Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 647030, 9 pages
http://dx.doi.org/10.1155/2014/647030

Research Article

Explicit Determinants of the RFPrLrR Circulant and RLPrFrL Circulant Matrices Involving Some Famous Numbers

Tingting Xu,[1,2] Zhaolin Jiang,[1] and Ziwu Jiang[1]

[1] *Department of Mathematics, Linyi University, Linyi, Shandong 276005, China*
[2] *School of Mathematical Sciences, Shandong Normal University, Jinan 250014, China*

Correspondence should be addressed to Zhaolin Jiang; jzh1208@sina.com

Received 28 April 2014; Accepted 5 June 2014; Published 19 June 2014

Academic Editor: Tongxing Li

Circulant matrices may play a crucial role in solving various differential equations. In this paper, the techniques used herein are based on the inverse factorization of polynomial. We give the explicit determinants of the RFPrLrR circulant matrices and RLPrFrL circulant matrices involving Fibonacci, Lucas, Pell, and Pell-Lucas number, respectively.

1. Introduction

It has been found out that circulant matrices play an important role in solving differential equations in various fields such as Lin and Yang discretized the partial integrodifferential equation (PIDE) in pricing options with the preconditioned conjugate gradient (PCG) method, where constructed the circulant preconditioners. By using the FFT, the cost for each linear system is $O(n\log n)$ where n is the size of the system in [1]. Lei and Sun [2] proposed the preconditioned CGNR (PCGNR) method with a circulant preconditioner to solve such Toeplitz-like systems. Kloeden et al. adopted the simplest approximation schemes for (1) in [3] with the Euler method, which reads (5) in [3]. They exploited that the covariance matrix of the increments can be embedded in a circulant matrix. The total loops can be done by fast Fourier transformation, which leads to a total computational cost of $O(m\log m) = O(n\log n)$. By using a Strang-type block-circulant preconditioner, Zhang et al. [4] speeded up the convergent rate of boundary-value methods. In [5], the resulting dense linear system exhibits so much structure that it can be solved very efficiently by a circulant preconditioned conjugate gradient method. Ahmed et al. used coupled map lattices (CML) as an alternative approach to include spatial effects in FOS. Consider the 1-system CML (10) in [6]. They claimed that the system is stable if all the eigenvalues of the circulant matrix satisfy (2) in [6]. Wu and Zou in [7] discussed the existence and approximation of solutions of asymptotic or periodic boundary-value problems of mixed functional differential equations. They focused on (5.13) in [7] with a circulant matrix, whose principal diagonal entries are zeroes.

Circulant matrix family have important applications in various disciplines including image processing, communications, signal processing, encoding, and preconditioner. They have been put on firm basis with the work of Davis [8] and Jiang and Zhou [9]. The circulant matrices, long a fruitful subject of research, have in recent years been extended in many directions [10–13]. The $f(x)$-circulant matrices are another natural extension of this well-studied class and can be found in [14–20]. The $f(x)$-circulant matrix has a wide application, especially on the generalized cyclic codes in [14]. The properties and structures of the $x^n - rx - r$-circulant matrices, which are called RFPrLrR circulant matrices, are better than those of the general $f(x)$-circulant matrices, so there are good algorithms for determinants.

There are many interests in properties and generalization of some special matrices with famous numbers. Jaiswal evaluated some determinants of circulant whose elements are the generalized Fibonacci numbers [21]. Dazheng gave the determinant of the Fibonacci-Lucas quasicyclic matrices [22]. Lind presented the determinants of circulant and skew circulant involving Fibonacci numbers in [23]. Shen et al. [24] discussed the determinant of circulant matrix involving

Fibonacci and Lucas numbers. Akbulak and Bozkurt [25] gave the norms of Toeplitz involving Fibonacci and Lucas numbers. The authors [26, 27] discussed some properties of Fibonacci and Lucas matrices. Stanimirović et al. gave generalized Fibonacci and Lucas matrix in [28]. Z. Zhang and Y. Zhang [29] investigated the Lucas matrix and some combinatorial identities.

Firstly, we introduce the definitions of the RFPrLrR circulant matrices and RLPrFrL circulant matrices and properties of the related famous numbers. Then, we present the main results and the detailed process.

2. Definition and Lemma

Definition 1. A row first-plus-$r$last r-right (RFPrLrR) circulant matrix with the first row $(a_0, a_1, \ldots, a_{n-1})$, denoted by RFP$r$LRcirc$_rfr(a_0, a_1, \ldots, a_{n-1})$, means a square matrix of the form

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ ra_{n-1} & a_0 + ra_{n-1} & \cdots & a_{n-2} \\ ra_{n-2} & ra_{n-1} + ra_{n-2} & \cdots & a_{n-3} \\ \cdots & \cdots & \cdots & \cdots \\ ra_1 & ra_2 + ra_1 & \cdots & a_0 + ra_{n-1} \end{pmatrix}. \quad (1)$$

Note that the RFPrLrR circulant matrix is a $x^n - rx - r$ circulant matrix, which is neither an extension nor special case of the circulant matrix [8]. They are two completely different kinds of special matrices.

We define $\Theta_{(r,r)}$ as the basic RFPrLrR circulant matrix; that is,

$$\Theta_{(r,r)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ r & r & 0 & \cdots & 0 & 0 \end{pmatrix}_{n \times n} = \text{RFP}r\text{LRcirc}_r\text{fr}\,(0, 1, 0, \ldots, 0). \quad (2)$$

Both the minimal polynomial and the characteristic polynomial of $\Theta_{(r,r)}$ are $g(x) = x^n - rx - r$, which has only simple roots, denoted by ε_k $(k = 1, 2, \ldots, n)$. In addition, $\Theta_{(r,r)}$ satisfies $\Theta^j_{(r,r)} = \text{RFP}r\text{LRcirc}_r\text{fr}(\underbrace{0, \ldots, 0}_{j}, 1, \underbrace{0, \ldots, 0}_{n-j-1})$ and $\Theta^n_{(r,r)} = rI_n + r\Theta_{(r,r)}$. Then a matrix A can be written in the form

$$A = f\left(\Theta_{(r,r)}\right) = \sum_{i=0}^{n-1} a_i \Theta^i_{(r,r)}, \quad (3)$$

if and only if A is a RFPrLrR circulant matrix, where the polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^i$ is called the representer of the RFPrLrR circulant matrix A.

Since $\Theta_{(r,r)}$ is nonderogatory, then A is a RFMrLrR circulant matrix if and only if A commutes with $\Theta_{(r,r)}$; that is, $A\Theta_{(r,r)} = \Theta_{(r,r)}A$. Because of the representation, RFMrLrR circulant matrices have very nice structure and the algebraic properties also can be easily attained. Moreover, the product of two RFMrLrR circulant matrices and the inverse A^{-1} are again RFMrLrR circulant matrices.

Definition 2. A row last-plus-$r$first r-left (RLPrFrL) circulant matrix with the first row $(a_0, a_1, \ldots, a_{n-1})$, denoted by RLP$r$FLcirc$_rfr(a_0, a_1, \ldots, a_{n-1})$, means a square matrix of the form

$$B = \begin{pmatrix} a_0 & \cdots & a_{n-2} & a_{n-1} \\ a_1 & \cdots & a_{n-1} + ra_0 & ra_0 \\ a_2 & \cdots & ra_0 + ra_1 & ra_1 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1} + ra_0 & \cdots & ra_{n-3} + ra_{n-2} & ra_{n-2} \end{pmatrix}. \quad (4)$$

Let $A = \text{RLP}r\text{FLcirc}_r\text{fr}(a_0, a_1, \ldots, a_{n-1})$ and $B = \text{RFP}r\text{LRcirc}_r\text{fr}(a_{n-1}, a_{n-2}, \ldots, a_0)$. By explicit computation, we find

$$A = B\hat{I}_n, \quad (5)$$

where $\hat{I}_n$ is the backward identity matrix of the form

$$\hat{I}_n = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}. \quad (6)$$

The Fibonacci, Lucas, Pell, and the Pell-Lucas sequences [30–36] are defined by the following recurrence relations, respectively:

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1}, \quad \text{where } F_0 = 0,\ F_1 = 1, \\ L_{n+1} &= L_n + L_{n-1}, \quad \text{where } L_0 = 2,\ L_1 = 1, \\ P_{n+1} &= 2P_n + P_{n-1}, \quad \text{where } P_0 = 0,\ P_1 = 1, \\ Q_{n+1} &= 2Q_n + Q_{n-1}, \quad \text{where } Q_0 = 2,\ Q_1 = 2. \end{aligned} \quad (7)$$

The first few values of these sequences are given by the following table ($n \geq 0$):

n	0	1	2	3	4	5	6	7
F_n	0	1	1	2	3	5	8	13
L_n	2	1	3	4	7	11	18	29
P_n	0	1	2	5	12	29	70	169
Q_n	2	2	6	14	34	82	198	478

(8)

The sequences $\{F_n\}$, $\{L_n\}$, $\{P_n\}$, and $\{Q_n\}$ are given by the Binet formulae

$$\begin{aligned} F_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n, \\ P_n &= \frac{\alpha_1^n - \beta_1^n}{\alpha_1 - \beta_1}, \qquad Q_n = \alpha_1^n + \beta_1^n, \end{aligned} \quad (9)$$

where α, β are the roots of the characteristic equation $x^2 - x - 1 = 0$ and α_1, β_1 are the roots of the characteristic equation $x^2 - 2x - 1 = 0$.

By Proposition 5.1 in [14], we deduce the following lemma.

Lemma 3. *Let* $A = RFPrLRcirc_r fr(a_0, \ldots, a_{n-1})$*; then the eigenvalues of* A *are*

$$f(\varepsilon_k) = \sum_{i=0}^{n-1} \left(a_i \varepsilon_k^i\right), \tag{10}$$

and in addition,

$$\det A = \prod_{k=1}^{n} \sum_{i=0}^{n-1} \left(a_i \varepsilon_k^i\right), \tag{11}$$

where ε_k $(k = 1, 2, \ldots, n)$ *are the roots of the equation*

$$x^n - rx - r = 0. \tag{12}$$

Lemma 4. *Consider*

$$\begin{aligned}
&\prod_{k=1}^{n} \left(c + \varepsilon_k b + \varepsilon_k^2 a\right) \\
&\quad = c^n - rc\left[(as)^{n-1} + (at)^{n-1}\right] \\
&\qquad - r\left[(as)^n + (at)^n\right] + r^2 a^{n-1}(c - b + a),
\end{aligned} \tag{13}$$

where

$$s = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad t = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \tag{14}$$

and ε_k $(k = 1, 2, \ldots, n)$ *satisfy* (12), $a, b, c \in \mathbf{R}$, $a \neq 0$.

Proof. Consider

$$\begin{aligned}
\prod_{k=1}^{n} \left(c + \varepsilon_k b + \varepsilon_k^2 a\right) &= a^n \prod_{k=1}^{n} \left(\varepsilon_k^2 + \frac{b}{a}\varepsilon_k + \frac{c}{a}\right) \\
&= a^n \prod_{k=1}^{n} (\varepsilon_k - s)(\varepsilon_k - t) \\
&= a^n \prod_{k=1}^{n} (s - \varepsilon_k)(t - \varepsilon_k),
\end{aligned} \tag{15}$$

while

$$\begin{gathered}
s + t = -\frac{b}{a}, \qquad st = \frac{c}{a}, \\
s = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad t = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.
\end{gathered} \tag{16}$$

Since ε_k $(k = 1, 2, \ldots, n)$ satisfy (12), we must have

$$x^n - rx - r = \prod_{k=1}^{n} (x - \varepsilon_k). \tag{17}$$

So

$$\begin{aligned}
&\prod_{k=1}^{n} \left(c + \varepsilon_k b + \varepsilon_k^2 a\right) \\
&\quad = a^n (s^n - sr - r)(t^n - tr - r) \\
&\quad = a^n \left[(st)^n - rst\left(s^{n-1} + t^{n-1}\right) - r\left(s^n + t^n\right)\right] \\
&\qquad + a^n \left[r^2 (s + t + st + 1)\right] \\
&\quad = a^n \left[\left(\frac{c}{a}\right)^n - r\frac{c}{a}\left(s^{n-1} + t^{n-1}\right) - r\left(s^n + t^n\right)\right] \\
&\qquad + a^n \left[r^2 \left(\frac{c}{a} - \frac{b}{a} + 1\right)\right] \\
&\quad = c^n - rc\left[(as)^{n-1} + (at)^{n-1}\right] - r\left[(as)^n + (at)^n\right] \\
&\qquad + r^2 a^{n-1} (c - b + a).
\end{aligned} \tag{18}$$

□

3. Determinant of the RFPrLrR and RLPrFrL Circulant Matrices with the Fibonacci Numbers

Theorem 5. *Let* $\mathbb{A} = RFPrLRcirc_r fr(F_0, F_1, \ldots, F_{n-1})$*. Then*

$$\begin{aligned}
\det \mathbb{A} = &\frac{(-rF_n)^n - (-r)^{n+1} F_{n-1}^{n-1}}{1 - rL_{n-1} - rL_n + (-1)^{n-1} r^2} \\
&+ \frac{(-r)^{n+1} F_{n-1}^{n-1} F_n \left(g_1^{n-1} + h_1^{n-1}\right)}{1 - rL_{n-1} - rL_n + (-1)^{n-1} r^2} \\
&+ \frac{(-r)^{n+1} F_{n-1}^{n} \left(g_1^n + h_1^n\right)}{1 - rL_{n-1} - rL_n + (-1)^{n-1} r^2},
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
g_1 = &\frac{(rF_n + rF_{n-1} - 1)}{-2rF_{n-1}} \\
&+ \frac{\sqrt{r^2 (F_n - F_{n-1})^2 - 2r(F_n + F_{n+1})}}{-2rF_{n-1}}, \\
h_1 = &\frac{(rF_n + rF_{n-1} - 1)}{-2rF_{n-1}} \\
&- \frac{\sqrt{r^2 (F_n - F_{n-1})^2 - 2r(F_n + F_{n+1})}}{-2rF_{n-1}}.
\end{aligned} \tag{20}$$

Proof. The matrix $\mathbb{A}$ can be written as

$$\mathbb{A} = \begin{pmatrix}
F_0 & F_1 & \cdots & F_{n-1} \\
rF_{n-1} & F_0 + rF_{n-1} & \cdots & F_{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
rF_2 & rF_3 + rF_2 & \cdots & F_1 \\
rF_1 & rF_2 + rF_1 & \cdots & F_0 + rF_{n-1}
\end{pmatrix}_{n \times n}. \tag{21}$$

Using Lemma 3, the determinant of $\mathbb{A}$ is

$$\begin{aligned}\det \mathbb{A} &= \prod_{k=1}^{n}\left(F_0 + F_1\varepsilon_k + \cdots + F_{n-1}\varepsilon_k^{n-1}\right)\\ &= \prod_{k=1}^{n}\left(\frac{\alpha-\beta}{\alpha-\beta}\varepsilon_k + \cdots + \frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\varepsilon_k^{n-1}\right)\\ &= \prod_{k=1}^{n}\frac{-rF_{n-1}\varepsilon_k^2 + \left(1 - rF_{n-1} - rF_n\right)\varepsilon_k - rF_n}{1-\varepsilon_k-\varepsilon_k^2}.\end{aligned} \tag{22}$$

Using Lemma 4, we obtain

$$\begin{aligned}\det \mathbb{A} = {} & \frac{\left(-rF_n\right)^n - (-r)^{n+1}F_{n-1}^{n-1}}{1 - rL_{n-1} - rL_n + (-1)^{n-1}r^2}\\ & + \frac{(-r)^{n+1}F_{n-1}^{n-1}F_n\left(g_1^{n-1} + h_1^{n-1}\right)}{1 - rL_{n-1} - rL_n + (-1)^{n-1}r^2}\\ & + \frac{(-r)^{n+1}F_{n-1}^{n}\left(g_1^{n} + h_1^{n}\right)}{1 - rL_{n-1} - rL_n + (-1)^{n-1}r^2},\end{aligned} \tag{23}$$

where

$$\begin{aligned}g_1 = {} & \frac{\left(rF_n + rF_{n-1} - 1\right)}{-2rF_{n-1}}\\ & + \frac{\sqrt{r^2\left(F_n - F_{n-1}\right)^2 - 2r\left(F_n + F_{n+1}\right)}}{-2rF_{n-1}},\\ h_1 = {} & \frac{\left(rF_n + rF_{n-1} - 1\right)}{-2rF_{n-1}}\\ & - \frac{\sqrt{r^2\left(F_n - F_{n-1}\right)^2 - 2r\left(F_n + F_{n+1}\right)}}{-2rF_{n-1}}.\end{aligned} \tag{24}$$

□

Using the method in Theorem 5 similarly, we also have the following.

Theorem 6. *Let $\mathbb{A}' = RFPrLRcirc_r fr(F_{n-1}, \ldots, F_0)$. Then*

$$\begin{aligned}\det \mathbb{A}' = {} & \frac{\left(r - F_{n-1}\right)^n - r\left(r - F_{n-1}\right)\left(F_n - r\right)^{n-1}}{(-1)^n + rL_{n-1} - rL_n + r^2}\\ & - \frac{r\left(F_n - r\right)^n}{(-1)^n + rL_{n-1} - rL_n + r^2}.\end{aligned} \tag{25}$$

Theorem 7. *Let $\mathbb{F} = RLPrFLcirc_r fr(F_0, \ldots, F_{n-1})$. Then*

$$\begin{aligned}\det \mathbb{F} = {} & \frac{\left(r - F_{n-1}\right)^n - r\left(r - F_{n-1}\right)\left(F_n - r\right)^{n-1}}{(-1)^n + rL_{n-1} - rL_n + r^2}\\ & \times (-1)^{n(n-1)/2}\\ & - \frac{r\left(F_n - r\right)^n}{(-1)^n + rL_{n-1} - rL_n + r^2}(-1)^{n(n-1)/2}.\end{aligned} \tag{26}$$

Proof. The matrix $\mathbb{F}$ can be written as

$$\begin{aligned}\mathbb{F} &= \begin{pmatrix} F_0 & \cdots & F_{n-2} & F_{n-1}\\ F_1 & \cdots & F_{n-1} + rF_0 & rF_0\\ F_2 & \cdots & rF_0 + rF_1 & rF_1\\ \cdots & \cdots & \cdots & \cdots\\ F_{n-1} + rF_0 & \cdots & rF_{n-3} + rF_{n-2} & rF_{n-2}\end{pmatrix}\\ &= \begin{pmatrix} F_{n-1} & F_{n-2} & \cdots & F_0\\ rF_0 & F_{n-1} + rF_0 & \cdots & F_1\\ \vdots & \vdots & \ddots & \vdots\\ rF_{n-3} & rF_{n-4} + rF_{n-3} & \cdots & F_{n-2}\\ rF_{n-2} & rF_{n-3} + rF_{n-2} & \cdots & F_{n-1} + rF_0\end{pmatrix}\\ &\quad\times \begin{pmatrix} 0 & 0 & \cdots & 0 & 1\\ 0 & \cdots & 0 & 1 & 0\\ \vdots & \therefore & \therefore & \therefore & \vdots\\ 0 & 1 & 0 & \cdots & 0\\ 1 & 0 & 0 & \cdots & 0\end{pmatrix} = \mathbb{A}'\Gamma.\end{aligned} \tag{27}$$

Hence, we have

$$\det \mathbb{F} = \det \mathbb{A}' \det \Gamma, \tag{28}$$

where $\mathbb{A}' = \mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}(F_{n-1}, F_{n-2}, \ldots, F_0)$ and its determinant is obtained from Theorem 6,

$$\begin{aligned}\det \mathbb{A}' = {} & \frac{\left(r - F_{n-1}\right)^n - r\left(r - F_{n-1}\right)\left(F_n - r\right)^{n-1}}{(-1)^n + rL_{n-1} - rL_n + r^2}\\ & - \frac{r\left(F_n - r\right)^n}{(-1)^n + rL_{n-1} - rL_n + r^2}.\end{aligned} \tag{29}$$

In addition,

$$\det \Gamma = (-1)^{n(n-1)/2}, \tag{30}$$

so

$$\begin{aligned}\det \mathbb{F} = {} & \frac{\left(r - F_{n-1}\right)^n - r\left(r - F_{n-1}\right)\left(F_n - r\right)^{n-1}}{(-1)^n + rL_{n-1} - rL_n + r^2}\\ & \times (-1)^{n(n-1)/2}\\ & - \frac{r\left(F_n - r\right)^n}{(-1)^n + rL_{n-1} - rL_n + r^2}(-1)^{n(n-1)/2}.\end{aligned} \tag{31}$$

□

4. Determinant of the RFMrLrR and RLMrFrL Circulant Matrices with the Lucas Numbers

Theorem 8. *Let $\mathbb{B} = RFPrLRcirc_r fr(L_0, L_1, \ldots, L_{n-1})$. Then*

$$\begin{aligned}\det \mathbb{B} = {} & \frac{\left(2 - rL_n\right)^n}{1 - rL_{n-1} - rL_n + (-1)^{n-1}r^2}\\ & + \frac{(-r)^n L_{n-1}^{n-1}\left(2 - rL_n\right)\left(g_2^{n-1} + h_2^{n-1}\right)}{1 - rL_{n-1} - rL_n + (-1)^{n-1}r^2}\\ & - \frac{(-r)^n L_{n-1}^{n-1}\left[rL_{n-1}\left(g_2^n + h_2^n\right) - 3r\right]}{1 - rL_{n-1} - rL_n + (-1)^{n-1}r^2},\end{aligned} \tag{32}$$

where

$$g_2 = \frac{1 + rL_{n-1} + rL_n}{-2rL_{n-1}} + \frac{\sqrt{r^2(L_n - L_{n-1})^2 + 10rL_{n-1} + 2rL_n + 1}}{-2rL_{n-1}},$$
$$h_2 = \frac{1 + rL_{n-1} + rL_n}{-2rL_{n-1}} - \frac{\sqrt{r^2(L_n - L_{n-1})^2 + 10rL_{n-1} + 2rL_n + 1}}{-2rL_{n-1}}. \quad (33)$$

Proof. The matrix $\mathbb{B}$ can be written as

$$\mathbb{B} = \begin{pmatrix} L_0 & L_1 & \cdots & L_{n-1} \\ rL_{n-1} & L_1 + rL_{n-1} & \cdots & L_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ rL_2 & rL_3 + rL_2 & \cdots & L_1 \\ rL_1 & rL_2 + rL_1 & \cdots & L_0 + rL_{n-1} \end{pmatrix}. \quad (34)$$

Using Lemma 3, we have

$$\begin{aligned} \det \mathbb{B} &= \prod_{k=1}^{n} \left(L_0 + L_1\varepsilon_k + \cdots + L_{n-1}\varepsilon_k^{n-1}\right) \\ &= \prod_{k=1}^{n} \left[2 + (\alpha + \beta)\varepsilon_k + \cdots + \left(\alpha^{n-1} + \beta^{n-1}\right)\varepsilon_k^{n-1}\right] \\ &= \prod_{k=1}^{n} \frac{-rL_{n-1}\varepsilon_k^2 - (1 + rL_n + rL_{n-1})\varepsilon_k - 2 + rL_n}{1 - \varepsilon_k - \varepsilon_k^2}. \end{aligned} \quad (35)$$

According to Lemma 4, we obtain

$$\begin{aligned} &\prod_{k=1}^{n} \left[-rL_{n-1}\varepsilon_k^2 - (1 + rL_n + rL_{n-1})\varepsilon_k + 2 - rL_n\right] \\ &= (2 - rL_n)^n + (-r)^n L_{n-1}^{n-1}(2 - rL_n)\left(g_2^{n-1} + h_2^{n-1}\right) \\ &\quad - (-r)^n L_{n-1}^{n-1}\left[rL_{n-1}\left(g_2^n + h_2^n\right) - 3r\right]. \end{aligned} \quad (36)$$

Then, we get

$$\begin{aligned} \det \mathbb{B} &= \frac{(2 - rL_n)^n}{1 - rL_{n-1} - rL_n + (-1)^{n-1}r^2} \\ &\quad + \frac{(-r)^n L_{n-1}^{n-1}(2 - rL_n)\left(g_2^{n-1} + h_2^{n-1}\right)}{1 - rL_{n-1} - rL_n + (-1)^{n-1}r^2} \\ &\quad - \frac{(-r)^n L_{n-1}^{n-1}\left[rL_{n-1}\left(g_2^n + h_2^n\right) - 3r\right]}{1 - rL_{n-1} - rL_n + (-1)^{n-1}r^2}, \end{aligned} \quad (37)$$

where

$$g_2 = \frac{1 + rL_{n-1} + rL_n}{-2rL_{n-1}} + \frac{\sqrt{r^2(L_n - L_{n-1})^2 + 10rL_{n-1} + 2rL_n + 1}}{-2rL_{n-1}},$$
$$h_2 = \frac{1 + rL_{n-1} + rL_n}{-2rL_{n-1}} - \frac{\sqrt{r^2(L_n - L_{n-1})^2 + 10rL_{n-1} + 2rL_n + 1}}{-2rL_{n-1}}. \quad (38)$$

□

Using the method in Theorem 8 similarly, we also have the following.

Theorem 9. *Let $\mathbb{B}' = RFPrLRcirc_r fr(L_{n-1}, \ldots, L_0)$. Then*

$$\begin{aligned} \det \mathbb{B}' &= \frac{(-r - L_{n-1})^n}{(-1)^n + rL_{n-1} - rL_n + r^2} \\ &\quad + \frac{2^{n-1}r^n (r + L_{n-1})\left(g_3^{n-1} + h_3^{n-1}\right)}{(-1)^n + rL_{n-1} - rL_n + r^2} \\ &\quad + \frac{2^{n-1}r^n\left[-2r\left(g_3^n + h_3^n\right) + r(L_n - L_{n-1})\right]}{(-1)^n + rL_{n-1} - rL_n + r^2}, \end{aligned} \quad (39)$$

where

$$g_3 = \frac{L_n - r + \sqrt{(r - L_n)^2 + 8r(r + L_{n-1})}}{4r},$$
$$h_3 = \frac{L_n - r - \sqrt{(r - L_n)^2 + 8r(r + L_{n-1})}}{4r}. \quad (40)$$

Theorem 10. *Let $\mathbb{L} = RLPrFLcirc_r fr(L_0, L_1, \ldots, L_{n-1})$. Then*

$$\begin{aligned} \det \mathbb{L} &= \frac{(-r - L_{n-1})^n}{(-1)^n + rL_{n-1} - rL_n + r^2}(-1)^{n(n-1)/2} \\ &\quad + \frac{2^{n-1}r^n (r + L_{n-1})\left(g_3^{n-1} + h_3^{n-1}\right)}{(-1)^n + rL_{n-1} - rL_n + r^2}(-1)^{n(n-1)/2} \\ &\quad + \frac{2^{n-1}r^n\left[-2r\left(g_3^n + h_3^n\right) + r(L_n - L_{n-1})\right]}{(-1)^n + rL_{n-1} - rL_n + r^2} \\ &\quad \times (-1)^{n(n-1)/2}, \end{aligned} \quad (41)$$

where

$$g_3 = \frac{L_n - r + \sqrt{(r - L_n)^2 + 8r(r + L_{n-1})}}{4r},$$
$$h_3 = \frac{L_n - r - \sqrt{(r - L_n)^2 + 8r(r + L_{n-1})}}{4r}. \quad (42)$$

Proof. The matrix $\mathbb{L}$ can be written as

$$\mathbb{L}=\begin{pmatrix} L_0 & \cdots & L_{n-2} & L_{n-1} \\ L_1 & \cdots & L_{n-1}+rL_0 & rL_0 \\ \vdots & \ddots & \vdots & \vdots \\ L_{n-2} & \cdots & rL_{n-4}+rL_{n-3} & rL_{n-3} \\ L_{n-1}-rL_0 & \cdots & rL_{n-3}+rL_{n-2} & rL_{n-2} \end{pmatrix} = \begin{pmatrix} L_{n-1} & L_{n-2} & \cdots & L_0 \\ rL_0 & L_{n-1}+rL_0 & \cdots & L_1 \\ \vdots & \vdots & \ddots & \vdots \\ rL_{n-3} & rL_{n-4}+rL_{n-3} & \cdots & L_{n-2} \\ rL_{n-2} & rL_{n-3}+rL_{n-2} & \cdots & L_{n-1}+rL_0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & ⋰ & ⋰ & ⋰ & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} = \mathbb{B}'\Gamma. \tag{43}$$

Thus, we have

$$\det\mathbb{L} = \det\mathbb{B}' \det\Gamma, \tag{44}$$

where matrix $\mathbb{B}' = \mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}(L_{n-1},\dots,L_0)$ and its determinant can be obtained from Theorem 9,

$$\begin{aligned}\det B' &= \frac{(-r-L_{n-1})^n}{(-1)^n + rL_{n-1} - rL_n + r^2} \\ &\quad + \frac{2^{n-1}r^n\left(r+L_{n-1}\right)\left(g_3^{n-1}+h_3^{n-1}\right)}{(-1)^n + rL_{n-1} - rL_n + r^2} \\ &\quad + \frac{2^{n-1}r^n\left[-2r\left(g_3^n+h_3^n\right)+r\left(L_n-L_{n-1}\right)\right]}{(-1)^n + rL_{n-1} - rL_n + r^2},\end{aligned} \tag{45}$$

where

$$\begin{aligned} g_3 &= \frac{L_n - r + \sqrt{(r-L_n)^2 + 8r\left(r+L_{n-1}\right)}}{4r}, \\ h_3 &= \frac{L_n - r - \sqrt{(r-L_n)^2 + 8r\left(r+L_{n-1}\right)}}{4r}. \end{aligned} \tag{46}$$

In addition,

$$\det\Gamma = (-1)^{n(n-1)/2}, \tag{47}$$

so the determinant of matrix $\mathbb{L}$ is

$$\begin{aligned}\det\mathbb{L} &= \frac{(-r-L_{n-1})^n}{(-1)^n + rL_{n-1} - rL_n + r^2}(-1)^{n(n-1)/2} \\ &\quad + \frac{2^{n-1}r^n\left(r+L_{n-1}\right)\left(g_3^{n-1}+h_3^{n-1}\right)}{(-1)^n + rL_{n-1} - rL_n + r^2}(-1)^{n(n-1)/2} \\ &\quad + \frac{2^{n-1}r^n\left[-2r\left(g_3^n+h_3^n\right)+r\left(L_n-L_{n-1}\right)\right]}{(-1)^n + rL_{n-1} - rL_n + r^2} \\ &\quad \times (-1)^{n(n-1)/2},\end{aligned} \tag{48}$$

where

$$\begin{aligned} g_3 &= \frac{L_n - r + \sqrt{(r-L_n)^2 + 8r\left(r+L_{n-1}\right)}}{4r}, \\ h_3 &= \frac{L_n - r - \sqrt{(r-L_n)^2 + 8r\left(r+L_{n-1}\right)}}{4r}. \end{aligned} \tag{49}$$

□

5. Determinants of the RFPrLrR and RLPrFrL Circulant Matrix with the Pell Numbers

Theorem 11. *If* $\mathbb{C} = RFPrLRcirc_rfr(P_0, P_1, \dots, P_{n-1})$, *then*

$$\begin{aligned}\det\mathbb{C} &= \frac{(-rP_n)^n}{1 - rQ_{n-1} - rQ_n + 2(-1)^{n-1}r^2} \\ &\quad + \frac{\left[P_n\left(g_4^{n-1}+h_4^{n-1}\right) + P_{n-1}\left(g_4^n+h_4^n\right) - 1\right]}{1 - rQ_{n-1} - rQ_n + 2(-1)^{n-1}r^2} \\ &\quad \times (-r)^{n+1}P_{n-1}^{n-1},\end{aligned} \tag{50}$$

where

$$\begin{aligned} g_4 &= \frac{rP_{n-1} + rP_n - 1}{-2rP_{n-1}} \\ &\quad + \frac{\sqrt{r^2\left(P_n-P_{n-1}\right)^2 - 2r\left(P_n+P_{n-1}\right)+1}}{-2rP_{n-1}}, \\ h_4 &= \frac{rP_{n-1} + rP_n - 1}{-2rP_{n-1}} \\ &\quad - \frac{\sqrt{r^2\left(P_n-P_{n-1}\right)^2 - 2r\left(P_n+P_{n-1}\right)+1}}{-2rP_{n-1}}. \end{aligned} \tag{51}$$

Proof. The matrix $\mathbb{C}$ can be written as

$$\mathbb{C} = \begin{pmatrix} P_0 & P_1 & \cdots & P_{n-1} \\ rP_{n-1} & P_0 + rP_{n-1} & P_1 & P_{n-2} \\ \vdots & rP_{n-1}+rP_{n-2} & \ddots & \vdots \\ rP_2 & \vdots & \cdots & P_1 \\ rP_1 & rP_2 + rP_1 & \cdots & P_0 + rP_{n-1} \end{pmatrix}_{n\times n}. \tag{52}$$

Using Lemma 3, the determinant of $\mathbb{C}$ is

$$\begin{aligned}\det\mathbb{C} &= \prod_{k=1}^{n}\left(P_0 + P_1\varepsilon_k + \cdots + P_{n-1}\varepsilon_k^{n-1}\right) \\ &= \prod_{k=1}^{n}\left(\frac{\alpha_1-\beta_1}{\alpha_1-\beta_1}\varepsilon_k + \cdots + \frac{\alpha_1^{n-1}-\beta_1^{n-1}}{\alpha_1-\beta_1}\varepsilon_k^{n-1}\right) \\ &= \prod_{k=1}^{n}\frac{-rP_{n-1}\varepsilon_k^2 + \left(1 - rP_{n-1} - rP_n\right)\varepsilon_k - rP_n}{1 - 2\varepsilon_k - \varepsilon_k^2}.\end{aligned} \tag{53}$$

According to Lemma 4, we can get

$$\det \mathbb{C} = \frac{(-rP_n)^n}{1 - rQ_{n-1} - rQ_n + 2(-1)^{n-1}r^2} + \frac{\left[P_n\left(g_4^{n-1} + h_4^{n-1}\right) + P_{n-1}\left(g_4^n + h_4^n\right) - 1\right]}{1 - rQ_{n-1} - rQ_n + 2(-1)^{n-1}r^2} \times (-r)^{n+1} P_{n-1}^{n-1}, \tag{54}$$

where

$$\begin{aligned} g_4 &= \frac{rP_{n-1} + rP_n - 1}{-2rP_{n-1}} + \frac{\sqrt{r^2(P_n - P_{n-1})^2 - 2r(P_n + P_{n-1}) + 1}}{-2rP_{n-1}}, \\ h_4 &= \frac{rP_{n-1} + rP_n - 1}{-2rP_{n-1}} - \frac{\sqrt{r^2(P_n - P_{n-1})^2 - 2r(P_n + P_{n-1}) + 1}}{-2rP_{n-1}}. \end{aligned} \tag{55}$$

□

Using the method in Theorem 11 similarly, we also have the following.

Theorem 12. *If* $\mathbb{C}' = RFPrLRcirc_r fr(P_{n-1}, P_{n-2}, \ldots, P_0)$*, then*

$$\det \mathbb{C}' = \frac{(r - P_{n-1})^n - (P_n - r)^{n-1}(rP_n - rP_{n-1})}{(-1)^n + rQ_{n-1} - rQ_n + 2r^2}. \tag{56}$$

Theorem 13. *If* $\mathbb{P} = RLPrFLcirc_r fr(P_0, P_1, \ldots, P_{n-1})$*, then one has*

$$\det \mathbb{P} = \frac{(r - P_{n-1})^n - (P_n - r)^{n-1}(rP_n - rP_{n-1})}{(-1)^n + rQ_{n-1} - rQ_n + 2r^2} \times (-1)^{n(n-1)/2}. \tag{57}$$

Proof. The matrix $\mathbb{P}$ can be written as

$$\begin{aligned} \mathbb{P} &= \begin{pmatrix} P_0 & \cdots & P_{n-2} & P_{n-1} \\ P_1 & \iddots & P_{n-1} + rP_0 & rP_0 \\ \vdots & \iddots & rP_0 + rP_1 & \vdots \\ P_{n-2} & \iddots & \vdots & rP_{n-3} \\ P_{n-1} + rP_0 & \cdots & rP_{n-3} + rP_{n-2} & rP_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} P_{n-1} & P_{n-2} & \cdots & P_0 \\ rP_0 & P_{n-1} + rP_0 & \cdots & P_1 \\ \vdots & \vdots & \ddots & \vdots \\ rP_{n-2} & rP_{n-3} + rP_{n-2} & \cdots & P_{n-1} + rP_0 \end{pmatrix} \\ &\quad \times \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \end{aligned} \tag{58}$$

Then we can get

$$\det \mathbb{P} = \det \mathbb{C}' \det \Gamma, \tag{59}$$

where $\mathbb{C}' = \text{RFPLRcircfr}(P_{n-1}, P_{n-2}, \ldots, P_0)$ and its determinant could be obtained through Theorem 12; namely,

$$\det \mathbb{C}' = \frac{(r - P_{n-1})^n - (P_n - r)^{n-1}(rP_n - rP_{n-1})}{(-1)^n + rQ_{n-1} - rQ_n + 2r^2}, \tag{60}$$

$$\det \Gamma = (-1)^{n(n-1)/2}. \tag{61}$$

So

$$\begin{aligned} \det \mathbb{P} &= \det \mathbb{C}' \det \Gamma \\ &= \frac{(r - P_{n-1})^n - (P_n - r)^{n-1}(rP_n - rP_{n-1})}{(-1)^n + rQ_{n-1} - rQ_n + 2r^2} \times (-1)^{n(n-1)/2}. \end{aligned} \tag{62}$$

□

6. Determinants of the RFP*r*L*r*R and RLP*r*F*r*L Circulant Matrix with the Pell-Lucas Numbers

Theorem 14. *If* $\mathbb{D} = RFPrLRcirc_r fr(Q_0, Q_1, \ldots, Q_{n-1})$*, then one has*

$$\det \mathbb{D} = \frac{(2 - rQ_n)^n}{1 - rQ_{n-1} - rQ_n + 2(-1)^{n-1}r^2} + \frac{(2 - rQ_n)\left(g_5^{n-1} + h_5^{n-1}\right) - rQ_{n-1}\left(g_5^n + h_5^n\right) - 4r}{1 - rQ_{n-1} - rQ_n + 2(-1)^{n-1}r^2} \times (-r)^n Q_{n-1}^{n-1}, \tag{63}$$

where

$$\begin{aligned} g_5 &= \frac{2 + rQ_{n-1} + rQ_n}{-2rQ_{n-1}} + \frac{\sqrt{r^2(Q_n - Q_{n-1})^2 + 12rQ_{n-1} + 4rQ_n + 4}}{-2rQ_{n-1}}, \\ h_5 &= \frac{2 + rQ_{n-1} + rQ_n}{-2rQ_{n-1}} - \frac{\sqrt{r^2(Q_n - Q_{n-1})^2 + 12rQ_{n-1} + 4rQ_n + 4}}{-2rQ_{n-1}}. \end{aligned} \tag{64}$$

Proof. The method is similar to Theorem 11. □

Certainly, we can get the following theorem.

Theorem 15. *If* $\mathbb{D}' = RFPrLRcirc_r fr(Q_{n-1}, \ldots, Q_1, Q_0)$*, then one gets*

$$\det \mathbb{D}' = \frac{(-2r - rQ_{n-1})^n - 2^{n-1} r^n K}{(-1)^n + rQ_{n-1} - rQ_n + 2r^2}, \quad (65)$$

where

$$K = (-2r - Q_{n-1})(g_6^{n-1} + h_6^{n-1}) + 2r(g_6^n + h_6^n) - r(Q_n - Q_{n-1}),$$

$$g_6 = \frac{Q_n + \sqrt{Q_n^2 + 8rQ_{n-1} + 16r^2}}{4r}, \quad (66)$$

$$h_6 = \frac{Q_n - \sqrt{Q_n^2 + 8rQ_{n-1} + 16r^2}}{4r}.$$

Theorem 16. *If* $\mathbb{Q} = RLPrLFcirc_r fr(Q_0, Q_1, \ldots, Q_{n-1})$*, then*

$$\det \mathbb{Q} = \frac{(-2r - rQ_{n-1})^n - 2^{n-1} r^n K}{(-1)^n + rQ_{n-1} - rQ_n + 2r^2}(-1)^{n(n-1)/2}, \quad (67)$$

where

$$K = (-2r - Q_{n-1})(g_6^{n-1} + h_6^{n-1}) + 2r(g_6^n + h_6^n) - r(Q_n - Q_{n-1}),$$

$$g_6 = \frac{Q_n + \sqrt{Q_n^2 + 8rQ_{n-1} + 16r^2}}{4r}, \quad (68)$$

$$h_6 = \frac{Q_n - \sqrt{Q_n^2 + 8rQ_{n-1} + 16r^2}}{4r}.$$

7. Conclusion

The determinant problems of the RFPrLrR circulant matrices and RLPrFrL circulant matrices involving the Fibonacci, Lucas, Pell, and Pell-Lucas number are considered in this paper. The explicit determinants are presented by using some terms of these numbers.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research is supported by the Development Project of Science & Technology of Shandong Province (Grant no. 2012GGX10115) and NSFC (Grant no. 11301252) and the AMEP of Linyi University, China.

References

[1] F.-R. Lin and H.-X. Yang, “A fast stationary iterative method for a partial integro-differential equation in pricing options,” *Calcolo*, vol. 50, no. 4, pp. 313–327, 2013.

[2] S.-L. Lei and H.-W. Sun, “A circulant preconditioner for fractional diffusion equations,” *Journal of Computational Physics*, vol. 242, pp. 715–725, 2013.

[3] P. E. Kloeden, A. Neuenkirch, and R. Pavani, “Multilevel Monte Carlo for stochastic differential equations with additive fractional noise,” *Annals of Operations Research*, vol. 189, pp. 255–276, 2011.

[4] C. Zhang, H. Chen, and L. Wang, “Strang-type preconditioners applied to ordinary and neutral differential-algebraic equations,” *Numerical Linear Algebra with Applications*, vol. 18, no. 5, pp. 843–855, 2011.

[5] E. W. Sachs and A. K. Strauss, “Efficient solution of a partial integro-differential equation in finance,” *Applied Numerical Mathematics. An IMACS Journal*, vol. 58, no. 11, pp. 1687–1703, 2008.

[6] E. Ahmed and A. S. Elgazzar, “On fractional order differential equations model for nonlocal epidemics,” *Physica A: Statistical Mechanics and its Applications*, vol. 379, no. 2, pp. 607–614, 2007.

[7] J. Wu and X. Zou, “Asymptotic and periodic boundary value problems of mixed FDEs and wave solutions of lattice differential equations,” *Journal of Differential Equations*, vol. 135, no. 2, pp. 315–357, 1997.

[8] P. J. Davis, *Circulant Matrices*, John Wiley & Sons, New York, NY, USA, 1979.

[9] Z. L. Jiang and Z. X. Zhou, *Circulant Matrices*, Chengdu Technology University, Chengdu, China, 1999.

[10] Z. L. Jiang, “On the minimal polynomials and the inverses of multilevel scaled factor circulant matrices,” *Abstract and Applied Analysis*, vol. 2014, Article ID 521643, 10 pages, 2014.

[11] Z. Jiang, T. Xu, and F. Lu, “Isomorphic operators and functional equations for the skew-circulant algebra,” *Abstract and Applied Analysis*, vol. 2014, Article ID 418194, 8 pages, 2014.

[12] Z. Jiang, Y. Gong, and Y. Gao, “Invertibility and explicit inverses of circulant-type matrices with k-Fibonacci and k-Lucas numbers,” *Abstract and Applied Analysis*, vol. 2014, Article ID 238953, 9 pages, 2014.

[13] J. Li, Z. Jiang, and F. Lu, “Determinants, norms, and the spread of circulant matrices with tribonacci and generalized Lucas numbers,” *Abstract and Applied Analysis*, vol. 2014, Article ID 381829, 9 pages, 2014.

[14] D. Chillag, “Regular representations of semisimple algebras, separable field extensions, group characters, generalized circulants, and generalized cyclic codes,” *Linear Algebra and its Applications*, vol. 218, pp. 147–183, 1995.

[15] Z.-L. Jiang and Z.-B. Xu, “Efficient algorithm for finding the inverse and the group inverse of FLS r-circulant matrix,” *Journal of Applied Mathematics & Computing*, vol. 18, no. 1-2, pp. 45–57, 2005.

[16] Z. L. Jiang and D. H. Sun, “Fast algorithms for solving the inverse problem of Ax = b,” in *Proceedings of the 8th International Conference on Matrix Theory and Its Applications in China*, pp. 121C–124C, 2008.

[17] J. Li, Z. Jiang, and N. Shen, “Explicit determinants of the Fibonacci RFPLR circulant and Lucas RFPLR circulant matrix,” *JP Journal of Algebra, Number Theory and Applications*, vol. 28, no. 2, pp. 167–179, 2013.

[18] Z. P. Tian, “Fast algorithm for solving the first-plus last-circulant linear system,” *Journal of Shandong University: Natural Science*, vol. 46, no. 12, pp. 96–103, 2011.

[19] N. Shen, Z. L. Jiang, and J. Li, “On explicit determinants of the RFMLR and RLMFL circulant matrices involving certain

famous numbers," *WSEAS Transactions on Mathematics*, vol. 12, no. 1, pp. 42–53, 2013.

[20] Z. Tian, "Fast algorithms for solving the inverse problem of $AX = b$ in four different families of patterned matrices," *Far East Journal of Applied Mathematics*, vol. 52, no. 1, pp. 1–12, 2011.

[21] D. V. Jaiswal, "On determinants involving generalized Fibonacci numbers," *The Fibonacci Quarterly: Official Organ of the Fibonacci Association*, vol. 7, pp. 319–330, 1969.

[22] L. Dazheng, "Fibonacci-Lucas quasi-cyclic matrices," *The Fibonacci Quarterly: The Official Journal of the Fibonacci Association*, vol. 40, no. 3, pp. 280–286, 2002.

[23] D. A. Lind, "A Fibonacci circulant," *The Fibonacci Quarterly: Official Organ of the Fibonacci Association*, vol. 8, no. 5, pp. 449–455, 1970.

[24] S.-Q. Shen, J.-M. Cen, and Y. Hao, "On the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers," *Applied Mathematics and Computation*, vol. 217, no. 23, pp. 9790–9797, 2011.

[25] M. Akbulak and D. Bozkurt, "On the norms of Toeplitz matrices involving Fibonacci and Lucas numbers," *Hacettepe Journal of Mathematics and Statistics*, vol. 37, no. 2, pp. 89–95, 2008.

[26] G.-Y. Lee, J.-S. Kim, and S.-G. Lee, "Factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices," *The Fibonacci Quarterly: The Official Journal of the Fibonacci Association*, vol. 40, no. 3, pp. 203–211, 2002.

[27] M. Miladinović and P. Stanimirović, "Singular case of generalized Fibonacci and Lucas matrices," *Journal of the Korean Mathematical Society*, vol. 48, no. 1, pp. 33–48, 2011.

[28] P. Stanimirović, J. Nikolov, and I. Stanimirović, "A generalization of Fibonacci and Lucas matrices," *Discrete Applied Mathematics: The Journal of Combinatorial Algorithms, Informatics and Computational Sciences*, vol. 156, no. 14, pp. 2606–2619, 2008.

[29] Z. Zhang and Y. Zhang, "The Lucas matrix and some combinatorial identities," *Indian Journal of Pure and Applied Mathematics*, vol. 38, no. 5, pp. 457–465, 2007.

[30] F. Yilmaz and D. Bozkurt, "Hessenberg matrices and the Pell and Perrin numbers," *Journal of Number Theory*, vol. 131, no. 8, pp. 1390–1396, 2011.

[31] J. Blümlein, D. J. Broadhurst, and J. A. M. Vermaseren, "The multiple zeta value data mine," *Computer Physics Communications*, vol. 181, no. 3, pp. 582–625, 2010.

[32] M. Janjić, "Determinants and recurrence sequences," *Journal of Integer Sequences*, vol. 15, no. 3, pp. 1–2, 2012.

[33] M. Elia, "Derived sequences, the Tribonacci recurrence and cubic forms," *The Fibonacci Quarterly: The Official Journal of the Fibonacci Association*, vol. 39, no. 2, pp. 107–115, 2001.

[34] R. Melham, "Sums involving Fibonacci and Pell numbers," *Portugaliae Mathematica*, vol. 56, no. 3, pp. 309–317, 1999.

[35] Y. Yazlik and N. Taskara, "A note on generalized k-Horadam sequence," *Computers & Mathematics with Applications*, vol. 63, no. 1, pp. 36–41, 2012.

[36] E. Kılıç, "The generalized Pell (p, i)-numbers and their Binet formulas, combinatorial representations, sums," *Chaos, Solitons & Fractals*, vol. 40, no. 4, pp. 2047–2063, 2009.