rightarrow H$. Find $x \in \mathrm{VI}(\Xi, B) = \{x \in \Xi : \langle Bx, y-x\rangle \geq 0, \forall y \in \Xi\}$, where $\Xi := \mathrm{VI}(F(T), A) \neq \emptyset$.

In 2009, Yao et al. [19] considered the following two-step iterative algorithm with the initial guess $x_0 \in C$ which is chosen arbitrarily:

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n) T y_n, \\ y_n &= \beta_n S x_n + (1-\beta_n) x_n, \quad \forall n \geq 0, \end{aligned} \tag{1}$$

where $\alpha_n, \beta_n \in (0,1)$ satisfies certain assumptions. Let S, T be two nonexpansive mappings and let $f : C \to C$ be a contraction mapping. Then, they proved that the above iterative sequence $\{x_n\}$ converges strongly to fixed point.

Next, Iiduka [17] introduced a monotone variational inequality with variational inequality constraint over the fixed point set of a nonexpansive mapping; the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_1 \in H$, is chosen arbitrarily:

$$\begin{aligned} y_n &= T\left(x_n - \lambda_n A_1 x_n\right), \\ x_{n+1} &= y_n - \mu\alpha_n A_2 y_n, \quad \forall n \geq 0, \end{aligned} \tag{2}$$

where $\alpha_n \in (0,1]$ and $\lambda_n \in (0, 2\alpha]$ satisfy certain conditions, $A_1 : H \to H$ is an inverse-strongly monotone, $A_2 : H \to H$ is a strongly monotone and Lipschitz continuous, and $T : H \to H$ is a nonexpansive mapping; then the strongly convergence analysis of the sequence generated by (2) is proved under some appropriate conditions.

In 2011, Yao et al. [20] studied the hierarchical problem over the fixed point set. Let the sequences $\{x_n\}$ be generated by these two following algorithms:

implicit algorithm $x_t = TP_C[I - t(A - \gamma f)]x_t, \forall t \in (0,1)$

explicit algorithm $x_{n+1} = \beta_n x_n + (1-\beta_n)TP_C[I - \alpha_n(A - \gamma f)]x_n, \forall n \geq 0$.

They illustrated that these two algorithms converge strongly to the unique solution of the variational inequality which is to find $x^* \in F(T)$ such that

$$\langle (A - \gamma f) x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T), \tag{3}$$

where $A : C \to H$ is a strongly positive linear bounded operator, $f : C \to H$ is a ρ-contraction, and $T : C \to C$ is a nonexpansive mapping satisfying some conditions.

Very recently, Ceng et al. [21] studied the following new algorithms. For $x_0 \in C$ is chosen arbitrarily, they defined a sequence $\{x_n\}$ by

$$\begin{aligned} &x_{n+1} \\ &= P_C\left[\lambda_n\gamma\left(\alpha_n f\left(x_n\right) + \left(1 - \alpha_n\right) Sx_n\right) + \left(I - \lambda_n\mu F\right) Tx_n\right], \\ &\qquad \forall n \geq 0, \end{aligned} \tag{4}$$

where the mappings S, T are nonexpansive mappings with $F(T) \neq \emptyset$. Let $F : C \to H$ be a Lipschitzian and strongly monotone operator and let $f : C \to H$ be a contraction mapping satisfying some appropriate conditions. They proved that the proposed algorithms strongly converge to the minimum norm fixed point of T.

In this paper, we consider a new iterative algorithm for solving the triple hierarchical problem over the solution set of the variational inequality problem and the fixed point set of a nonexpansive mapping which contain algorithms (1) and (4) as follows:

$$\begin{aligned} y_n &= P_C\left[\beta_n Sx_n + \left(1 - \beta_n\right) x_n\right], \\ x_{n+1} &= \gamma\lambda_n\phi\left(x_n\right) + \left(I - \lambda_n\mu F\right) Ty_n, \quad \forall n \geq 0, \end{aligned} \tag{5}$$

where the mappings S, T are nonexpansive mappings with $F(T) \neq \emptyset$. Let $F : C \to H$ be a Lipschitzian and strongly monotone operator, and let $\phi : H \to H$ be a contraction mapping satisfying some mild conditions. Find a point $x^* \in F(T)$ such that

$$\langle (I - S) x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{6}$$

This solution set of (6) is denoted by $\Omega := VI(F(T), S)$. The strong convergence for the proposed algorithms to the solution is solved under some appropriate assumptions. Our results improve the results of Ceng et al. [21], Iiduka [17], Yao et al. [19], Yao et al. [20], and some authors.

2. Preliminaries

Let C be a nonempty closed convex subset of H. There holds the following inequality in an inner product space $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \forall x, y \in H$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\left\|x - P_C x\right\| \leq \left\|x - y\right\|, \quad \forall y \in C. \tag{7}$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \left\|P_C x - P_C y\right\|^2, \tag{8}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{9}$$

$$\left\|x - y\right\|^2 \geq \left\|x - P_C x\right\|^2 + \left\|y - P_C x\right\|^2, \tag{10}$$

for all $x \in H, y \in C$. Let B be a monotone mapping of C into H. In the context of the variational inequality problem the characterization of projection (9) implies the following:

$$u \in VI\left(C, B\right) \Longleftrightarrow u = P_C\left(u - \lambda Bu\right), \quad \lambda > 0. \tag{11}$$

It is also known that H satisfies the Opial's condition [22]; that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|$ holds for every $y \in H$ with $x \neq y$.

Lemma 1 (see [23]). *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero; that is, $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ imply $x = Tx$.*

Lemma 2 (see [24]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 3 (see [10]). *Let $B : H \to H$ be β-strongly monotone and L-Lipschitz continuous and let $\mu \in (0, 2\beta/L^2)$. For $\lambda \in [0,1]$, define $T_\lambda : H \to H$ by $T_\lambda(x) := x - \lambda\mu B(x)$ for all $x \in H$. Then, for all $x, y \in H$, $\|T_\lambda(x) - T_\lambda(y)\| \leq (1-\lambda\tau)\|x - y\|$ hold, where $\tau := 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \in (0,1]$.*

Lemma 4 (see [25]). *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n) a_n + \delta_n, \quad \forall n \geq 0, \tag{12}$$

where $\{\gamma_n\} \subset (0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathscr{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty}(\delta_n/\gamma_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Strong Convergence Theorem

In this section, we introduce an iterative algorithm of triple hierarchical for solving monotone variational inequality problems for κ-Lipschitzian and η-strongly monotone operators over the solution set of variational inequality problems and the fixed point set of a nonexpansive mapping.

Theorem 5. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H. *Let* $F : C \to C$ *be* κ*-Lipschitzian and* η*-strongly monotone operators with constant* κ *and* $\eta > 0$, *respectively, and let* $\phi : C \to C$ *be a* ρ*-contraction with coefficient* $\rho \in [0,1)$. *Let* $T : C \to C$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$, *and let* $S : H \to H$ *be a nonexpansive mapping. Let* $0 < \mu < 2\eta/\kappa^2$ *and* $0 < \gamma < \tau$, *where* $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)}$. *Suppose that* $\{x_n\}$ *is a sequence generated by the following algorithm where* $x_0 \in C$ *is chosen arbitrarily:*

$$\begin{aligned} y_n &= P_C\left[\beta_n S x_n + (1-\beta_n) x_n\right], \\ x_{n+1} &= \gamma\lambda_n \phi(x_n) + (I - \lambda_n \mu F) T y_n, \quad \forall n \geq 0, \end{aligned} \tag{13}$$

where $\{\beta_n\}, \{\lambda_n\}, \subset (0,1)$ *satisfy the following conditions:*

(C1): $\beta_n \leq k\lambda_n$;

(C2): $\lim_{n\to\infty}\lambda_n = 0$, $\lim_{n\to\infty}((\lambda_n - \lambda_{n-1})/\lambda_n) = 0$, $\sum_{n=0}^{\infty}\lambda_n = \infty$;

(C3): $\lim_{n\to\infty}((\beta_n - \beta_{n-1})/\beta_n) = 0$.

Then $\{x_n\}$ *converges strongly to* $x^* \in \Omega$, *which is the unique solution of another variational inequality:*

$$\langle (\mu F - \gamma\phi) x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega, \tag{14}$$

where $\Omega := VI(F(T), S) \neq \emptyset$.

Proof. We will divide the proof into four steps.

Step 1. We will show that $\{x_n\}$ is bounded. Indeed, for any $x^* \in F(T)$, we have

$$\begin{aligned} &\|y_n - x^*\| \\ &= \|P_C[\beta_n S x_n + (1-\beta_n)x_n] - P_C x^*\| \\ &\leq \|\beta_n S x_n + (1-\beta_n)x_n - x^*\| \\ &= \|\beta_n(Sx_n - Sx^*) + (1-\beta_n)(x_n - x^*) + \beta_n(Sx^* - x^*)\| \\ &\leq \beta_n\|x_n - x^*\| + (1-\beta_n)\|x_n - x^*\| + \beta_n\|Sx^* - x^*\| \\ &\leq \|x_n - x^*\| + \beta_n\|Sx^* - x^*\|. \end{aligned} \tag{15}$$

From (13), we deduce that

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &= \|\gamma\lambda_n\phi(x_n) + (I - \lambda_n\mu F)Ty_n - x^*\| \\ &= \|\gamma\lambda_n(\phi(x_n) - \phi(x^*)) + (I - \lambda_n\mu F)(Ty_n - x^*) \\ &\qquad + \lambda_n(\gamma\phi(x^*) - \mu F x^*)\| \\ &\leq \gamma\lambda_n\|\phi(x_n) - \phi(x^*)\| + (I - \lambda_n\mu F)\|Ty_n - x^*\| \\ &\quad + \lambda_n\|\gamma\phi(x^*) - \mu F x^*\| \\ &\leq \gamma\rho\lambda_n\|x_n - x^*\| + (1 - \lambda_n\tau)\|y_n - x^*\| \\ &\quad + \lambda_n\|\gamma\phi(x^*) - \mu F x^*\|. \end{aligned} \tag{16}$$

Substituting (15) into (16), we obtain

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &\leq \gamma\rho\lambda_n\|x_n - x^*\| \\ &\quad + (1-\lambda_n\tau)\{\|x_n - x^*\| + \beta_n\|Sx^* - x^*\|\} \\ &\quad + \lambda_n\|\gamma\phi(x^*) - \mu F x^*\| \\ &\leq \gamma\rho\lambda_n\|x_n - x^*\| + (1-\lambda_n\tau)\|x_n - x^*\| \\ &\quad + \beta_n\|Sx^* - x^*\| + \lambda_n\|\gamma\phi(x^*) - \mu F x^*\| \\ &\leq [1 - \lambda_n(\tau - \gamma\rho)]\|x_n - x^*\| + k\lambda_n\|Sx^* - x^*\| \\ &\quad + \lambda_n\|\gamma\phi(x^*) - \mu F x^*\| \\ &\leq [1 - \lambda_n(\tau - \gamma\rho)]\|x_n - x^*\| \\ &\quad + \lambda_n(k\|Sx^* - x^*\| + \|\gamma\phi(x^*) - \mu F x^*\|) \\ &\leq \max\left\{\|x_n - x^*\| + \frac{1}{\tau - \gamma\rho}\right. \\ &\qquad\qquad \left.\times (k\|Sx^* - x^*\| + \|\gamma\phi(x^*) - \mu F x^*\|)\right\}. \end{aligned} \tag{17}$$

By induction, it follows that

$$\begin{aligned}\|x_n - x^*\| &\le \max\left\{\|x_0 - x^*\| + \frac{1}{\tau - \gamma\rho}\right. \\ &\left.\quad\times (k\|Sx^* - x^*\| + \|\gamma\phi(x^*) - \mu F x^*\|)\right\}, \\ &\qquad n \ge 0.\end{aligned} \tag{18}$$

Therefore, $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{Ty_n\}$, $\{Sx_n\}$, $\{\phi(x_n)\}$, and $\{FT(y_n)\}$.

Step 2. We will show that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Setting $v_n := \beta_n Sx_n + (1-\beta_n)x_n$, we obtain

$$\begin{aligned}\|v_n - v_{n-1}\| &= \|\beta_n Sx_n + (1-\beta_n)x_n - \beta_{n-1}Sx_{n-1} - (1-\beta_{n-1})x_{n-1}\| \\ &= \|\beta_n(Sx_n - Sx_{n-1}) + (\beta_n - \beta_{n-1})Sx_{n-1} \\ &\quad + (1-\beta_n)(x_n - x_{n-1}) + (\beta_{n-1} - \beta_n)x_{n-1}\| \\ &\le \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|x_{n-1}\|) \\ &\quad + (1-\beta_n)\|x_n - x_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|x_{n-1}\|),\end{aligned} \tag{19}$$

which implies that

$$\begin{aligned}\|y_n - y_{n-1}\| = \|P_C v_n - P_C v_{n-1}\| &\le \|v_n - v_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|x_{n-1}\|).\end{aligned} \tag{20}$$

It follows from (13) that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\gamma\lambda_n\phi(x_n) + (I - \lambda_n\mu F)Ty_n - \gamma\lambda_{n-1}\phi(x_{n-1}) \\ &\quad - (I - \lambda_{n-1}\mu F)Ty_{n-1}\| \\ &= \|\gamma\lambda_n(\phi(x_n) - \phi(x_{n-1})) + (\lambda_n - \lambda_{n-1})\gamma\phi(x_{n-1}) \\ &\quad + (I - \lambda_n\mu F)Ty_n - (I - \lambda_{n-1}\mu F)Ty_{n-1}\| \\ &\le \gamma\rho\lambda_n\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\gamma\|\phi(x_{n-1})\| \\ &\quad + \|(I - \lambda_n\mu F)Ty_n - (I - \lambda_n\mu F)Ty_{n-1} \\ &\quad + (I - \lambda_n\mu F)Ty_{n-1} - (I - \lambda_{n-1}\mu F)Ty_{n-1}\| \\ &\le \gamma\rho\lambda_n\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\gamma\|\phi(x_{n-1})\| \\ &\quad + (1 - \lambda_n\tau)\|y_n - y_{n-1}\| + |\lambda_n - \lambda_{n-1}|\mu\|FTy_{n-1}\| \\ &\le \gamma\rho\lambda_n\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}| \\ &\quad \times(\gamma\|\phi(x_{n-1})\| + \mu\|FTy_{n-1}\|) \\ &\quad + (1 - \lambda_n\tau)\{\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \\ &\qquad \times(\|Sx_{n-1}\| + \|x_{n-1}\|)\} \\ &\le [1 - \lambda_n(\tau - \gamma\rho)]\|x_n - x_{n-1}\| \\ &\quad + |\lambda_n - \lambda_{n-1}|(\gamma\|\phi(x_{n-1})\| + \mu\|FTy_{n-1}\|) \\ &\quad + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|x_{n-1}\|) \\ &= [1 - \lambda_n(\tau - \gamma\rho)]\|x_n - x_{n-1}\| \\ &\quad + \left(\frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} + \frac{|\beta_n - \beta_{n-1}|}{\lambda_n}\right)\lambda_n M_1 \\ &\le [1 - \lambda_n(\tau - \gamma\rho)]\|x_n - x_{n-1}\| \\ &\quad + \left(\frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} + \frac{k|\beta_n - \beta_{n-1}|}{\beta_n}\right)\lambda_n M_1,\end{aligned} \tag{21}$$

where M_1 is a constant such that

$$\sup_{n\ge0}\{(\gamma\|\phi(x_n)\| + \mu\|FTy_n\|), (\|Sx_n\| + \|x_n\|)\} \le M_1. \tag{22}$$

Hence, conditions (C2) and (C3) allow us to apply Lemma 4; then we get

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{23}$$

By (21), we get

$$\begin{aligned}\frac{\|x_{n+1} - x_n\|}{\lambda_n} &\le [1 - \lambda_n(\tau - \gamma\rho)]\frac{\|x_n - x_{n-1}\|}{\lambda_n} \\ &\quad + \frac{|\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}|}{\lambda_n}M_1 \\ &= [1 - \lambda_n(\tau - \gamma\rho)]\frac{\|x_n - x_{n-1}\|}{\lambda_{n-1}} \\ &\quad + [1 - \lambda_n(\tau - \gamma\rho)]\left(\frac{\|x_n - x_{n-1}\|}{\lambda_n} - \frac{\|x_n - x_{n-1}\|}{\lambda_{n-1}}\right) \\ &\quad + \frac{|\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}|}{\lambda_n}M_1 \\ &\le [1 - \lambda_n(\tau - \gamma\rho)]\frac{\|x_n - x_{n-1}\|}{\lambda_{n-1}} \\ &\quad + \lambda_n\|x_n - x_{n-1}\|\frac{1}{\lambda_n}\left|\frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}}\right| \\ &\quad + M_1\lambda_n\frac{|\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}|}{\lambda_n^2}.\end{aligned} \tag{24}$$

Using the conditions (C2) and (C3), we can apply Lemma 4 to conclude that

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\lambda_n}=0. \tag{25}$$

By (13), we compute

$$\begin{aligned}\|x_{n+1}-Ty_n\| &= \|\gamma\lambda_n\phi(x_n)+(I-\lambda_n\mu F)Ty_n-Ty_n\| \\ &= \|\gamma\lambda_n\phi(x_n)+Ty_n-\lambda_n\mu FTy_n-Ty_n\| \\ &\le \lambda_n\|\gamma\phi(x_n)-\mu FTy_n\|.\end{aligned} \tag{26}$$

From the condition (C2), we note that $\lim_{n\to\infty}\|x_{n+1}-Ty_n\|=0$. At the same time, from (13), we also have

$$\begin{aligned}\|y_n-x_n\| &= \|P_C[\beta_nSx_n+(1-\beta_n)x_n]-P_Cx_n\| \\ &\le \|\beta_nSx_n+(1-\beta_n)x_n-x_n\| \\ &\le \beta_n\|Sx_n-x_n\|.\end{aligned} \tag{27}$$

By the conditions (C1) and (C2), we note that $\lim_{n\to\infty}\|y_n-x_n\|=0$. Consider

$$\begin{aligned}\|y_n-Ty_n\| &\le \|y_n-x_n\|+\|x_n-x_{n+1}\| \\ &\quad+\|x_{n+1}-Ty_n\|\longrightarrow 0.\end{aligned} \tag{28}$$

From (23), (26), and (27), we obtain

$$\lim_{n\to\infty}\|y_n-Ty_n\|=0. \tag{29}$$

We set $v_n=\beta_nSx_n+(1-\beta_n)x_n$; then we get

$$\begin{aligned}\|y_n-v_n\| &= \|P_Cv_n-v_n\| \\ &\le \|v_n-v_n\|\longrightarrow 0,\quad \text{as } n\longrightarrow\infty.\end{aligned} \tag{30}$$

From (13), we have

$$\begin{aligned}\|Ty_n-Tx_n\| &= \|TP_C[\beta_nSx_n+(1-\beta_n)x_n]-TP_Cx_n\| \\ &\le \|\beta_nSx_n+(1-\beta_n)x_n-x_n\| \\ &\le \beta_n\|Sx_n-x_n\|.\end{aligned} \tag{31}$$

By the conditions (C1) and (C2) again, we note that $\lim_{n\to\infty}\|Ty_n-Tx_n\|=0$. Consider

$$\|x_n-Tx_n\|\le\|x_n-y_n\|+\|y_n-Ty_n\|+\|Ty_n-Tx_n\|\longrightarrow 0. \tag{32}$$

From (29), $\lim_{n\to\infty}\|x_n-y_n\|=0$, and $\lim_{n\to\infty}\|Ty_n-Tx_n\|=0$, we obtain

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{33}$$

Step 3. We will show that $\limsup_{n\to\infty}\langle\mu Fx^*-\gamma\phi(x^*),x_n-x^*\rangle\le 0$. Rewrite (13) as

$$\begin{aligned}x_{n+1} &= \gamma\lambda_n\phi(x_n)+(I-\mu\lambda_nF)Ty_n \\ &\quad-v_n+\beta_nSx_n+(1-\beta_n)x_n.\end{aligned} \tag{34}$$

We observe that

$$\begin{aligned}&x_n-x_{n+1} \\ &= x_n-\gamma\lambda_n\phi(x_n) \\ &\quad-(I-\mu\lambda_nF)Ty_n+v_n-\beta_nSx_n-x_n+\beta_nx_n \\ &= \lambda_n(\mu F-\gamma\phi)x_n \\ &\quad-\lambda_n\mu Fx_n-(I-\mu\lambda_nF)Ty_n+(I-\mu\lambda_nF)y_n \\ &\quad-(I-\mu\lambda_nF)y_n+v_n+\beta_n(I-S)x_n \\ &= \lambda_n(\mu F-\gamma\phi)x_n+\lambda_n\mu(Fy_n-Fx_n)+(y_n-Ty_n) \\ &\quad-\mu\lambda_nF(y_n-Ty_n)+(v_n-y_n)+\beta_n(I-S)x_n \\ &= \lambda_n(\mu F-\gamma\phi)x_n+\lambda_n\mu(Fy_n-Fx_n)+(y_n-Ty_n) \\ &\quad-\mu\lambda_nF(y_n-Ty_n)+\lambda_n(y_n-Ty_n) \\ &\quad-\lambda_n(y_n-Ty_n)+(v_n-y_n)+\beta_n(I-S)x_n \\ &= \lambda_n(\mu F-\gamma\phi)x_n+\lambda_n\mu(Fy_n-Fx_n) \\ &\quad+\lambda_n(I-\mu F)(y_n-Ty_n)+(1-\lambda_n)(y_n-Ty_n) \\ &\quad+(v_n-y_n)+\beta_n(I-S)x_n.\end{aligned} \tag{35}$$

Set

$$z_n=\frac{x_n-x_{n+1}}{\lambda_n},\quad\forall n\ge 0. \tag{36}$$

We note from (35) that

$$\begin{aligned}z_n = &(\mu F-\gamma\phi)x_n+\mu(Fy_n-Fx_n)+(I-\mu F)(y_n-Ty_n) \\ &+\frac{1-\lambda_n}{\lambda_n}(y_n-Ty_n) \\ &+\frac{1}{\lambda_n}(v_n-y_n)+\frac{\beta_n}{\lambda_n}(I-S)x_n.\end{aligned} \tag{37}$$

This yields that, for each $x^*\in F(T)$,

$$\begin{aligned}&\langle z_n,x_n-x^*\rangle \\ &= \langle(\mu F-\gamma\phi)x_n,x_n-x^*\rangle+\mu\langle(Fy_n-Fx_n),x_n-x^*\rangle \\ &\quad+\langle(I-\mu F)y_n-(I-\mu F)Ty_n,x_n-x^*\rangle \\ &\quad+\frac{1-\lambda_n}{\lambda_n}\langle y_n-Ty_n,x_n-x^*\rangle \\ &\quad+\frac{1}{\lambda_n}\langle v_n-y_n,x_n-x^*\rangle+\frac{\beta_n}{\lambda_n}\langle(I-S)x_n,x_n-x^*\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle (\mu F - \gamma\phi) x^*, x_n - x^* \rangle \\
&\quad + \langle (\mu F - \gamma\phi) x_n - (\mu F - \gamma\phi) x^*, x_n - x^* \rangle \\
&\quad + \mu \langle (F y_n - F x_n), x_n - x^* \rangle \\
&\quad + \langle (I - \mu F) y_n - (I - \mu F) T y_n, x_n - x^* \rangle \\
&\quad + \frac{1 - \lambda_n}{\lambda_n} \langle y_n - T y_n, x_n - x^* \rangle + \frac{1}{\lambda_n} \langle v_n - y_n, x_n - x^* \rangle \\
&\quad + \frac{\beta_n}{\lambda_n} \langle (I - S) x_n, x_n - x^* \rangle .
\end{aligned} \tag{38}
$$

In view of (38), $\langle (\mu F - \gamma\phi) x_n - (\mu F - \gamma\phi) x^*, x_n - x^* \rangle$ is nonnegative due to the monotonicity of $\mu F - \gamma\phi$. From (38), we derive that

$$
\begin{aligned}
\langle z_n, x_n - x^* \rangle \geq{}& \langle (\mu F - \gamma\phi) x^*, x_n - x^* \rangle \\
&+ \mu \langle (F y_n - F x_n), x_n - x^* \rangle \\
&+ \langle (I - \mu F) y_n - (I - \mu F) T y_n, x_n - x^* \rangle \\
&+ \frac{1 - \lambda_n}{\lambda_n} \langle y_n - T y_n, x_n - x^* \rangle \\
&+ \frac{1}{\lambda_n} \langle v_n - y_n, x_n - x^* \rangle \\
&+ \frac{\beta_n}{\lambda_n} \langle (I - S) x_n, x_n - x^* \rangle .
\end{aligned} \tag{39}
$$

Since (29) implies $\|(I - \mu F) y_n - (I - \mu F) T y_n\| \to 0$, as $n \to \infty$, from (25), then we get $z_n \to 0$. Using (C1) and (30), $\|y_n - x_n\| \to 0$, as $n \to \infty$ and $\{x_n\}$ is bounded. We obtain from (39) that

$$
\limsup_{n \to \infty} \langle (\mu F - \gamma\phi) x^*, x_n - x^* \rangle \leq 0, \quad \forall x^* \in F(T). \tag{40}
$$

Since the sequence $\{x_n\}$ is bounded, we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$
\begin{aligned}
&\limsup_{n \to \infty} \langle (\mu F - \gamma\phi) x^*, x_n - x^* \rangle \\
&\quad = \limsup_{j \to \infty} \langle (\mu F - \gamma\phi) x^*, x_{n_j} - x^* \rangle
\end{aligned} \tag{41}
$$

and $x_{n_j} \rightharpoonup \tilde{x}$. From (33), by the demiclosed principle of the nonexpansive mapping, it follows that $\tilde{x} \in F(T)$. Then

$$
\begin{aligned}
&\limsup_{j \to \infty} \langle (\mu F - \gamma\phi) x^*, x_{n_j} - x^* \rangle \\
&\quad = \langle (\mu F - \gamma\phi) x^*, \tilde{x} - x^* \rangle \leq 0.
\end{aligned} \tag{42}
$$

Step 4. Finally, we will prove $x_{n+1} \to x^*$. From (13), we note that

$$
\begin{aligned}
\|y_n - x^*\|^2 &= \|P_C[\beta_n S x_n + (1 - \beta_n) x_n] - P_C x^*\|^2 \\
&\leq \|[\beta_n S x_n + (1 - \beta_n) x_n] - x^*\|^2 \\
&\leq \|\beta_n (S x_n - S x^*) + (1 - \beta_n)(x_n - x^*) \\
&\qquad + \beta_n (S x^* - x^*)\|^2 \\
&\leq \|\beta_n (S x_n - S x^*) + (1 - \beta_n)(x_n - x^*)\|^2 \\
&\quad + 2\beta_n \langle S x^* - x^*, y_n - x^* \rangle \\
&\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \|x_n - x^*\|^2 \\
&\quad + 2\beta_n \langle S x^* - x^*, y_n - x^* \rangle \\
&\leq \|x_n - x^*\|^2 + 2\beta_n \|S x^* - x^*\| \|y_n - x^*\| .
\end{aligned} \tag{43}
$$

Using (43), we compute

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\gamma \lambda_n \phi(x_n) + (I - \lambda_n \mu F) T y_n - x^*\|^2 \\
&= \|\gamma \lambda_n (\phi(x_n) - \phi(x^*)) \\
&\qquad + (I - \lambda_n \mu F) T y_n - (I - \lambda_n \mu F) x^* \\
&\qquad + (I - \lambda_n \mu F) x^* - x^* + \gamma \lambda_n \phi(x^*)\|^2 \\
&= \|\gamma \lambda_n (\phi(x_n) - \phi(x^*)) + (I - \lambda_n \mu F)(T y_n - x^*) \\
&\qquad + \lambda_n (\gamma \phi(x^*) - \mu F x^*)\|^2 \\
&\leq \|\gamma \lambda_n (\phi(x_n) - \phi(x^*)) + (I - \lambda_n \mu F)(T y_n - x^*)\|^2 \\
&\quad + 2\lambda_n \langle \gamma \phi(x^*) - \mu F x^*, x_{n+1} - x^* \rangle \\
&\leq \gamma^2 \lambda_n^2 \|\phi(x_n) - \phi(x^*)\|^2 + (1 - \lambda_n \tau)^2 \|T y_n - x^*\|^2 \\
&\quad + 2\lambda_n \langle \gamma \phi(x^*) - \mu F x^*, x_{n+1} - x^* \rangle \\
&\quad + 2 \langle \gamma \lambda_n (\phi(x_n) - \phi(x^*)), (I - \mu \lambda_n F)(T y_n - x^*) \rangle \\
&\leq \gamma^2 \rho^2 \lambda_n^2 \|x_n - x^*\|^2 + \left(1 - 2\lambda_n \tau + \lambda_n^2 \tau^2\right) \|y_n - x^*\|^2 \\
&\quad + 2\lambda_n \langle \gamma \phi(x^*) - \mu F x^*, x_{n+1} - x^* \rangle \\
&\quad + 2\gamma \lambda_n \langle \phi(x_n) - \phi(x^*), (I - \mu \lambda_n F) T y_n - (I - \mu \lambda_n F) x^* \rangle \\
&= \gamma^2 \rho^2 \lambda_n^2 \|x_n - x^*\|^2 + \left(1 - 2\lambda_n \tau + \lambda_n^2 \tau^2\right) \|y_n - x^*\|^2 \\
&\quad + 2\lambda_n \langle \gamma \phi(x^*) - \mu F x^*, x_{n+1} - x^* \rangle \\
&\quad + 2\gamma \lambda_n \langle \phi(x_n) - \phi(x^*), (T y_n - x^*) - \mu \lambda_n F(T y_n - x^*) \rangle
\end{aligned}
$$

$$\begin{aligned}
&= \gamma^2\rho^2\lambda_n^2\|x_n - x^*\|^2 + \left(1 - 2\lambda_n\tau + \lambda_n^2\tau^2\right)\|y_n - x^*\|^2 \\
&\quad + 2\lambda_n\langle\gamma\phi(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\
&\quad + 2\gamma\lambda_n\langle\phi(x_n) - \phi(x^*), T y_n - x^*\rangle \\
&\quad - 2\gamma\lambda_n\langle\phi(x_n) - \phi(x^*), \mu\lambda_n F(T y_n - x^*)\rangle \\
&\le \gamma^2\rho^2\lambda_n^2\|x_n - x^*\|^2 + \left(1 - 2\lambda_n\tau + \lambda_n^2\tau^2\right) \\
&\quad \times \left\{\|x_n - x^*\|^2 + 2\beta_n\|Sx^* - x^*\|\|y_n - x^*\|\right\} \\
&\quad + 2\lambda_n\langle\gamma\phi(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\
&\quad + 2\gamma\rho\lambda_n\|x_n - x^*\|\|T y_n - x^*\| \\
&\quad - 2\gamma\rho\mu\lambda_n^2\|x_n - x^*\|\|F(T y_n - x^*)\| \\
&\le \left[1 - \lambda_n\left(2\tau - \lambda_n\tau^2 - \lambda_n\gamma^2\rho^2\right)\right]\|x_n - x^*\|^2 \\
&\quad + 2\varepsilon_n\lambda_n\|Sx^* - x^*\|\|y_n - x^*\| \\
&\quad + 2\lambda_n\langle\gamma\phi(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\
&\quad + 2\gamma\rho\lambda_n\|x_n - x^*\|\|T y_n - x^*\| \\
&\quad - 2\gamma\rho\mu\lambda_n^2\|x_n - x^*\|\|F(T y_n - x^*)\|.
\end{aligned} \tag{44}$$

Since $\{x_n\}$, $\{Ty_n\}$, and $\{FTy_n\}$ are all bounded, we can choose a constant $M_2 > 0$ such that

$$\begin{aligned}
&\sup_{n\ge 0}\frac{1}{2\tau - \lambda_n\tau^2 - \lambda_n\gamma^2\rho^2} \\
&\quad \times \left\{2\gamma\rho\mu\|x_n - x^*\|\|F(T y_n - x^*)\|\right\} \le M_2.
\end{aligned} \tag{45}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \le{}& \left[1 - \lambda_n\left(2\tau - \lambda_n\tau^2 - \lambda_n\gamma^2\rho^2\right)\right]\|x_n - x^*\|^2 \\
&+ \lambda_n\left(2\tau - \lambda_n\tau^2 - \lambda_n\gamma^2\rho^2\right)\delta_n,
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
\delta_n ={}& \frac{2\varepsilon_n}{2\tau - \lambda_n\tau^2 - \lambda_n\gamma^2\rho^2}\|Sx^* - x^*\|\|y_n - x^*\| \\
&+ \frac{2}{2\tau - \lambda_n\tau^2 - \lambda_n\gamma^2\rho^2}\langle\gamma\phi(x^*) - \mu F x^*, x_{n+1} - x^*\rangle \\
&+ \frac{2}{2\tau - \lambda_n\tau^2 - \lambda_n\gamma^2\rho^2}\gamma\rho\|x_n - x^*\|\|T y_n - x^*\| \\
&- \lambda_n M_2.
\end{aligned} \tag{47}$$

Now, applying Lemma 4 and (35), we conclude that $x_n \to x^*$. This completes the proof. □

Corollary 6. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F : C \to C$ be κ-Lipschitzian and η-strongly monotone operators with constant κ and $\eta > 0$, respectively. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \ne \emptyset$, and let $S : H \to H$ be a nonexpansive mapping. Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0 \in C$ arbitrarily:*

$$x_{n+1} = (I - \lambda_n\mu F) T P_C\left[\beta_n S x_n + (1 - \beta_n)x_n\right], \quad \forall n \ge 0, \tag{48}$$

where $\{\beta_n\}, \{\lambda_n\} \subset (0, 1)$ satisfy the following conditions (C1)–(C3). Then $\{x_n\}$ converges strongly to $x^* \in \Omega$, which is the unique solution of variational inequality:

$$\langle(I - \mu F)x^*, x - x^*\rangle \ge 0, \quad \forall x \in \Omega, \tag{49}$$

where $\Omega := VI(F(T), S) \ne \emptyset$.

Proof. Putting $\phi \equiv 0$ in Theorem 5, we can obtain the desired conclusion immediately. □

Corollary 7. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $\phi : H \to H$ be a ρ-contraction with coefficient $\rho \in [0, 1)$, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \ne \emptyset$ and $S : H \to H$ a nonexpansive mapping. Suppose $\{x_n\}$ is a sequence generated by the following algorithm, $x_0 \in C$, arbitrarily:*

$$\begin{aligned}
y_n &= P_C\left[\beta_n S x_n + (1 - \beta_n)x_n\right], \\
x_{n+1} &= \lambda_n\phi(x_n) + (1 - \lambda_n)T y_n, \quad \forall n \ge 0,
\end{aligned} \tag{50}$$

where $\{\beta_n\}, \{\lambda_n\} \subset (0, 1)$ satisfy the following conditions (C1)–(C3). Then $\{x_n\}$ converges strongly to $x^* \in \Omega$, which is the unique solution of variational inequality:

$$\langle(I - \phi)x^*, x - x^*\rangle \ge 0, \quad \forall x \in \Omega, \tag{51}$$

where $\Omega := VI(F(T), S) \ne \emptyset$.

Proof. Putting $\gamma = 1$, $\mu = 2$, and $F \equiv I/2$ in Theorem 5, we can obtain the desired conclusion immediately. □

Corollary 8. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \ne \emptyset$ and let $S : H \to H$ be a nonexpansive mapping. Suppose $\{x_n\}$ is a sequence generated by the following algorithm, $x_0 \in C$, arbitrarily:*

$$x_{n+1} = (1 - \lambda_n) T P_C\left[\beta_n S x_n + (1 - \beta_n)x_n\right], \quad \forall n \ge 0, \tag{52}$$

where $\{\beta_n\}, \{\lambda_n\} \subset (0, 1)$ satisfy the following conditions (C1)–(C3). Then $\{x_n\}$ converges strongly to $x^* \in F(T)$, which is the unique solution of variational inequality:

$$\langle(I - S)x^*, x - x^*\rangle \ge 0, \quad \forall x \in F(T). \tag{53}$$

Proof. Putting $\phi \equiv 0$ in Corollary 7, we can obtain the desired conclusion immediately. □

Corollary 9. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $\phi : H \to H$ be a ρ-contraction with coefficient $\rho \in [0,1)$, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $S : C \to C$ a nonexpansive mapping. Suppose $\{x_n\}$ is a sequence generated by the following algorithm, $x_0 \in C$, arbitrarily:*

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n) T [\beta_n S x_n + (1-\beta_n) x_n], \quad \forall n \geq 0, \tag{54}$$

where $\{\beta_n\}, \{\lambda_n\} \subset (0,1)$ satisfy the following conditions (C1)–(C3). Then $\{x_n\}$ converges strongly to $x^* \in F(T)$, which is the unique solution of variational inequality:

$$\langle (I-S) x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{55}$$

Proof. Putting $P_C \equiv I$ in Corollary 7, we can obtain the desired conclusion immediately. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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