Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 657858, 7 pages
http://dx.doi.org/10.1155/2014/657858

Research Article

ON Modified $(\alpha - \eta)$-Contractive Mappings

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Received 6 May 2014; Accepted 6 July 2014; Published 21 July 2014

Academic Editor: Abdul Latif

Hussain et al. (2013) established new fixed point results in complete metric space. In this paper, we prove fixed point results of α-admissible mappings with respect to η, for modified contractive condition in complete metric space. An example is given to show the validity of our work. Our results generalize/improve several recent and classical results existing in the literature.

1. Preliminaries and Scope

The study of fixed point problems in nonlinear analysis has emerged as a powerful and very important tool in the last 60 years. Particularly, the technique of fixed point theory has been applicable to many diverse fields of sciences such as engineering, chemistry, biology, physics, and game theory. Over the years, fixed point theory has been generlized in many directions by several mathematicians (see [1–36]).

In 1973, Geraghty [12] studied different contractive conditions and established some useful fixed point theorems.

In 2012, Samet et al. [33] introduced a concept of $\alpha-\psi$-contractive type mappings and established various fixed point theorems for mappings in complete metric spaces. Afterwards Karapinar and Samet [10] refined the notions and obtained various fixed point results. Hussain et al. [17] extended the concept of α-admissible mappings and obtained useful fixed point theorems. Subsequently, Abdeljawad [4] introduced pairs of α-admissible mappings satisfying new sufficient contractive conditions different from those in [17, 33] and proved fixed point and common fixed point theorems. Lately, Salimi et al. [32] modified the concept of $\alpha-\psi$-contractive mappings and established fixed point results.

We define Ω the family of nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$, and $\psi(0) = 0$ for each $t > 0$ where ψ^n is the nth term of ψ.

Lemma 1 (see [32]). *If $\psi \in \Omega$, then $\psi(t) < t$ for all $t > 0$.*

Definition 2 (see [33]). Let (X, d) be a metric space and let $S : X \to X$ be a given mapping. We say that S is an $\alpha-\psi$-contractive mapping if there exist two functions $\alpha : X\times X \to [0,+\infty)$ and $\psi \in \Omega$ such that

$$\alpha(x, y)\, d(Sx, Sy) \le \psi(d(x, y)), \quad (1)$$

for all $x, y \in X$.

Definition 3 (see [33]). Let $S : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. One says that S is α-admissible if $x, y \in X$, $\alpha(x, y) \ge 1 \Rightarrow \alpha(Sx, Sy) \ge 1$.

Example 4. Consider $X = [0,\infty)$. Define $S : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by $Sx = 2x$, for all $x, y \in X$ and

$$\alpha(x, y) = \begin{cases} e^{y/x} & \text{if } x \ge y,\ x \ne 0 \\ 0 & \text{if } x < y. \end{cases} \quad (2)$$

Then S is α-admissible.

Definition 5 (see [32]). Let $S : X \to X$ and let $\alpha, \eta : X\times X \to [0,+\infty)$ be two functions. One says that S is α-admis-sible mapping with respect to η if $x, y \in X$, $\alpha(x, y) \ge \eta(x, y) \Rightarrow \alpha(Sx, Sy) \ge \eta(Sx, Sy)$. Note that if one takes $\eta(x, y) = 1$, then this definition reduces to definition [33]. Also if we take $\alpha(x, y) = 1$, then one says that S is an η-subadmissible mapping.

2. Main Results

In this section, we prove fixed point theorems for α-admissible mappings with respect to η, satisfying modified $(\alpha-\eta)$-contractive condition in complete metric space.

Theorem 6. *Let (X, d) be a complete metric space and let S is α-admissible mappings with respect to η. Assume that there exists a function $\beta : [0,+\infty) \to [0,1)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ such that*

$$(d(Sx, Sy) + l)^{\alpha(x,Sx)\alpha(y,Sy)} \leq (\beta(d(x,y))\, d(x,y) + l)^{\eta(x,Sx)\eta(y,Sy)} \quad (3)$$

for all $x, y \in X$ where $l \geq 1$; then suppose that one of the following holds:

(i) *S is continuous;*

(ii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then*

$$\alpha(p, Sp) \geq \eta(p, Sp). \quad (4)$$

If there exists $x_0, x_1 \in X$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$, then S has a unique fixed point.

Proof. Let $x_0 \in X$ and define

$$x_{n+1} = Sx_n, \quad \forall n \geq 0. \quad (5)$$

We will assume that $x_n \neq x_{n+1}$ for each n. Otherwise, there exists an n such that $x_n = x_{n+1}$. Then $x_n = Sx_n$ and x_n is a fixed point of S. Since $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$ and S is α-admissible mapping with respect to η, we have

$$\alpha(x_1, x_2) = \alpha(Sx_0, Sx_1) \geq \eta(Sx_0, Sx_1) = \eta(x_1, x_2). \quad (6)$$

By continuing in this way, we have

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \quad (7)$$

for all $n \in \mathbb{N} \cup \{0\}$. From (7), we have

$$\alpha(x_{n-1}, x_n)\, \alpha(x_n, x_{n+1}) \geq \eta(x_{n-1}, x_n)\, \eta(x_n, x_{n+1}). \quad (8)$$

Thus applying the inequality (3), with $x = x_{k-1}$ and $y = x_k$, we obtain

$$\begin{aligned}(d(x_k, x_{k+1}) + l)^{\eta(x_{k-1},Sx_{k-1})\eta(x_k,Sx_k)} &= (d(Sx_{k-1}, Sx_k) + l)^{\eta(x_{k-1},Sx_{k-1})\eta(x_k,Sx_k)} \\ &\leq (d(Sx_{k-1}, Sx_k) + l)^{\alpha(x_{k-1},Sx_{k-1})\alpha(x_k,Sx_k)} \\ &\leq (\beta(d(x_{k-1}, x_k))\, d(x_{k-1}, x_k) + l)^{\eta(x_{k-1},Sx_{k-1})\eta(x_k,Sx_k)}\end{aligned} \quad (9)$$

which implies that

$$d(x_k, x_{k+1}) \leq \beta(d(x_{k-1}, x_k))\, d(x_{k-1}, x_k). \quad (10)$$

We suppose that

$$d(x_k, x_{k+1}) \leq d(x_{k-1}, x_k). \quad (11)$$

Then we prove that $d(x_{k-1}, x_k) \to 0$. It is clear that $\{d(x_{k-1}, x_k)\}$ is a decreasing sequence. Therefore, there exists some positive number ϱ such that $\lim_{n\to\infty} d(x_k, x_{k+1}) = \varrho$. Now we will prove that $\varrho = 0$. From (10), we have

$$\frac{d(x_k, x_{k+1})}{d(x_{k-1}, x_k)} \leq \beta(d(x_{k-1}, x_k)) \leq 1. \quad (12)$$

Now by taking limit $k \to \infty$, we have

$$\begin{gathered} 1 = \frac{d}{d} = \frac{\lim_{k\to\infty} d(x_k, x_{k+1})}{\lim_{k\to\infty} d(x_{k-1}, x_k)} \leq \beta(d(x_{k-1}, x_k)) \leq 1, \\ \lim_{k\to\infty} \beta(d(x_{k-1}, x_k)) = 1. \end{gathered} \quad (13)$$

By using property of β function, we have $\lim_{k\to\infty} d(x_{k-1}, x_k) = 0$. Thus

$$\lim_{k\to\infty} d(x_k, x_{k+1}) = 0. \quad (14)$$

Now we prove that sequence $\{x_n\}$ is Cauchy sequence. Suppose on contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that, for all positive integers k, we have $n_k > m_k > k$,

$$\begin{gathered} d(x_{m_k}, x_{n_k}) \geq \epsilon, \\ d(x_{m_k}, x_{n_{k-1}}) < \epsilon. \end{gathered} \quad (15)$$

By the triangle inequality, we have

$$\begin{aligned}\epsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) \\ &< \epsilon + d(x_{n_{k-1}}, x_{n_k})\end{aligned} \quad (16)$$

for all $k \in \mathbb{N}$. Now taking limit as $k \to +\infty$ in (16) and using (14), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \quad (17)$$

Again using triangle inequality, we have

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{n_{k+1}}) \\ &\quad + d(x_{n_{k+1}}, x_{n_k}), \\ d(x_{m_{k+1}}, x_{n_{k+1}}) &\leq d(x_{m_{k+1}}, x_{m_k}) + d(x_{m_k}, x_{n_k}) \\ &\quad + d(x_{n_k}, x_{n_{k+1}}). \end{aligned} \quad (18)$$

Taking limit as $k \to +\infty$ and using (14) and (17), we obtain

$$\lim_{k\to+\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon. \quad (19)$$

By using (3), (17), and (19), we have

$$\begin{aligned}&\left(d\left(x_{m_{k+1}},x_{n_{k+1}}\right)+l\right)^{\eta(x_{m_k},Sx_{m_k})\eta(x_{n_k},Sx_{n_k})}\\&\quad\le\left(d\left(x_{m_{k+1}},x_{n_{k+1}}\right)+l\right)^{\alpha(x_{m_k},Sx_{m_k})\alpha(x_{n_k},Sx_{n_k})}\\&\quad\le\left(d\left(Sx_{m_k},Tx_{n_k}\right)+l\right)^{\alpha(x_{m_k},Sx_{m_k})\alpha(x_{n_k},Sx_{n_k})}\\&\quad\le\left(\beta\left(d\left(x_{m_k},x_{n_k}\right)\right)d\left(x_{m_k},x_{n_k}\right)+l\right)^{\eta(x_{m_k},Sx_{m_k})\eta(x_{n_k},Sx_{n_k})}\end{aligned}\tag{20}$$

which implies that

$$d\left(x_{m_{k+1}},x_{n_{k+1}}\right)\le\beta\left(d\left(x_{m_k},x_{n_k}\right)\right)d\left(x_{m_k},x_{n_k}\right).\tag{21}$$

Therefore, we have

$$\frac{d\left(x_{m_{k+1}},x_{n_{k+1}}\right)}{d\left(x_{m_k},x_{n_k}\right)}\le\beta\left(d\left(x_{m_k},x_{n_k}\right)\right)\le1.\tag{22}$$

Now taking limit as $k\to+\infty$ in (22), we get

$$\lim_{n\to\infty}\beta\left(d\left(x_{m_k},x_{n_k}\right)\right)=1.\tag{23}$$

Hence $\lim_{k\to\infty}d(x_{m_k},x_{n_k})=0<\epsilon$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete so there exists $p\in X$ such that $x_n\to p$. Now we prove that $p=Sp$. Suppose (i) holds; that is, S is continuous, so we get

$$Sp=S\lim_{n\to\infty}x_n=\lim_{n\to\infty}Sx_n=\lim_{n\to\infty}x_{n+1}=p.\tag{24}$$

Thus $p=Sp$. Now we suppose that (ii) holds. Since

$$\alpha\left(x_n,x_{n+1}\right)\ge\eta\left(x_n,x_{n+1}\right)\tag{25}$$

for all $n\in\mathbb{N}\cup\{0\}$. By the hypotheses of (ii), we have

$$\alpha\left(p,Sp\right)\alpha\left(x_k,Sx_k\right)\ge\eta\left(p,Sp\right)\eta\left(x_k,Sx_k\right).\tag{26}$$

Using the triangle inequality and (3), we have

$$\begin{aligned}&\left(d\left(Sp,x_{k+1}\right)+l\right)^{\eta(p,Sp)\eta(x_k,Sx_k)}\\&\quad=\left(d\left(Sp,Sx_k\right)+l\right)^{\eta(p,Sp)\eta(x_k,Sx_k)}\\&\quad\le\left(d\left(Sp,Sx_k\right)+l\right)^{\alpha(p,Sp)\alpha(x_k,Sx_k)}\\&\quad\le\left(\beta\left(d\left(p,x_k\right)\right)d\left(p,x_k\right)+l\right)^{\eta(p,Sp)\eta(x_k,Sx_k)}\end{aligned}\tag{27}$$

which implies that

$$d\left(Sp,x_{k+1}\right)\le\beta\left(d\left(p,x_k\right)\right)d\left(p,x_k\right).\tag{28}$$

Letting $k\to\infty$ then we have $d(p,Sp)=0$. Thus $p=Sp$. Let there exists q to be another fixed point of $Sq\in X$, s.t $q=Sq$;

$$\begin{aligned}&(d(p,q)+l)^{\eta(p,Sp)\eta(q,Sq)}\\&\quad=\left(d\left(Sp,Sq\right)+l\right)^{\eta(p,Sp)\eta(q,Sq)}\\&\quad\le\left(d\left(Sp,Sq\right)+l\right)^{\alpha(p,Sp)\alpha(q,Sq)}\\&\quad\le(\beta(d(p,q))d(p,q)+l)^{\eta(p,Sp)\eta(q,Sq)}\end{aligned}\tag{29}$$

which implies that

$$d\left(p,q\right)+l\le\beta\left(d\left(p,q\right)\right)d\left(p,q\right)+l.\tag{30}$$

By the property of β function, $\beta(d(p,q))=1$, implies $d(p,q)=0$; then we have $p=q$. Hence S has a unique fixed point. □

If $\eta(x,y)=1$ in Theorem 6, we get the following corollary.

Corollary 7 (see [17]). *Let (X,d) be a complete metric space and let S be α-admissible mapping. Assume that there exists a function $\beta:[0,+\infty)\to[0,1)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n)\to1$ implies $t_n\to0$ such that*

$$\left(d\left(Sx,Sy\right)+l\right)^{\alpha(x,Sx)\alpha(y,Sy)}\le\beta\left(d\left(x,y\right)\right)d\left(x,y\right)+l,\tag{31}$$

for all $x,y\in X$, where $l\ge1$. Suppose that either

(i) *S is continuous, or*

(ii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\ge1$ for all $n\in\mathbb{N}\cup\{0\}$ and $x_n\to p\in X$ as $n\to+\infty$, then*

$$\alpha\left(p,Sp\right)\ge1.\tag{32}$$

If there exists $x_0,x_1\in X$ such that $\alpha(x_0,x_1)\ge1$; then S has a fixed point.

If $\alpha(x,y)=1$ in Theorem 6, we get the following corollary.

Corollary 8. *Let (X,d) be a complete metric space and let S be η-subadmissible mapping. Assume that there exists a function $\beta:[0,+\infty)\to[0,1)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n)\to1$ implies $t_n\to0$ such that*

$$\left(d\left(Sx,Sy\right)+l\right)\le\left(\beta\left(d\left(x,y\right)\right)d\left(x,y\right)+l\right)^{\eta(x,Sx)\eta(y,Sy)}\tag{33}$$

for all $x,y\in X$ where $l\ge1$; then suppose that one of the following holds:

(i) *S is continuous;*

(ii) *if $\{x_n\}$ is a sequence in X such that $\eta(x_n,x_{n+1})\le1$ for all $n\in\mathbb{N}\cup\{0\}$ and $x_n\to p\in X$ as $n\to+\infty$, then*

$$\eta\left(p,Sp\right)\le1.\tag{34}$$

If there exists $x_0,x_1\in X$ such that $\eta(x_0,x_1)\le1$, then S has a fixed point.

Example 9. Let $X=[0,\infty)$ with usual metric $d(x,y)=|x-y|$ for all $x,y\in X$ and $S:X\to X$, $\alpha:X\times X\to[0,\infty)$ and $\beta:[0,+\infty)\to[0,1]$ for all $x,y\in X$ be defined by

$$\begin{aligned}&Sx=\begin{cases}0&\text{if } x\in[0,1]\\\sqrt{x}&\text{if } x\in(1,5]\end{cases},\\&\alpha\left(x,y\right)=\begin{cases}1&\text{if } x\ge y\\0&\text{if } x<y\end{cases},\\&\beta\left(t\right)=\frac{1}{\sqrt{t}},\quad\beta\left(0\right)\in[0,1].\end{aligned}\tag{35}$$

We prove that Corollary 7 can be applied to S. Let $x, y \in X$; clearly $Sx \leq x$ and $Sy \leq y$, then S of α-admissible mapping $\alpha(x, y) \geq 1$, and $\alpha(x, Sx) \geq 1$, $\alpha(y, Sy) \geq 1$, and $\alpha(x, Sx)\alpha(y, Sy) \geq 1$ imply that

$$\begin{aligned}
&(d(Sx, Sy) + l)^{\alpha(x,Sx)\alpha(y,Sy)} \\
&\quad = Sx - Sy + l = \sqrt{x} - \sqrt{y} + l \leq \frac{x - y}{\sqrt{x} + \sqrt{y}} + l \\
&\quad \leq \frac{2(x - y)}{3\sqrt{x - y}} + l = \beta(d(x, y))(d(x, y)) + l.
\end{aligned} \tag{36}$$

If $\alpha(x, Sx)\alpha(y, Sy) = 0$, then we have

$$\begin{aligned}
&(d(Sx, Sy) + l)^{\alpha(x,Sx)\alpha(y,Sy)} \\
&\quad = 1 \leq \beta(d(x, y))(d(x, y)) + l.
\end{aligned} \tag{37}$$

Let $x = 5$ and $y = 2$; then

$$\begin{aligned}
&d(S5, S2)^{\alpha(5,S5)\alpha(3,S3)} \\
&\quad = 0.8218 \\
&\quad \leq \beta(d(5, 3))(d(5, 3)) \\
&\quad = 1.4142.
\end{aligned} \tag{38}$$

Theorem 10. *Let (X, d) be a complete metric space and let S be α-admissible mappings with respect to η. Assume that there exists a function $\beta : [0, +\infty) \to [0, 1)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ such that*

$$\begin{aligned}
&\alpha(x, Sx)\alpha(y, Sy) d(Sx, Sy) \\
&\quad \leq \eta(x, Sx)\eta(y, Sy)\beta(d(x, y)) d(x, y)
\end{aligned} \tag{39}$$

for all $x, y \in X$; then suppose that one of the following holds:

(i) *S is continuous;*

(ii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then*

$$\alpha(p, Sp) \geq \eta(p, Sp). \tag{40}$$

If there exists $x_0, x_1 \in X$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$, then S has a fixed point.

Proof. Let $x_0 \in X$ and define

$$x_{n+1} = Sx_n, \quad \forall n \geq 0. \tag{41}$$

We will assume that $x_n \neq x_{n+1}$ for each n. Otherwise, there exists an n such that $x_n = x_{n+1}$. Then $x_n = Sx_n$ and x_n is a fixed point of S. Since $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$ and S is α-admissible mapping with respect to η, we have

$$\begin{aligned}
\alpha(x_1, x_2) &= \alpha(Sx_0, Sx_1) \\
&\geq \eta(Sx_0, Sx_1) = \eta(x_1, x_2).
\end{aligned} \tag{42}$$

By continuing in this way, we have

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \tag{43}$$

for all $n \in \mathbb{N} \cup \{0\}$. From (43), we have

$$\alpha(x_{n-1}, x_n)\alpha(x_n, x_{n+1}) \geq \eta(x_{n-1}, x_n)\eta(x_n, x_{n+1}). \tag{44}$$

Thus applying the inequality (39), with $x = x_{k-1}$ and $y = x_k$, we obtain

$$\begin{aligned}
&\eta(x_{k-1}, Sx_{k-1})\eta(x_k, Sx_k) d(x_k, x_{k+1}) \\
&\quad = \eta(x_{k-1}, Sx_{k-1})\eta(x_k, Sx_k) d(Sx_{k-1}, Sx_k) \\
&\quad \leq \alpha(x_{k-1}, Sx_{k-1})\alpha(x_k, Sx_k) d(Sx_{k-1}, Sx_k) \\
&\quad \leq \eta(x_{k-1}, Sx_{k-1})\eta(x_k, Sx_k)\beta(d(x_{k-1}, x_k)) \\
&\qquad \times d(x_{k-1}, x_k)
\end{aligned} \tag{45}$$

which implies that

$$d(x_k, x_{k+1}) \leq \beta(d(x_{k-1}, x_k)) d(x_{k-1}, x_k). \tag{46}$$

We suppose that

$$d(x_k, x_{k+1}) \leq d(x_{k-1}, x_k). \tag{47}$$

Then we prove that $d(x_{k-1}, x_k) \to 0$. It is clear that $\{d(x_{k-1}, x_k)\}$ is a decreasing sequence. Therefore, there exists some positive number ϱ such that $\lim_{n\to\infty} d(x_k, x_{k+1}) = \varrho$. Now we will prove that $\varrho = 0$. From (47), we have

$$\frac{d(x_k, x_{k+1})}{d(x_{k-1}, x_k)} \leq \beta(d(x_{k-1}, x_k)) \leq 1. \tag{48}$$

Now by taking limit $k \to \infty$, we have

$$\begin{aligned}
1 = \frac{d}{d} = \frac{\lim_{k\to\infty} d(x_k, x_{k+1})}{\lim_{k\to\infty} d(x_{k-1}, x_k)} &\leq \beta(d(x_{k-1}, x_k)) \leq 1, \\
\lim_{k\to\infty} \beta(d(x_{k-1}, x_k)) &= 1.
\end{aligned} \tag{49}$$

By using property of β function, we have $\lim_{k\to\infty} d(x_{k-1}, x_k) = 0$. Thus

$$\lim_{k\to\infty} d(x_k, x_{k+1}) = 0. \tag{50}$$

Now we prove that sequence $\{x_n\}$ is Cauchy sequence. Suppose on contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ such that, for all positive integers k, we have $n_k > m_k > k$,

$$\begin{aligned}
d(x_{m_k}, x_{n_k}) &\geq \epsilon, \\
d(x_{m_k}, x_{n_{k-1}}) &< \epsilon.
\end{aligned} \tag{51}$$

By the triangle inequality, we have

$$\begin{aligned}
\epsilon &\leq d(x_{m_k}, x_{n_k}) \\
&\leq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) \\
&< \epsilon + d(x_{n_{k-1}}, x_{n_k})
\end{aligned} \tag{52}$$

for all $k \in \mathbb{N}$. Now taking limit as $k \to +\infty$ in (52) and using (50), we have

$$\lim_{k\to\infty} d\left(x_{m_k}, x_{n_k}\right) = \epsilon. \tag{53}$$

Again using triangle inequality, we have

$$\begin{aligned} d\left(x_{m_k}, x_{n_k}\right) &\le d\left(x_{m_k}, x_{m_{k+1}}\right) + d\left(x_{m_{k+1}}, x_{n_{k+1}}\right) \\ &\quad + d\left(x_{n_{k+1}}, x_{n_k}\right), \\ d\left(x_{m_{k+1}}, x_{n_{k+1}}\right) &\le d\left(x_{m_{k+1}}, x_{m_k}\right) + d\left(x_{m_k}, x_{n_k}\right) \\ &\quad + d\left(x_{n_k}, x_{n_{k+1}}\right). \end{aligned} \tag{54}$$

Taking limit as $k \to +\infty$ and using (50) and (53), we obtain

$$\lim_{k\to+\infty} d\left(x_{m_{k+1}}, x_{n_{k+1}}\right) = \epsilon. \tag{55}$$

By using (39), (53), and (55), we have

$$\begin{aligned} &\eta\left(x_{m_k}, Sx_{m_k}\right)\eta\left(x_{n_k}, Sx_{n_k}\right) d\left(x_{m_{k+1}}, x_{n_{k+1}}\right) \\ &\quad\le \alpha\left(x_{m_k}, Sx_{m_k}\right)\alpha\left(x_{n_k}, Sx_{n_k}\right) d\left(x_{m_{k+1}}, x_{n_{k+1}}\right) \\ &\quad\le \alpha\left(x_{m_k}, Sx_{m_k}\right)\alpha\left(x_{n_k}, Sx_{n_k}\right) d\left(Sx_{m_k}, Tx_{n_k}\right) \\ &\quad\le \eta\left(x_{m_k}, Sx_{m_k}\right)\eta\left(x_{n_k}, Sx_{n_k}\right)\beta\left(d\left(x_{m_k}, x_{n_k}\right)\right) \\ &\qquad\times d\left(x_{m_k}, x_{n_k}\right) \end{aligned} \tag{56}$$

which implies that

$$d\left(x_{m_{k+1}}, x_{n_{k+1}}\right) \le \beta\left(d\left(x_{m_k}, x_{n_k}\right)\right) d\left(x_{m_k}, x_{n_k}\right). \tag{57}$$

Therfore, we have

$$\frac{d\left(x_{m_{k+1}}, x_{n_{k+1}}\right)}{d\left(x_{m_k}, x_{n_k}\right)} \le \beta\left(d\left(x_{m_k}, x_{n_k}\right)\right) \le 1. \tag{58}$$

Now taking limit as $k \to +\infty$ in (58), we get

$$\lim_{n\to\infty}\beta\left(d\left(x_{m_k}, x_{n_k}\right)\right) = 1. \tag{59}$$

Hence $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = 0 < \epsilon$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete so there exists $p \in X$ such that $x_n \to p$. Now we prove that $p = Sp$. Suppose (i) holds; that is, S is continuous, so we get

$$Sp = S\lim_{n\to\infty} x_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} x_{n+1} = p. \tag{60}$$

Thus $p = Sp$. Now we suppose that (ii) holds. Since

$$\alpha\left(x_n, x_{n+1}\right) \ge \eta\left(x_n, x_{n+1}\right) \tag{61}$$

for all $n \in \mathbb{N} \cup \{0\}$. By the hypotheses of (ii), we have

$$\alpha\left(p, Sp\right)\alpha\left(x_k, Sx_k\right) \ge \eta\left(p, Sp\right)\eta\left(x_k, Sx_k\right). \tag{62}$$

Using the triangle inequality and (39), we have

$$\begin{aligned} &\eta\left(p, Sp\right)\eta\left(x_k, Sx_k\right) d\left(Sp, x_{k+1}\right) \\ &\quad= \eta\left(p, Sp\right)\eta\left(x_k, Sx_k\right) d\left(Sp, Sx_k\right) \\ &\quad\le \alpha\left(p, Sp\right)\alpha\left(x_k, Sx_k\right) d\left(Sp, Sx_k\right) \\ &\quad\le \eta\left(p, Sp\right)\eta\left(x_k, Sx_k\right)\beta\left(d\left(p, x_k\right)\right) d\left(p, x_k\right), \end{aligned} \tag{63}$$

which implies that

$$d\left(Sp, x_{k+1}\right) \le \beta\left(d\left(p, x_k\right)\right) d\left(p, x_k\right). \tag{64}$$

Letting $k \to \infty$, we have $d(p, Sp) = 0$. Thus $p = Sp$. Let there exists q to be another fixed point of $Sq \in X$, s.t $q = Sq$;

$$\begin{aligned} &\eta\left(p, Sp\right)\eta\left(q, Sq\right) d\left(Sp, Sq\right) \\ &\quad\le \alpha\left(p, Sp\right)\alpha\left(q, Sq\right) d\left(Sp, Sq\right) \\ &\quad\le \eta\left(p, Sp\right)\eta\left(q, Sq\right)\beta\left(d\left(p, q\right)\right) d\left(p, q\right), \end{aligned} \tag{65}$$

implies

$$d\left(Sp, Sq\right) \le \beta\left(d\left(p, q\right)\right) d\left(p, q\right). \tag{66}$$

By the property of β function, $\beta(d(p, q)) = 1$ implies $d(p, q) = 0$; then we have $p = q$. Hence S has a unique fixed point. □

If $\eta(x, y) = 1$ in Theorem 10, we get the following corollary.

Corollary 11 (see [17]). *Let (X, d) be a complete metric space and let S be α-admissible mapping. Assume that there exists a function $\beta : [0, +\infty) \to [0, 1)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ such that*

$$\alpha\left(x, Sx\right)\alpha\left(y, Sy\right) d\left(Sx, Sy\right) \le \beta\left(d\left(x, y\right)\right) d\left(x, y\right) \tag{67}$$

for all $x, y \in X$. Suppose that either

(i) *S is continuous, or*

(ii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then*

$$\alpha\left(p, Sp\right) \ge 1. \tag{68}$$

If there exists $x_0, x_1 \in X$ such that $\alpha(x_0, x_1) \ge 1$, then S has a fixed point. Our results are more general than those in [17, 32, 33] and improve several results existing in the literature.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors contributed equally and significantly to writing this paper. All authors read and approved the final paper.

Acknowledgments

Marwan Amin Kutbi gratefully acknowledges the support from the Deanship of Scientific Research (DSR) at King Abdulaziz University (KAU) during this research. The authors thank the editor and the referees for their valuable comments and suggestions which improved greatly the quality of this paper.

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