Research Article

Discontinuous Mixed Covolume Methods for Linear Parabolic Integrodifferential Problems

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The semidiscrete and fully discrete discontinuous mixed covolume schemes for the linear parabolic integrodifferential problems on triangular meshes are proposed. The error analysis of the semidiscrete and fully discrete discontinuous mixed covolume scheme is presented and the optimal order error estimate in discontinuous $H(\mathrm{div})$ and first-order error estimate in L^2 are obtained with the lowest order Raviart-Thomas mixed element space.

1. Introduction

We consider the following linear parabolic integrodifferential problems

$$q_t(\mathbf{x}, t) - \nabla \cdot \left\{ A(\mathbf{x}, t) \nabla q(\mathbf{x}, t) + \int_0^t B(\mathbf{x}, t, \tau) \nabla q(\mathbf{x}, t, \tau) \, d\tau \right\}$$

$$= f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times (0, T],$$

$$q(\mathbf{x}, 0) = q_0(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

$$q(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times (0, T],$$

$$(1)$$

where $\Omega \in \mathbf{R}^2$ is a bounded convex polygonal domain with the boundary $\partial\Omega$, $\mathbf{x} = (x, y)$, q is an unknown function, A is a symmetric, bounded matrix function, B is a bounded matrix function, $q_0(\mathbf{x})$ and $f(\mathbf{x}, t)$ are known functions, and $f(\mathbf{x}, t) \in L^2(\Omega)$. Furthermore, we assume that the matrix $M = A^{-1}$ and M_t are two bounded matrix functions.

Here and in what follows, we will not write the independent $\mathbf{x}$, t for any functions unless it is necessary.

For the parabolic integrodifferential problems many numerical methods were proposed, such as the finite element methods in [1], H1-Galerkin mixed finite element methods in [2], finite element approximation with a weakly singular kernel in [3], expanded mixed finite element methods in [4], and expanded mixed covolume method in [5].

Because the discontinuous Galerkin method has the advantages of a high order of accuracy, high parallelizability, localizability, and easy handling of complicated geometries it has been used to solve elliptic problems and convection-diffusion problems by many researchers; see [6–11]. The discontinuous finite volume method in recent years was used to solve elliptic problems, Stokes problems, and parabolic problems in [12–14]. In [15] the discontinuous mixed covolume methods for elliptic problems were demonstrated by Yang and Jiang. Zhu and Jiang extended the discontinuous mixed covolume methods to parabolic problems in [16]. The goal of this paper is to extend the discontinuous mixed covolume methods in the linear parabolic integrodifferential problems.

The rest of this paper is organized as follows. In Section 2, some notations are introduced and the semidiscrete and the fully discrete discontinuous mixed covolume schemes for the integrodifferential equations (1) are established. In Section 3, the existence and uniqueness for the semidiscrete and the fully discrete discontinuous mixed covolume approximations are proven. We defined a generalized discontinuous mixed covolume elliptic projection in Section 4. We prove the optimal error estimations in both H^1 and L^2 norms of semidiscrete and the fully discrete discontinuous mixed covolume methods in Sections 5 and 6.

Throughout this paper, the letter C denotes a generic positive constant independent of the mesh parameter and may stand for different values at its different appearances.

2. Discontinuous Mixed Covolume Formulation

Let $\mathbf{w} = -A\nabla q$ and $D = BM$, and rewrite (1) as the system of first-order partial differential equations

$$M\mathbf{w} + \nabla q = 0, \quad (\mathbf{x}, t) \in \Omega \times (0, T],$$

$$q_t + \nabla \cdot \mathbf{w} + \int_0^t \nabla \cdot (D\mathbf{w}) \, d\tau = f, \quad (\mathbf{x}, t) \in \Omega \times (0, T],$$

$$q(\mathbf{x}, 0) = q_0(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

$$q(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times (0, T]. \tag{2}$$

We will use the standard definitions for the Sobolev spaces $H^s(K)$ and their associated inner products $(\cdot, \cdot)_{s,K}$, norms $\|\cdot\|_{s,K}$ and seminorms $|\cdot|_{s,K}$ in [17]. The space $H^0(K)$ coincides with $L^s(K)$, in which the norm and the inner product are denoted by $\|\cdot\|_K$ and $(\cdot, \cdot)_K$, respectively.

Let $\mathcal{T}_h = \{K\}$ be a triangulation of the domain Ω. As usual, we assume the triangles K to be shape-regular. For a given triangulation $\mathcal{T}_h$, we construct a dual mesh $\mathcal{T}_h^*$ based upon the primal partition $\mathcal{T}_h$. Each triangle in $\mathcal{T}_h$ can be divided into three subtriangles by connecting the barycenter Q of the triangle to their corner nodes A_i $(i = 1, 2, 3)$. Then we define the dual partition $\mathcal{T}_h^*$ to be the union of the triangles shown in Figure 1. Let $P_k(T)$ consist of all the polynomials functions of degree less than or equal to k defined on T. We define the finite-dimensional trial function space for velocity on $\mathcal{T}_h$ by

$$U_h := \left\{ \mathbf{u} \in L^2(\Omega)^2 : \mathbf{u}|_K \in P_1(K)^2, \forall K \in \mathcal{T}_h \right\}. \tag{3}$$

Define the finite-dimensional test function space V_h for velocity associated with the dual partition $\mathcal{T}_h^*$ as

$$V_h := \left\{ \mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_T \in P_0(T)^2, \forall T \in \mathcal{T}_h^* \right\}. \tag{4}$$

Let H_h be the finite-dimensional space for pressure

$$H_h := \left\{ p \in L^2(\Omega) : p|_K \in P_0(K), \forall K \in \mathcal{T}_h \right\}. \tag{5}$$

Let Γ denote the union of the boundary of the triangles K of $\mathcal{T}_h$ and $\Gamma^0 : \Gamma \backslash \partial\Omega$. The traces of functions in V_h and H_h are double valued on Γ^0. Let e be an interior edge shared by two triangles K_1 and K_2 in T_h. Define the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ on e pointing exterior to K_1 and K_2, respectively. Next, we introduce some traces operators that we will use in our numerical formulation. We define the average $\{\cdot\}$ and jump $[\cdot]$ on e for scalar q and vector $\mathbf{v}$, respectively:

$$\{p\} = \frac{1}{2}\left(p|_{\partial K_1} + p|_{\partial K_2}\right), \qquad [p] = p|_{\partial K_1}\mathbf{n}_1 + p|_{\partial K_2}\mathbf{n}_2,$$

$$\{\mathbf{u}\} = \frac{1}{2}\left(\mathbf{u}|_{\partial K_1} + \mathbf{u}|_{\partial K_2}\right), \qquad [\mathbf{u}] = \mathbf{u}|_{\partial K_1} \cdot \mathbf{n}_1 + \mathbf{u}|_{\partial K_2} \cdot \mathbf{n}_2; \tag{6}$$

if e is an edge on the boundary of Ω, we set

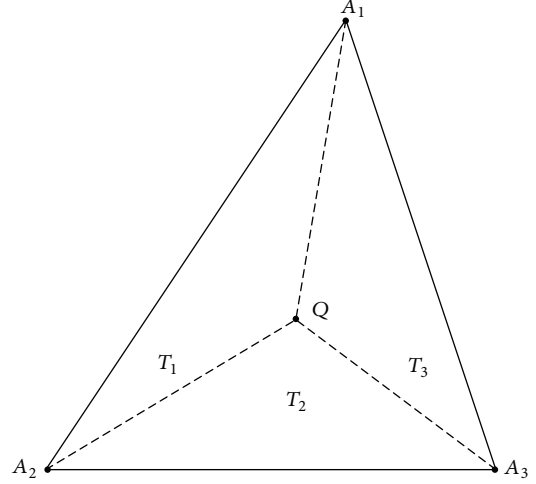$$\{p\} = p, \qquad [\mathbf{u}] = \mathbf{u} \cdot \mathbf{n}, \tag{7}$$

FIGURE 1: Element $T \in \mathcal{T}_h^*$ for a triangular mesh.

where $\mathbf{n}$ is the outward unit normal. We do not require either of the quantities $[p]$ or $\{\mathbf{u}\}$ on boundary edges, and we leave them undefined.

Multiplying the first and second equations in system (2) by $\mathbf{v} \in V_h$ and $p \in H_h$, respectively, and using the integration by parts formula in the equation, we have

$$\sum_{T \in \mathcal{T}_h^*} \int_T M\mathbf{w} \cdot \mathbf{v} \, dx + \sum_{T \in \mathcal{T}_h^*} \int_{\partial T} q\mathbf{v} \cdot \mathbf{n} \, ds = 0, \quad \forall \mathbf{v} \in V_h,$$

$$\sum_{K \in \mathcal{T}_h} \int_K q_t p \, dx + \sum_{K \in \mathcal{T}_h} \int_K \nabla \cdot \mathbf{w} p \, dx$$

$$+ \sum_{K \in \mathcal{T}_h} \int_K \int_0^t \nabla \cdot (D\mathbf{w}) \, p \, d\tau \, dx = (f, p), \quad \forall p \in H_h, \tag{8}$$

where $\mathbf{n}$ is the outward normal vector on ∂T. Let $T_j \in \mathcal{T}_h^*$ $(j = 1, 2, 3)$ be the triangles in $K \in \mathcal{T}_h$. Then we have

$$\sum_{T \in \mathcal{T}_h^*} \int_{\partial T} q\mathbf{v} \cdot \mathbf{n} \, ds = \sum_{K \in \mathcal{T}_h} \sum_{j=1}^3 \int_{A_{j+1}QA_j} q\mathbf{v} \cdot \mathbf{n} \, ds$$

$$+ \sum_{K \in \mathcal{T}_h} \int_{\partial K} q\mathbf{v} \cdot \mathbf{n} \, ds, \quad \forall \mathbf{v} \in V_h, \tag{9}$$

where $A_4 = A_1$. A straightforward computation gives

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} q\mathbf{v} \cdot \mathbf{n} \, ds$$

$$= \sum_{e \in \Gamma^0} \int_e [q] \cdot \{\mathbf{v}\} \, ds + \sum_{e \in \Gamma} \int_e \{q\} \cdot [\mathbf{v}] \, ds, \quad \forall \mathbf{v} \in V_h. \tag{10}$$

Let $\int_\Gamma q ds = \sum_{e \in \Gamma} \int_e q ds$. Using the above formula and the fact that $[q] = 0$ for $q \in H^1(\Omega)$ on Γ^0, (9) becomes

$$\sum_{T \in \mathcal{T}_h^*} \int_{\partial T} q \mathbf{v} \cdot \mathbf{n} ds$$

$$= \sum_{K \in \mathcal{T}_h} \sum_{j=1}^{3} \int_{A_{j+1} Q A_j} q \mathbf{v} \cdot \mathbf{n} ds + \sum_{e \in \Gamma} \int_e \{q\} [\mathbf{v}] ds, \quad \forall \mathbf{v} \in V_h. \tag{11}$$

Then, system (8) can be rewritten as in the following:

$$\sum_{T \in \mathcal{T}_h^*} \int_T M \mathbf{w} \cdot \mathbf{v} dx + \sum_{K \in \mathcal{T}_h} \sum_{j=1}^{3} \int_{A_{j+1} Q A_j} q \mathbf{v} \cdot \mathbf{n} ds$$

$$+ \sum_{e \in \Gamma} \int_e \{q\} [\mathbf{v}] ds = 0, \quad \forall \mathbf{v} \in V_h,$$

$$\sum_{K \in \mathcal{T}_h} \int_K q_t p dx + \sum_{K \in \mathcal{T}_h} \int_K \nabla \cdot \mathbf{w} p dx$$

$$+ \sum_{K \in \mathcal{T}_h} \int_K \int_0^t \nabla \cdot (D\mathbf{w}) p d\tau dx = (f, p), \quad \forall p \in H_h. \tag{12}$$

Let $U(h) = U_h + H^2(\Omega)^2$. Define a mapping $\gamma : U(h) \to V_h$ as

$$\gamma \mathbf{u}|_T = \frac{1}{h_e} \int_e \mathbf{u}|_T ds, \quad T \in \mathcal{T}_h^*, \tag{13}$$

where h_e is the length of the edge e. For $\mathbf{u} = (u_1, u_2) \in U(h)$, γu_i ($i = 1, 2$) is defined as

$$\gamma u_i|_T = \frac{1}{h_e} \int_e u_i|_T ds, \quad T \in \mathcal{T}_h^*, \ (i = 1, 2). \tag{14}$$

Then the system (12) is equivalent to

$$\sum_{T \in \mathcal{T}_h^*} \int_T M \mathbf{w} \cdot \gamma \mathbf{v} dx + \sum_{K \in \mathcal{T}_h} \sum_{j=1}^{3} \int_{A_{j+1} Q A_j} q \gamma \mathbf{v} \cdot \mathbf{n} ds$$

$$+ \sum_{e \in \Gamma} \int_e \{q\} [\gamma \mathbf{v}] ds = 0, \quad \forall \mathbf{v} \in U(h),$$

$$\sum_{K \in \mathcal{T}_h} \int_K q_t p dx + \sum_{K \in \mathcal{T}_h} \int_K \nabla \cdot \mathbf{w} p dx$$

$$+ \sum_{K \in \mathcal{T}_h} \int_K \int_0^t \nabla \cdot (D\mathbf{w}) p d\tau dx = (f, p), \quad \forall p \in H_h. \tag{15}$$

Let

$$a_0(\mathbf{u}, \mathbf{v}) := \sum_{T \in \mathcal{T}_h^*} \int_T M \mathbf{u} \cdot \gamma \mathbf{v} dx,$$

$$b(\mathbf{v}, p) := \sum_{K \in \mathcal{T}_h} \sum_{j=1}^{3} \int_{A_{j+1} Q A_j} p \gamma \mathbf{v} \cdot \mathbf{n} ds + \sum_{e \in \Gamma} \int_e \{p\} [\gamma \mathbf{v}] ds,$$

$$c_0(\mathbf{u}, p) := \sum_{K \in \mathcal{T}_h} \int_K \nabla \cdot \mathbf{u} p dx,$$

$$\tilde{c}_0(\mathbf{u}, p) := \sum_{K \in \mathcal{T}_h} \int_K \nabla \cdot (D\mathbf{u}) p dx. \tag{16}$$

Using the above bilinear forms, it is clear that system (15) can be rewritten as in the following:

$$a_0(\mathbf{w}, \mathbf{v}) + b(\mathbf{v}, q) = 0, \quad \forall \mathbf{v} \in U(h),$$

$$(q_t, p) + c_0(\mathbf{w}, p) + \int_0^t \tilde{c}_0(\mathbf{w}, p) d\tau = (f, p), \quad \forall p \in H_h. \tag{17}$$

In order to define our numerical schemes, we introduce the bilinear forms as follows:

$$a(\mathbf{u}, \mathbf{v}) := a_0(\mathbf{u}, \mathbf{v}) + \alpha \sum_{e \in \Gamma} \frac{1}{h_e} \int_e M [\mathbf{u}] [\mathbf{v}] ds,$$

$$c(\mathbf{u}, p) := c_0(\mathbf{u}, p) - \int_\Gamma \{p\} [\gamma \mathbf{u}] ds, \tag{18}$$

$$\tilde{c}(\mathbf{u}, p) := \tilde{c}_0(\mathbf{u}, p) - \int_\Gamma \{p\} [\gamma \mathbf{u}] ds,$$

where $\alpha > 0$ is a parameter to be determined later. For the exact solution $(\mathbf{w}, q)$ of system (2), we have

$$a_0(\mathbf{w}, \mathbf{u}) = a(\mathbf{w}, \mathbf{u}), \quad \forall \mathbf{u} \in U_h,$$

$$c_0(\mathbf{w}, p) = c(\mathbf{w}, p), \quad \forall p \in H_h,$$

$$\tilde{c}_0(\mathbf{w}, p) = \tilde{c}(\mathbf{w}, p), \quad \forall p \in H_h. \tag{19}$$

Therefore, it follows from (17) that

$$a(\mathbf{w}, \mathbf{v}) + b(\mathbf{v}, q) = 0, \quad \forall \mathbf{v} \in U_h,$$

$$(q_t, p) + c(\mathbf{w}, p) + \int_0^t \tilde{c}(\mathbf{w}, p) d\tau = (f, p), \quad \forall p \in H_h. \tag{20}$$

The discontinuous mixed covolume scheme for (2) reads as follows. Seek $(\mathbf{w}_h, q_h) \in U_h \times H_h$ such that

$$a(\mathbf{w}_h, \mathbf{v}) + b(\mathbf{v}, q_h) = 0, \quad \forall \mathbf{v} \in U_h,$$

$$(q_{ht}, p) + c(\mathbf{w}_h, p) + \int_0^t \tilde{c}(\mathbf{w}_h, p) d\tau = (f, p), \quad \forall p \in H_h, \tag{21}$$

where $q_h(0) = \tilde{q}_h(0)$, $\mathbf{w}_h(0) = \widetilde{\mathbf{w}}_h(0)$, $\tilde{q}_h(0)$, $\widetilde{\mathbf{w}}_h(0)$ will be given in Section 4.

Let $N > 0$ be a positive integer; let $0 = t^0 < t^1 < \cdots < t^j < \cdots < t^N = T$ be a subdivision of time. $t^j = j\Delta t$ $(0 \le j \le N)$, $\Delta t = T/N$. We use the backward Euler difference quotient

$$\partial_t q_h^j = \frac{q_h^j - q_h^{j-1}}{\Delta t} \quad (j = 1, 2, \ldots, N) \tag{22}$$

to approximate the differential quotient $\partial q_h^j / \partial t$ $(j = 1, 2, \ldots, N)$ and the numerical integration $\Delta t \sum_{k=0}^{j-1} \nabla \cdot (D^k \mathbf{w}_h^k)$ to approximate the integration $\int_0^{t_j} \nabla \cdot (D\mathbf{w}_h) d\tau$ $(j = 1, 2, \ldots, N)$ in the semidiscrete scheme; then we obtain the backward Euler fully discrete discontinuous mixed covolume scheme for the problem (1): find $(\mathbf{w}_h^j, q_h^j) \in U_h \times H_h$ $(j = 1, 2, \ldots, N)$, such that

$$a\left(\mathbf{w}_h^j, \mathbf{v}\right) + b\left(\mathbf{v}, q_h^j\right) = 0, \quad \forall \mathbf{v} \in U_h,$$

$$\left(\partial_t q_h^j, p\right) + c\left(\mathbf{w}_h^j, p\right) + \tilde{c}\left(\Delta t \sum_{k=0}^{j-1} \mathbf{w}_h^k, p\right) = (f, p), \tag{23}$$

$$\forall p \in H_h, \qquad j = 1, 2, \ldots, N,$$

where $q_h^0 = \tilde{q}_h(0)$, $\mathbf{w}_h^0 = \widetilde{\mathbf{w}}_h(0)$, $\tilde{q}_h(0)$, $\widetilde{\mathbf{w}}_h(0)$ will be given in Section 4.

We define the following norms for $\mathbf{u} \in U(h)$:

$$\|\|\mathbf{u}\|\|_{\text{div}}^2 = \|\mathbf{u}\|^2 + \|\nabla_h \cdot \mathbf{u}\|^2 + \sum_{e \in \Gamma} \frac{1}{h_e} \int_e [\mathbf{u}]^2 ds,$$

$$\|\|\mathbf{u}\|\|_1^2 = \|\mathbf{u}\|^2 + |\mathbf{u}|_{1,h}^2 + \sum_{e \in \Gamma} \frac{1}{h_e} \int_e [\mathbf{u}]^2 ds, \tag{24}$$

$$\|\|\mathbf{u}\|\|^2 = \|\|\mathbf{u}\|\|_1^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |\mathbf{u}|_{2,K}^2,$$

where $\nabla_h \cdot \mathbf{u}$ is the function whose restriction to each element $K \in \mathcal{T}_h$ is equal to $\nabla \cdot \mathbf{u}$ and $|\mathbf{u}|_{1,h}^2 = \sum_{K \in \mathcal{T}_h} |\mathbf{u}|_{1,K}^2$.

We will introduce some useful Lemmas; for more details, see [6].

Lemma 1. *For* $\mathbf{u}, \mathbf{v} \in U(h)$, *we have*

$$a(\mathbf{u}, \mathbf{v}) \le C \|\|\mathbf{u}\|\|_{\text{div}} \|\|\mathbf{v}\|\|_{\text{div}}. \tag{25}$$

Lemma 2. *For* $(\mathbf{u}, p) \in U(h) \times L^2(\Omega)$, *we have*

$$b(\mathbf{u}, p) = -c(\mathbf{u}, p). \tag{26}$$

Lemma 3. *For* $(\mathbf{u}, p) \in U(h) \times L^2(\Omega)$, *we have*

$$b(\mathbf{u}, p) \le C \|\|\mathbf{u}\|\| \left(\|p\| + \left(\sum_{K \in \mathcal{T}_h} h_K^2 |p|_{1,K}^2 \right)^{1/2} \right); \tag{27}$$

if $(\mathbf{u}, p) \in U_h \times H_h$, *then*

$$b(\mathbf{u}, p) \le C \|\|\mathbf{u}\|\| \cdot \|p\|,$$

$$\tilde{c}(\mathbf{u}, p) \le C \|\|\mathbf{u}\|\| \cdot \|p\|. \tag{28}$$

Lemma 4. *Let* $Z_h = \{\mathbf{u} \mid \mathbf{u} \in U_h, c(\mathbf{u}, p) = 0, \forall p \in H_h\}$; *for any* $\mathbf{u} \in Z_h$, *there is a constant* C_0 *independent of h such that, for* α *is large enough,*

$$a(\mathbf{u}, \mathbf{u}) \ge C_0 \|\|\mathbf{u}\|\|_{\text{div}}^2. \tag{29}$$

3. Existence and Uniqueness for Discontinuous Mixed Covolume Approximations

In this section, we prove the discontinuous mixed covolume formulation has a unique solution in the finite element space $U_h \times H_h$.

Theorem 5. *The semidiscrete discontinuous mixed covolume scheme* (21) *has a unique solution in the space* $U_h \times H_h$.

Proof. Only prove that homogenous equation

$$a(\mathbf{w}_h, \mathbf{u}) + b(\mathbf{u}, q_h) = 0, \quad \forall \mathbf{u} \in U_h,$$

$$(q_{ht}, p) + c(\mathbf{w}_h, p) + \int_0^t \tilde{c}_0(\mathbf{w}_h, p) = 0, \quad \forall p \in H_h, \tag{30}$$

$$q_h(0) = 0, \qquad \mathbf{w}_h(0) = 0,$$

of (21) exists unique zero solution since the number of unknowns is the same as number of line equations.

By letting $\mathbf{v} = \mathbf{w}_h$ in the first formula of (30) and $p = q_h$ in the second formula of (30), using Lemma 2, the sum of (30) gives

$$a(\mathbf{w}_h, \mathbf{w}_h) + (q_{ht}, q_h) = -\int_0^t \tilde{c}_0(\mathbf{w}_h, q_h). \tag{31}$$

Using $(1/2)(d/dt)(q_h, q_h) = (q_{ht}, q_h)$ and Lemmas 3 and 4, we have that

$$C_0 \|\|\mathbf{w}_h\|\|_{\text{div}}^2 + \frac{1}{2} \frac{d}{dt} \|q_h\|^2$$

$$\le C \int_0^t \|\|\mathbf{w}_h\|\| \|q_h\| d\tau \le C \int_0^t \|\|\mathbf{w}_h\|\|_{\text{div}} \|q_h\| d\tau. \tag{32}$$

Using Hölder inequality and Gronwall Lemma, we get

$$C_0 \|\|\mathbf{w}_h\|\|_{\text{div}}^2 + \frac{1}{2} \frac{d}{dt} \|q_h\|^2 \le C \int_0^t \|q_h\|^2 d\tau. \tag{33}$$

Integrating the above formula, we get

$$2C_0 \int_0^t \|\|\mathbf{w}_h\|\|_{\text{div}}^2 d\tau + \|q_h\|^2$$

$$\le C \int_0^t \int_0^s \|q_h\|^2 d\tau \, ds \le C \int_0^t \|q_h\|^2 d\tau. \tag{34}$$

Then $\|q_h\| = 0$, $\|\|\mathbf{w}_h\|\|_{\text{div}} = 0$. So $q_h = 0$, $\mathbf{w}_h = \mathbf{0}$, $t \in (0, T]$. This completes the proof. $\square$

Theorem 6. *The fully discrete discontinuous mixed covolume method defined in* (23) *has a unique solution in the finite element space* $U_h \times H_h$ *if* Δt *is sufficiently small.*

Proof. Only prove that homogenous equation

$$a\left(\mathbf{w}_h^j, \mathbf{v}\right) + b\left(\mathbf{v}, q_h^j\right) = 0, \quad \forall \mathbf{v} \in U_h,$$

$$\left(\partial_t q_h^j, p\right) + c\left(\mathbf{w}_h^j, p\right) + \tilde{c}\left(\Delta t \sum_{k=0}^{j-1} \mathbf{w}_h^k, p\right) = 0, \quad (35)$$

$$\forall p \in H_h, \qquad j = 1, 2, \dots, N,$$

of (23) exists unique zero solution since the number of unknowns is the same as number of line equations.

By letting $\mathbf{v} = \mathbf{w}_h^j$ in the first formula of (35) and $p = q_h^j$ in the second formula of (35), using Lemma 2, the sum of (35) gives

$$a\left(\mathbf{w}_h^j, \mathbf{w}_h^j\right) + \left(\partial_t q_h^j, q_h^j\right) = -\tilde{c}\left(\Delta t \sum_{k=0}^{j-1} \mathbf{w}_h^k, q_h^j\right), \quad (36)$$

$$j = 1, 2, \dots, N.$$

Using Lemmas 3 and 4 and

$$\begin{aligned}
&\left(\partial_t q_h^j, q_h^j\right) \\
&= \frac{1}{\Delta t}\left(q_h^j - q_h^{j-1}, q_h^j\right) \\
&= \frac{1}{2\Delta t}\left[\left(q_h^j, q_h^j\right) - \left(q_h^{j-1}, q_h^{j-1}\right) + \left(q_h^j - q_h^{j-1}, q_h^j - q_h^{j-1}\right)\right] \\
&= \frac{1}{2\Delta t}\left[\left\|q_h^j\right\|^2 - \left\|q_h^{j-1}\right\|^2 + \left\|q_h^j - q_h^{j-1}\right\|^2\right] \\
&\geq \frac{1}{2\Delta t}\left[\left\|q_h^j\right\|^2 - \left\|q_h^{j-1}\right\|^2\right],
\end{aligned} \quad (37)$$

we have from (36) that

$$2C_0 \Delta t \left\|\mathbf{w}_h^j\right\|_{\text{div}}^2 + \left\|q_h^j\right\|^2 - \left\|q_h^{j-1}\right\|^2$$

$$\leq C\Delta t\left[(\Delta t)^2 \left\|\sum_{k=0}^{j-1}\mathbf{w}_h^k\right\|_{\text{div}}^2 + \left\|q_h^j\right\|^2\right], \quad j = 1, 2, \dots, N. \quad (38)$$

Adding the above inequality with j from 1 to i, using $q_h^0 = 0$ and the discrete Gronwall inequality, when Δt is sufficiently small, we have

$$2C_0 \Delta t \sum_{j=1}^{i} \left\|\mathbf{w}_h^j\right\|_{\text{div}}^2 + \left\|q_h^i\right\|^2 \leq 0 \quad (i = 1, 2, \dots, N). \quad (39)$$

Hence we have $\left\|q_h^i\right\|^2 = 0$ and $\left\|\mathbf{w}_h^i\right\|_{\text{div}}^2 = 0$ $(i = 1, 2, \dots, N)$; that is, $q_h^i = 0$ and $\mathbf{w}_h^i = \mathbf{0}$ $(i = 1, 2, \dots, N)$. This completes the proof. □

4. A Discontinuous Mixed Covolume Elliptic Projection

Define an operator π_K from $H^1(K)$ to $P_1(K)$ by requiring that, for any $\forall u \in H^1(K)$,

$$\int_{e_i} \pi_K u \, ds = \int_{e_i} u \, ds, \quad (i = 1, 2, 3), \quad (40)$$

where e_i $(i = 1, 2, 3)$ are the three sides of the element $K \in \mathcal{T}_h$. It was proved in [5] that

$$\left|\pi_K u - u\right|_{s,K} \leq h^{2-s}|u|_{2,K}, \quad \forall u \in H^2(K), \ (s = 0, 1, 2). \quad (41)$$

For any $\mathbf{u} \in H_0^1(\Omega)^2$, define $\Pi_1 \mathbf{u} \in U_h$ by

$$\left(\Pi_1 \mathbf{u}\right)_i\big|_K = \Pi_K u_i, \quad \forall K \in \mathcal{T}_h, \ (i = 1, 2). \quad (42)$$

Using the definition of Π_1 and integration by parts, we can show that

$$c\left(\mathbf{u} - \Pi_1 \mathbf{u}, p\right) = 0, \quad \forall p \in H_h. \quad (43)$$

It was proved in [6] that

$$\left\|\mathbf{w} - \Pi_1 \mathbf{w}\right\|_{\text{div}} \leq ch\|\mathbf{w}\|_2. \quad (44)$$

Let Π_2 be the projection from $L_0^2(\Omega)$ to the finite element space H_h.

Define a discontinuous mixed covolume elliptic projection by requiring that, finding $\widetilde{\mathbf{w}}_h, \tilde{q}_n : (0, t) \rightarrow U_h \times H_h$, such that

$$a\left(\mathbf{w} - \widetilde{\mathbf{w}}_h, \mathbf{v}\right) + b\left(\mathbf{v}, q - \tilde{q}_h\right) = 0, \quad \forall \mathbf{v} \in U_h,$$

$$c\left(\mathbf{w} - \widetilde{\mathbf{w}}_h, p\right) = 0, \quad \forall p \in H_h. \quad (45)$$

It was proved in [15] that the above formula has a unique solution and the error estimates in the following Theorem 7.

Theorem 7. *Let $(\widetilde{\mathbf{w}}_h, \tilde{q}_h) \in U_h \times H_h$ be the solution of (45) and $(\mathbf{w}, q) \in H^2(\Omega)^2 \times H^1(\Omega)$ the solution of (20). Then there exists a positive constant C independent of h such that*

$$\left\|\mathbf{w} - \widetilde{\mathbf{w}}_h\right\|_{\text{div}} + \left\|q - \tilde{q}_h\right\| \leq Ch\left(\|\mathbf{w}\|_2 + \|q\|_1\right). \quad (46)$$

Theorem 8. *Let $(\widetilde{\mathbf{w}}_h, \tilde{q}_h) \in U_h \times H_h$ be the solution of (45) and $(\mathbf{w}, q) \in H^2(\Omega)^2 \times H^1(\Omega)$ the solution of (20). Then there exists a positive constant C independent of h such that*

$$\left\|(\mathbf{w} - \widetilde{\mathbf{w}}_h)_t\right\|_{\text{div}} + \left\|(q - \tilde{q}_h)_t\right\|$$

$$\leq Ch\left(\|\mathbf{w}_t\|_2 + \|q_t\|_1 + \|\mathbf{w}\|_2 + \|q\|_1\right). \quad (47)$$

Differentiating each equation of (45) on t and using (43), (44) we can prove this theorem in the same way as [15].

5. Error Estimates for Semidiscrete Method

In this section, we will establish the error estimates in the $H(\mathrm{div})$ and L^2 norms for the semidiscrete discontinuous mixed covolume method.

Theorem 9. *Let $(\mathbf{w}_h, q_h) \in U_h \times H_h$ be the solution of (21) and $q_h(0) = \widetilde{q}_h(0)$, $(\mathbf{w}, q) \in H^2(\Omega)^2 \times H^1(\Omega)$ the solution of (2). Then there exists a positive constant C independent of h such that*

$$\vert\!\vert\!\vert \mathbf{w} - \mathbf{w}_h \vert\!\vert\!\vert_{\mathrm{div}} + \| q - q_h \|$$
$$\leq Ch \left[\int_0^t \left(\| \mathbf{w}_t \|_2 + \| q_t \|_1 + \| \mathbf{w} \|_2 + \| q \|_1 \right) d\tau + \| \mathbf{w} \|_2 + \| q \|_1 \right]. \tag{48}$$

Proof. Let $\xi = \widetilde{q}_h - q_h$, $\eta = \widetilde{\mathbf{w}}_h - \mathbf{w}_h$. Subtracting the two equations of (21) from those of (20), respectively, we have

$$a(\mathbf{w} - \mathbf{w}_h, \mathbf{v}) + b(\mathbf{v}, q - q_h) = 0, \quad \forall \mathbf{v} \in U_h,$$
$$\left(q_t - q_{h_t}, p \right) + c(\mathbf{w} - \mathbf{w}_h, p) + \int_0^t \widetilde{c}(\mathbf{w} - \mathbf{w}_h, p) \, d\tau = 0,$$
$$\forall p \in H_h. \tag{49}$$

Using (45), we have

$$a(\eta, \mathbf{v}) + b(\mathbf{v}, \xi) = 0, \quad \forall \mathbf{v} \in U_h,$$
$$\left(q_t - q_{h_t}, p \right) + c(\eta, p)$$
$$= -\int_0^t \widetilde{c}(\eta, p) \, d\tau - \int_0^t \widetilde{c}(\mathbf{w} - \widetilde{\mathbf{w}}_h, p) \, d\tau, \quad \forall p \in H_h. \tag{50}$$

Differentiating the first equation of (50) on t, we have that

$$a(\eta_t, \mathbf{v}) + b(\mathbf{v}, \xi_t) = 0, \quad \forall \mathbf{v} \in U_h. \tag{51}$$

By letting $p = \xi_t$ in the second formula of (50) and letting $\mathbf{v} = \eta$ in (51), using Lemma 2, the sum of them gives

$$a(\eta_t, \eta) + (\xi_t, \xi_t)$$
$$= \left((\widetilde{p}_h - p)_t, \xi_t \right) - \int_0^t \widetilde{c}(\eta, \xi_t) \, d\tau - \int_0^t \widetilde{c}(\mathbf{w} - \widetilde{\mathbf{w}}_h, \xi_t) \, d\tau. \tag{52}$$

Using

$$a(\eta_t, \eta) = \frac{1}{2} \frac{d}{dt} a(\eta, \eta) - \frac{1}{2} (M_t \eta, \eta) \tag{53}$$

and Lemmas 1 and 3 gives

$$\frac{1}{2} \frac{d}{dt} a(\eta, \eta) + \| \xi_t \|^2$$
$$\leq C \left[\| \xi_t \| \cdot \| (\widetilde{p}_h - p)_t \| + \vert\!\vert\!\vert \eta \vert\!\vert\!\vert_{\mathrm{div}}^2 \right.$$
$$\left. + \int_0^t \vert\!\vert\!\vert \eta \vert\!\vert\!\vert_{\mathrm{div}} \| \xi_t \| \, d\tau + \int_0^t \vert\!\vert\!\vert \mathbf{w} - \widetilde{\mathbf{w}}_h \vert\!\vert\!\vert_{\mathrm{div}} \| \xi_t \| \, d\tau \right]. \tag{54}$$

Multiplying the equation above with 2, integrating them from 0 to t and using Hölder inequality, ϵ-inequality, Gronwall inequality, Lemma 4, and (47), we can get

$$\vert\!\vert\!\vert \eta \vert\!\vert\!\vert_{\mathrm{div}}^2 + \frac{1}{C_0} \left(\int_0^t \| \xi_t \| \, d\tau \right)^2$$
$$\leq Ch^2 \left(\int_0^t \left(\| \mathbf{w}_t \|_2 + \| q_t \|_1 + \| \mathbf{w} \|_2 + \| q \|_1 \right) d\tau \right)^2, \tag{55}$$

so

$$\vert\!\vert\!\vert \eta \vert\!\vert\!\vert_{\mathrm{div}} \leq Ch \int_0^t \left(\| \mathbf{w}_t \|_2 + \| q_t \|_1 + \| \mathbf{w} \|_2 + \| q \|_1 \right) d\tau, \tag{56}$$

$$\int_0^t \| \xi_t \| \, d\tau \leq Ch \int_0^t \left(\| \mathbf{w}_t \|_2 + \| q_t \|_1 + \| \mathbf{w} \|_2 + \| q \|_1 \right) d\tau; \tag{57}$$

hence

$$\| \xi \| \leq \int_0^t \| \xi_t \| \, d\tau \leq Ch \int_0^t \left(\| \mathbf{w}_t \|_2 + \| q_t \|_1 + \| \mathbf{w} \|_2 + \| q \|_1 \right) d\tau. \tag{58}$$

Now, using the triangle inequality, (46), (56), and (58), we get

$$\vert\!\vert\!\vert \mathbf{w} - \mathbf{w}_h \vert\!\vert\!\vert_{\mathrm{div}} + \| q - q_h \|$$
$$\leq Ch \left[\int_0^t \left(\| \mathbf{w}_t \|_2 + \| q_t \|_1 + \| \mathbf{w} \|_2 + \| q \|_1 \right) d\tau + \| \mathbf{w} \|_2 + \| q \|_1 \right]. \tag{59}$$

The proof is complete. □

6. Error Estimates for Fully Discrete Method

Let $\xi^j = \widetilde{q}_h^j - q_h^j$, $\zeta^j = \widetilde{q}_h^j - q^j$, $\eta^j = \widetilde{\mathbf{w}}_h^j - \mathbf{w}_h^j$ $(j = 0, 1, \ldots, N)$, and then the error estimates for the backward Euler fully discrete discontinuous mixed covolume method in the $H(\mathrm{div})$ and L^2 norms are provided in next two theorems.

Theorem 10. *Let $(\mathbf{w}, q) \in H^2(\Omega)^2 \times H^1(\Omega)$ be the solution of (2) and $(\mathbf{w}_h^j, q_h^j) \in U_h \times H_h$ $(j = 1, 2, \ldots, N)$ the solution of (23) with $t = t_j$ $(j = 1, 2, \ldots, N)$, respectively. If $q_h^0 = \widetilde{q}_h(0) = q_0$, $\mathbf{w}_h^0 = \widetilde{\mathbf{w}}_h(0) = \mathbf{w}_0$, then there exists a positive constant C independent of h and Δt such that*

$$\max_{0 \leq i \leq N} \| q^i - q_h^i \| + \max_{0 \leq i \leq N} \vert\!\vert\!\vert \mathbf{w}^i - \mathbf{w}_h^i \vert\!\vert\!\vert_{\mathrm{div}}$$
$$\leq C\Delta t \left(\| q_{tt} \|_{L^\infty(L^2)} + \| \mathbf{w}_t \|_{L^\infty(H^2)} \right)$$
$$+ Ch \left(\| \mathbf{w}_t \|_{L^\infty(H^2)} + \| q_t \|_{L^\infty(H^1)} \right.$$
$$\left. + \| \mathbf{w} \|_{L^\infty(H^2)} + \| q \|_{L^\infty(H^1)} \right). \tag{60}$$

Proof. Subtracting the two equations of (23) from (20), respectively, with $t = t^j$ ($j = 0, 1, \ldots, N$), we can get the error equation:

$$a\left(\mathbf{w}^j - \mathbf{w}_h^j, \mathbf{v}\right) + b\left(\mathbf{v}, q^j - q_h^j\right) = 0, \quad \forall \mathbf{v} \in U_h,$$

$$\left(q_t^j - \partial_t q_h^j, p\right) + c\left(\mathbf{w}^j - \mathbf{w}_h^j, p\right) + \int_0^{t^j} \widetilde{c}\left(\mathbf{w}, p\right) d\tau$$

$$- \widetilde{c}\left(\Delta t \sum_{k=0}^{j-1} \mathbf{w}_h^k, p\right) = 0, \quad \forall p \in H_h, \ j = 1, 2, \ldots, N. \tag{61}$$

Choosing $\mathbf{v} = \eta^j$ ($j = 1, 2, \ldots, N$) and $p = \xi^j$ ($j = 1, 2, \ldots, N$) in the two equations of (61), adding them together, and using Lemma 2, discontinuous mixed covolume elliptic projection with $t = t^j$ ($j = 0, 1, \ldots, N$), we have

$$a\left(\eta^j, \eta^j\right) + \left(\partial_t \xi^j, \xi^j\right)$$

$$= \left(\partial_t q^j - q_t^j, \xi^j\right) + \left(\partial_t \zeta^j, \xi^j\right) - \int_0^{t^j} \widetilde{c}\left(\mathbf{w} - \widetilde{\mathbf{w}}_h, \xi^j\right) d\tau$$

$$- \widetilde{c}\left(\Delta t \sum_{k=0}^{j-1} \eta^k, \xi^j\right) - \int_0^{t^j} \widetilde{c}\left(\widetilde{\mathbf{w}}_h, \xi^j\right) d\tau + \widetilde{c}\left(\Delta t \sum_{k=0}^{j-1} \widetilde{\mathbf{w}}_h^k, \xi^j\right). \tag{62}$$

First, we estimate the left item of (62). Using Lemma 4, we have

$$a\left(\eta^j, \eta^j\right) \geq C_0 \left\|\left|\eta^j\right|\right\|_{\mathrm{div}}^2,$$

$$\left(\partial_t \xi^j, \xi^j\right)$$

$$= \frac{1}{\Delta t}\left(\xi^j - \xi^{j-1}, \xi^j\right)$$

$$= \frac{1}{2\Delta t}\left[\left(\xi^j, \xi^j\right) - \left(\xi^{j-1}, \xi^{j-1}\right) + \left(\xi^j - \xi^{j-1}, \xi^j - \xi^{j-1}\right)\right]$$

$$= \frac{1}{2\Delta t}\left[\left\|\xi^j\right\|^2 - \left\|\xi^{j-1}\right\|^2 + \left\|\xi^j - \xi^{j-1}\right\|^2\right]$$

$$> \frac{1}{2\Delta t}\left[\left\|\xi^j\right\|^2 - \left\|\xi^{j-1}\right\|^2\right]. \tag{63}$$

Then, we estimate the right item of (62). From

$$\left\|\partial_t q^j - q_t^j\right\|^2$$

$$= \left\|\frac{1}{\Delta t}\int_{t^{j-1}}^{t^j}(t^{j-1} - t)q_{tt}dt\right\|^2$$

$$\leq \int_\Omega \left(\frac{1}{\Delta t}\int_{t^{j-1}}^{t^j}\left(t^{j-1} - t\right)q_{tt}dt\right)^2 dx$$

$$\leq \frac{1}{(\Delta t)^2}\int_\Omega\left(\int_{t^{j-1}}^{t^j}\left(t^{j-1} - t\right)^2 dt \int_{t^{j-1}}^{t^j} q_{tt}^2 dt\right)dx$$

$$\leq C(\Delta t)^2 \left\|q_{tt}\right\|_{L^\infty(L^2)}^2, \tag{64}$$

we have

$$\left(\partial_t q^j - q_t^j, \xi^j\right) \leq C\left(\Delta t\right)\left\|q_{tt}\right\|_{L^\infty(L^2)}\left\|\xi^j\right\|,$$

$$\left\|\partial_t \zeta^j\right\|$$

$$= \left\|\frac{\zeta^j - \zeta^{j-1}}{\Delta t}\right\| = \left\|\frac{1}{\Delta t}\int_{t^{j-1}}^{t^j}\zeta_t dt\right\|$$

$$\leq \frac{1}{\Delta t}\int_{t^{j-1}}^{t^j}\left\|\zeta_t\right\|dt$$

$$\leq \frac{Ch}{\Delta t}\int_{t^{j-1}}^{t^j}\left(\left\|\mathbf{w}_t\right\|_2 + \left\|q_t\right\|_1 + \left\|\mathbf{w}\right\|_2 + \left\|q\right\|_1\right)dt$$

$$\leq Ch\left(\left\|\mathbf{w}_t\right\|_{L^\infty(H^2)} + \left\|q_t\right\|_{L^\infty(H^1)} + \left\|\mathbf{w}\right\|_{L^\infty(H^2)} + \left\|q\right\|_{L^\infty(H^1)}\right), \tag{65}$$

and therefore

$$\left(\partial_t \zeta^j, \xi^j\right)$$

$$\leq Ch\left(\left\|\mathbf{w}_t\right\|_{L^\infty(H^2)} + \left\|q_t\right\|_{L^\infty(H^1)} + \left\|\mathbf{w}\right\|_{L^\infty(H^2)} + \left\|q\right\|_{L^\infty(H^1)}\right)$$

$$\times \left\|\xi^j\right\|. \tag{66}$$

Using Lemma 3, we can get

$$-\int_0^{t^j}\widetilde{c}\left(\mathbf{w} - \widetilde{\mathbf{w}}_h, \xi^j\right)d\tau$$

$$\leq C\int_0^{t^j}\left\|\left|\mathbf{w} - \widetilde{\mathbf{w}}_h\right|\right\|_{\mathrm{div}}d\tau\left\|\xi^j\right\|$$

$$\leq Ch\left(\left\|\mathbf{w}\right\|_{L^\infty(H^2)} + \left\|p\right\|_{L^\infty(H^1)}\right)\left\|\xi^j\right\|,$$

$$-\widetilde{c}\left(\Delta t\sum_{k=0}^{j-1}\eta^k, \xi^j\right) \leq C\left(\Delta t\right)\sum_{k=0}^{j-1}\left\|\left|\eta^k\right|\right\|_{\mathrm{div}}\left\|\xi^j\right\|,$$

$$-\int_0^{t^j}\widetilde{c}\left(\widetilde{\mathbf{w}}_h, \xi^j\right)d\tau + \widetilde{c}\left(\Delta t\sum_{k=0}^{j-1}\widetilde{\mathbf{w}}_h^k, \xi^j\right)$$

$$= \widetilde{c}\left(\sum_{k=0}^{j-1}\int_{t^k}^{t^{k+1}}\left(\widetilde{\mathbf{w}}_h^k - \widetilde{\mathbf{w}}_h\left(\tau\right)\right)d\tau, \xi^j\right)$$

$$\leq C \left\| \sum_{k=0}^{j-1} \int_{t^k}^{t^{k+1}} \int_{t^k}^{\tau} \widetilde{\mathbf{w}}_{h_t} \, ds \, d\tau \right\| \left\| \xi^j \right\|$$

$$\leq C \left(\Delta t \right) \left\| \mathbf{w}_t \right\|_{L^\infty(H^2)} \left\| \xi^j \right\|.$$

$$\tag{67}$$

Substituting the estimations above into (62), multiplying them with Δt, adding them with j from 1 to i, and using $\xi^0 = 0$, we have

$$\left\| \xi^i \right\|^2 + 2 C_0 \Delta t \sum_{j=1}^{i} \left\| \left| \eta^j \right| \right\|_{\mathrm{div}}^2$$

$$\leq C (\Delta t)^2 \left(\left\| q_{tt} \right\|_{L^\infty(L^2)} + \left\| \mathbf{w}_t \right\|_{L^\infty(H^2)} \right)^2$$

$$+ C (\Delta t)^2 \sum_{k=0}^{j-1} \left\| \left| \eta^k \right| \right\|_{\mathrm{div}}^2 + C \Delta t \sum_{j=1}^{i} \left\| \xi^j \right\|^2$$

$$+ C h^2 \left(\left\| \mathbf{w}_t \right\|_{L^\infty(H^2)} + \left\| q_t \right\|_{L^\infty(H^1)} + \left\| \mathbf{w} \right\|_{L^\infty(H^2)} + \left\| q \right\|_{L^\infty(H^1)} \right)^2.$$

$$\tag{68}$$

Using the discrete Gronwall inequality, we have

$$\max_{0 \leq i \leq N} \left\| \xi^i \right\| + \max_{0 \leq i \leq N} \left\| \left| \eta^i \right| \right\|_{\mathrm{div}}$$

$$\leq C \Delta t \left(\left\| q_{tt} \right\|_{L^\infty(L^2)} + \left\| \mathbf{w}_t \right\|_{L^\infty(H^2)} \right)$$

$$+ C h \left(\left\| \mathbf{w}_t \right\|_{L^\infty(H^2)} + \left\| q_t \right\|_{L^\infty(H^1)} + \left\| \mathbf{w} \right\|_{L^\infty(H^2)} + \left\| q \right\|_{L^\infty(H^1)} \right).$$

$$\tag{69}$$

From the formula above and (46) and using the triangle inequality, we have

$$\max_{0 \leq i \leq N} \left\| q^i - q_h^i \right\| + \max_{0 \leq i \leq N} \left\| \left| \mathbf{w}^i - \mathbf{w}_h^i \right| \right\|_{\mathrm{div}}$$

$$\leq C \Delta t \left(\left\| q_{tt} \right\|_{L^\infty(L^2)} + \left\| \mathbf{w}_t \right\|_{L^\infty(H^2)} \right)$$

$$+ C h \left(\left\| \mathbf{w}_t \right\|_{L^\infty(H^2)} + \left\| q_t \right\|_{L^\infty(H^1)} + \left\| \mathbf{w} \right\|_{L^\infty(H^2)} + \left\| q \right\|_{L^\infty(H^1)} \right).$$

$$\tag{70}$$

This completes the proof. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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