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Research Article

$H^1 \cap L^p$ versus C^1 Local Minimizers

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We show that a local minimizer of Φ in the C^1 topology must be a local minimizer in the $H^1 \cap L^p$ topology, under suitable assumptions for the functional $\Phi = (1/2)\int_\Omega |\nabla u|^2 + (1/p)\int_\Omega |u|^p - \int_\Omega F(x,u)$ with supercritical exponent $p > 2^* = 2n/(n-2)$. This result can be used to establish a solution to the corresponding equation admitting sub- and supersolution. Hence, we extend the conclusion proved by Brezis and Nirenberg (1993), the subcritical and critical case.

1. Main Results for Supercritical Exponent

We consider the following functional:

$$\Phi = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{p}\int_\Omega |u|^p - \int_\Omega F(x,u), \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ with smooth boundary, supercritical exponent $p > 2^* = 2n/(n-2)$, and $F(x,u) = \int_0^u f(x,s)ds$ satisfies the growth condition:

$$|f(x,u)| \leqslant C\left(1 + |u|^\ell\right) \quad \text{with } \ell < p, \tag{2}$$

as well as the usual assumptions that f is measurable in x and continuous in u.

Our main results are the following.

Theorem 1. *Assuming that $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$ is a local minimizer of Φ in the C^1 topology, there is some $r > 0$, such that*

$$\Phi(u_0) \leqslant \Phi(u_0 + v), \quad \forall v \in C_0^1(\overline{\Omega}) \quad \text{with } \|v\|_{C^1} \leqslant r. \tag{3}$$

Then u_0 is also a local minimizer of Φ in the $H_0^1(\Omega) \cap L^p(\Omega)$ topology; that is, there exists $\epsilon_0 > 0$, such that

$$\Phi(u_0) \leqslant \Phi(u_0 + v), \quad \forall v \in H_0^1(\Omega) \cap L^p(\Omega) \quad \text{with } \|v\|_{H_0^1(\Omega)\cap L^p(\Omega)} \leqslant r, \tag{4}$$

where the topology $X \triangleq H_0^1(\Omega) \cap L^p(\Omega)$ given by $\|\cdot\|_X = \|\cdot\|_{H_0^1(\Omega)} + \|\cdot\|_{L^p(\Omega)}$.

It will be noted that an X neighbourhood is much bigger than a C_0^1 neighbourhood. The proof depends on the special structure of Φ, and the claim clearly would be false for a general function Φ.

In order to prove this theorem, the following preparatory steps are critical. We begin with a theorem concerning the topology of X.

Theorem 2. *Let X be defined as in the above theorem; then X is a reflexive and strictly convex Banach space with the duality $X^* \subset H^{-1}(\Omega) \oplus L^q(\Omega)((1/p) + (1/q) = 1)$.*

Proof of Theorem 2. Now we give a detailed proof for the reader's convenience.

At first we show that the definition of $\|\cdot\|_X$ is actually a norm. Obviously, separate points are as follows: if $\|x\|_X = 0$, that is, $\|x\|_{H_0^1(\Omega)} + \|x\|_{L^p(\Omega)} = 0$, then $x = 0$. And positive homogeneity is $\|\alpha x\|_X = \|\alpha x\|_{H_0^1(\Omega)} + \|\alpha x\|_{L^p(\Omega)} = |\alpha|[\|x\|_{H_0^1(\Omega)} + \|x\|_{L^p(\Omega)}] = |\alpha|\|x\|_X$. The triangle inequality is, for any $x, y \in X$,

$$\begin{aligned}\|x+y\|_X &= \|x+y\|_{H_0^1(\Omega)} + \|x+y\|_{L^p(\Omega)} \\ &\leqslant \left[\|x\|_{H_0^1(\Omega)} + \|x\|_{L^p(\Omega)}\right] \\ &\quad + \left[\|y\|_{H_0^1(\Omega)} + \|y\|_{L^p(\Omega)}\right] = \|x\|_X + \|y\|_X.\end{aligned} \tag{5}$$

And then it shows that the space X is complete; that is to say, any cauchy sequence $\{u_n\}$ in $\|\cdot\|_X$ will converge. From the definition of the norm $\|\cdot\|_X$, we know that u_n is also the cauchy sequence in $H_0^1(\Omega)$ and $L^p(\Omega)$. By the completion of the Banach space $H_0^1(\Omega)$ and $L^p(\Omega)$, we know that u_n will converge to u_1 in $H_0^1(\Omega)$ and u_n will converge to u_2 in $L^p(\Omega)$. And, since $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, we know that $u_n \to u_1$ in $L^2(\Omega)$ and also, due to $L^p(\Omega) \subseteq L^2(\Omega)$, we also know that $u_n \to u_2$ in $L^2(\Omega)$, and, based on the uniqueness of the limit in $L^2(\Omega)$, we have $u_1 = u_2$ (denoted by u). With this result, we have u_n converge to u in X, which implies that X is complete. Thus, X is a Banach space.

For strictly convex, which is based on the definition of the strictly convex of Banach space, we need to show that if $x \neq y$ and $\|x\|_{H_0^1(\Omega)} + \|x\|_{L^p(\Omega)} = \|y\|_{H_0^1(\Omega)} + \|y\|_{L^p(\Omega)} = 1$, then $\|x+y\|_{H_0^1(\Omega)} + \|x+y\|_{L^p(\Omega)} < 2$, which can be done by the following inequality:

$$\begin{aligned} &\|x+y\|_{H_0^1(\Omega)} + \|x+y\|_{L^p(\Omega)} \\ &\quad \leqslant \left(\|x\|_{H_0^1(\Omega)} + \|y\|_{H_0^1(\Omega)}\right) + \left(\|x\|_{L^p(\Omega)} + \|y\|_{L^p(\Omega)}\right) \\ &\quad = \left(\|x\|_{H_0^1(\Omega)} + \|x\|_{L^p(\Omega)}\right) + \left(\|y\|_{H_0^1(\Omega)} + \|y\|_{L^p(\Omega)}\right) \\ &\quad = 2. \end{aligned} \tag{6}$$

And the fact "=" in (6) is true if and only if $x = cy$ with the constant $c > 0$ in consequence of the uniformly convex space $H_0^1(\Omega)$ and $L^p(\Omega)(1 < p < \infty)$ (P97 [1]). And, combining with $\|x\|_X = \|y\|_X = 1$, we can get the constant $c = 1$, which contradicts the assumption $x \neq y$. Therefore, the Banach space X is strictly convex.

For the reflexive, we need the following lemmas (see P63, P105 [2]).

Lemma 3. *Let $X_1, \ldots, X_n$ be normed space. Then $X_1 \oplus \cdots \oplus X_n$ is a Banach spaces if and only if each X_j is a Banach space; furthermore, $X_1 \oplus \cdots \oplus X_n$ is reflexive if and only if each X_j is reflexive.*

Lemma 4. *Every closed subspace of a reflexive space is reflexive.*

Therefore, setting a space $E = H_0^1(\Omega) \oplus L^p(\Omega)$ with the norm $\|\cdot\|_X = \|\cdot\|_{H_0^1(\Omega)} + \|\cdot\|_{L^p(\Omega)}$. It follows from Lemma 3 that E is a reflexive Banach space. Obviously, our space $X = H_0^1(\Omega) \cap L^p(\Omega)(1 < p < \infty)$ with the norm $\|\cdot\|_X = \|\cdot\|_{H_0^1(\Omega)} + \|\cdot\|_{L^p(\Omega)}$ can be seen as a closed subspace of E by the embedded mapping $u \to (u, u)$ (denoting (u, u) by $i(u)$ in the following). Thus, based on Lemma 4, X is a reflexive Banach space.

For the dual, we need the following lemma (see P91 [2]).

Lemma 5. *Let $X_1, \ldots, X_n$ be normed spaces. Then there is an isometric isomorphism that identifies $(X_1 \oplus \cdots \oplus X_n)^*$ with $X_1^* \oplus \cdots \oplus X_n^*$, such that, if the element y^* of $(X_1 \oplus \cdots \oplus X_n)^*$ is identified with the element $x_1^*, \ldots, x_n^*$ of $X_1^* \oplus \cdots X_n^*$, then*

$$y^*(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j^* x_j \tag{7}$$

whenever $(x_1, \ldots, x_n) \in X_1 \oplus \cdots \oplus X_n$.

From the Lemma 5, we know that the dual space E^* of $E = H_0^1(\Omega) \oplus L^p(\Omega)$ will be $E^* = H^{-1}(\Omega) \oplus L^q(\Omega)$ with $(1/p) + (1/q) = 1$. And, if our space $X = H_0^1(\Omega) \cap L^p(\Omega)$ can be seen as a closed subspace of E, then, $X^* \subseteq H^{-1}(\Omega) \oplus L^q(\Omega)$ (in the sense of restriction). At the same time, from the Hahn-Banach theorem, we know that, for any $f \in X^*$, we can extend f to be a bounded linear functional $\tilde{f}$ on E, such that

$$\left\langle \tilde{f}, i(u) \right\rangle_{E^*, E} = \langle f, u \rangle_{X^*, X} \quad \forall u \in X. \tag{8}$$

And, from (7), we have

$$\left\langle \tilde{f}, i(u) \right\rangle_{E^*, E} = \langle f_1, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_2, u \rangle_{L^q(\Omega), L^p(\Omega)}. \tag{9}$$

Hence,

$$\begin{aligned} \langle f, u \rangle_{X^*, X} &= \left\langle \tilde{f}, i(u) \right\rangle_{E^*, E} \\ &= \langle f_1, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_2, u \rangle_{(L^q\Omega), L^p(\Omega)} \end{aligned} \tag{10}$$

which implies that $f \in E^*$. Therefore, $X^* \subset H^{-1}(\Omega) \oplus L^q(\Omega)$ and the proof of Theorem 2 is completely finished. □

Also, for the property of weak converge in X, we have the following.

Lemma 6. *If $u_n \rightharpoonup v$ in X as $n \to \infty$, then*

$$u_n \rightharpoonup v \quad \text{in } H_0^1(\Omega), \quad u_n \rightharpoonup v \quad \text{in } L^p(\Omega) \quad \text{as } n \longrightarrow \infty \tag{11}$$

$$u_n \longrightarrow v \quad \text{in } L^t(\Omega) \quad \forall 2 \leqslant t < p \quad \text{as } n \longrightarrow \infty \tag{12}$$

Proof. In fact, for (11), from Theorem 2, we know that, for any $f \in X^*$, there exists $f_1 \in H^{-1}(\Omega)$ and $f_2 \in L^q(\Omega)$, such that

$$\langle f, u \rangle_{X^*, X} = \langle f_1, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle f_2, u \rangle_{L^q(\Omega), L^p(\Omega)}. \tag{13}$$

Now, choosing $f_2 = 0$ in (13) (noting the fact that $H^{-1}(\Omega) \times \{0\} \subset X^*$) and combining with $u_n \rightharpoonup v$ in X, we know that, for any $f_1 \in H^{-1}(\Omega)$,

$$\begin{aligned} \langle f, u_n \rangle_{X^*, X} &= \langle f_1, u_n \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle 0, u_n \rangle_{L^q(\Omega), L^p(\Omega)} \\ &= \langle f_1, u_n \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \longrightarrow \langle f_1, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \end{aligned} \tag{14}$$

which implies that $u_n \rightharpoonup v$ in $H_0^1(\Omega)$. Similarly, choosing $f_1 = 0$ in (13) (also noting the fact that $\{0\} \times L^q(\Omega) \subset X^*$), we can get $u_n \rightharpoonup v$ in $L^p(\Omega)$ and finish the proof of (11).

And, for (12), from the interpolation inequality,

$$\|u-v\|_{L^t(\Omega)} \leqslant \|u-v\|_{L^2(\Omega)}^{\theta}\|u-v\|_{L^p(\Omega)}^{1-\theta} \tag{15}$$

and $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ with (11), it is easy to prove (12) and complete the proof of Lemma 6. □

For the operators generated by (1), we have, for any $p > 1$

Lemma 7. *Both of operators $-\Delta : u \rightarrow -\Delta u$ from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ and $F : u \rightarrow |u|^{p-2}u$ from $L^p(\Omega)$ to $L^{p'}(\Omega) = L^{p/(p-1)}(\Omega)$ are bijective, where $\|u\|_{H_0^1}^2 = \int_\Omega \nabla u \nabla u$.*

Proof. First, for any $u \in H_0^1$, it follows that $-\Delta u \in H^{-1}$ from

$$\langle -\Delta u, v\rangle_{H^{-1},H_0^1} = \int_\Omega \nabla u \nabla v, \quad \forall v \in H_0^1. \tag{16}$$

If $u \neq v \in H_0^1$, it follows from the maximum principle (see P179 Theorem 8.1 [3]) that $-\Delta u \neq -\Delta v \in H^{-1}$, which implies that it is an injection. Whereas, by Riesz's Lemma, we know that, for any $f \in H^{-1}$, there exists a $u \in H_0^1(\Omega)$, such that $\|f\|_{H^{-1}} = \|u\|_{H_0^1}$ and

$$\langle f, v\rangle_{H^{-1},H_0^1} = (u,v)_{H_0^1} = \int_\Omega \nabla u \nabla v = \int_\Omega -\Delta u v \quad \forall v \in H_0^1(\Omega) \tag{17}$$

which implies that $f = -\Delta u$ and $\|-\Delta u\|_{H^{-1}} = \|f\|_{H^{-1}} = \|u\|_{H_0^1}$. Hence $-\Delta : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is bijective (indeed, isometric).

Secondly, the map $F : u \rightarrow |u|^{p-2}u$ is clearly bounded, continuous, and also injective; namely, if $u \neq v \in L^p(\Omega)$, then $|u|^{p-2}u \neq |v|^{p-2}v \in L^{p'}$, which can be obtained by the following inequality $\langle |u|^{p-2}u - |v|^{p-2}v, u-v\rangle \geqslant (1/p)|u-v|^p$. For surjective, by applying the James Theorem in Banach space (see [4]) to the strictly convex space L^p and $L^{p'}$, for any $\|w\|_{L^{p'}} = 1$, there is only one unique supporting functional $\|u\|_{L^p} = 1$, such that $\langle w, u\rangle = 1$, which implies that $w = |u|^{p-2}u$. So F is bijective. □

For the regularity of solution of (1), we have the following.

Lemma 8. *Assuming $u_0 \in X$ satisfies in the weak sense*

$$\begin{aligned} -\Delta u + |u|^{r-2}u = f(x,u) \quad &\text{in } \Omega \\ u = 0 \quad &\text{on } \partial\Omega \end{aligned} \tag{18}$$

then one has $u_0 \in C^{1,\alpha}(\overline{\Omega})$, for all $\alpha < 1$.

Proof. Indeed, we set the corresponding evolution equation

$$\begin{aligned} u_t - \Delta u + |u|^{r-2}u = f(x,u) \quad &\text{in } \Omega \\ u = 0 \quad &\text{on } \partial\Omega \end{aligned} \tag{19}$$

and apply the same argument by the Moser iteration as Lemma 5.20 in [5], and, with the fact that the solution of (18) is the equilibrium point of (19), it is easy to show Lemma 8.

Now, we are in a position to prove Theorem 1. □

Proof. Suppose the conclusion (3) does not hold. Then

$$\forall \epsilon > 0, \quad \exists v_\epsilon \in B_\epsilon \quad \text{such that } \Phi(v_\epsilon) < \Phi(0) \tag{20}$$

where $B_\epsilon = \{u \in X; \|u\|_X \leqslant \epsilon\}$. □

Claim 1. $\min_{B_\epsilon}\Phi$ is achieved at some point (still denoted by v_ϵ).

Indeed, it is clear that there exists a constant C, such that $\|\Phi(u)\| \leqslant C$ for all $u \in B_\epsilon$. Hence, there exists a minimizing sequence $u_n \in B_\epsilon$, and there is by Lemma 6 a subsequence (also denoted by u_n) such that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, $L^p(\Omega)$. Combining with the lower semicontinuity of norm, we have $\liminf_{n\to\infty}\|u_n\|_{H_0^1(\Omega)} \geqslant \|u\|_{H_0^1(\Omega)}$, $\liminf_{n\to\infty}\|u_n\| \geqslant \|u\|_{L^p(\Omega)}$, and $\lim_{n\to\infty}F(x,u_n) \rightarrow F(x,u)$. Hence, $\liminf_{n\to\infty}\Phi(u_n) \geqslant \Phi(u)$ and Claim 1 is completely proved.

Now we will prove that $v_\epsilon \rightarrow 0$ in C^1, but (3) and (20) are contradictory (also see [6]). The corresponding Euler equation for v_ϵ involves a Lagrange multiplier $\mu_\epsilon \leqslant 0$; namely, v_ϵ satisfies

$$\left\langle \Phi'(v_\epsilon), \zeta\right\rangle_{X^*,X} = \mu_\epsilon \langle i(v_\epsilon), \zeta\rangle_{X^*,X}, \quad \forall \zeta \in X, \tag{21}$$

where $i(v_\epsilon) = -2\Delta v_\epsilon + p|v_\epsilon|^{p-2}v_\epsilon$ due to Lemma 7.

That is,

$$\begin{aligned} &\int_\Omega \nabla v_\epsilon \nabla\zeta + \int_\Omega |v_\epsilon|^{p-2}v_\epsilon\zeta - \int_\Omega f(x,v_\epsilon)\zeta \\ &\quad = 2\mu_\epsilon \int_\Omega \nabla v_\epsilon\nabla\zeta + \mu_\epsilon p\int_\Omega |v_\epsilon|^{p-2}v_\epsilon\zeta. \end{aligned} \tag{22}$$

This means that

$$-(1-2\mu_\epsilon)\Delta v_\epsilon + (1-p\mu_\epsilon)|v_\epsilon|^{p-2}v_\epsilon = f(x,v_\epsilon). \tag{23}$$

Recalling that $\mu_\epsilon \leqslant 0$ and combining with Lemma 8, one may bootstrap the bound $\|v_\epsilon\|_X \leqslant C$ to $\|v_\epsilon\|_{C^1} \leqslant C$ (independent of ϵ), since $v_\epsilon \rightarrow 0$ in X, $v_\epsilon \rightarrow 0$ in C^1 (by Ascoli). This concludes the proof.

2. Application of Theorem 1

Next, we present a simple and useful application of Theorem 1.

Considering Φ in Theorem 1 with f, such that, for some constant k,

$$f(x,u) + ku \text{ is nondecreasing } u \quad \text{for a.e. } x. \tag{24}$$

Assume that there are $C(\overline{\Omega})$ sub- and supersolutions $\underline{u}$ and $\overline{u}$; that is, in the distribution sense,

$$\begin{aligned} &-\Delta\underline{u} + |\underline{u}|^{p-2}\underline{u} - f(x,\underline{u}) \\ &\qquad \leqslant 0 \leqslant -\Delta\overline{u} + |\overline{u}|^{p-2}\overline{u} - f(x,\overline{u}) \quad \text{in } \Omega \\ &\qquad\qquad \underline{u} \leqslant 0 \leqslant \overline{u} \quad \text{on } \partial\Omega. \end{aligned} \tag{25}$$

Moreover, neither $\underline{u}$ nor $\overline{u}$ is a solution to (18).

Theorem 9. *Under the assumption* (2) *there is a solution* u_0 *to* (18), $\underline{u} < u_0 < \overline{u}$, *such that, in addition,* u_0 *is a local minimum of* Φ *in* X.

The proof relies on Theorem 1 as well as on the following.

Theorem 10. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with smooth boundary* $\partial\Omega$. *Let* $u \in L^{p-1}_{\mathrm{loc}}(\Omega)$ *and assume that, for some* $k \geqslant 0$, u *satisfies*

$$\begin{gathered} -\Delta u + |u|^{p-2}u + ku \geqslant 0 \quad \text{in } \Omega. \\ u \geqslant 0 \quad \text{on } \Omega \end{gathered} \tag{26}$$

Then either $u \equiv 0$ *or there exists* $\epsilon > 0$, *such that*

$$u(x) \geqslant \epsilon \operatorname{dist}(x, \partial\Omega) \quad \text{in } \Omega. \tag{27}$$

Moreover, if k *is replaced by the nonnegative continuous function* $c(x) \in C(\overline{\Omega})$, *then the conclusion is also valid.*

Proof. The measure $\mu = -\Delta u + |u|^{p-2}u + ku$ is nonnegative in Ω. We may assume $u \not\equiv 0$. □

Case 1. Consider $\mu \equiv 0$. In this case, $u \in C^{\infty}(\Omega)$ by induction applies to Lemma 8:

$$-\Delta u + |u|^{p-2}u + ku = 0, \quad u \geqslant 0 \text{ in } \Omega. \tag{28}$$

Since $u \not\equiv 0$, we have $u \geqslant \delta > 0$ in some closed ball B in Ω. Let Ω_j be an expanding sequence of subdomains of Ω with smooth boundaries and $\bigcup_j \Omega_j = \Omega$; suppose $B \subset \Omega_j$, for all j. Let h_j be the solution in $\Omega_j \setminus B$ of

$$\begin{gathered} -\Delta h_j + \left|h_j\right|^{p-2} h_j + kh_j = 0, h_j \geqslant 0 \quad \text{in } \Omega_j \setminus B \\ h_j = \delta \quad \text{on } \partial B \\ h_j = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{29}$$

In order to compare u with h_j, we need the following comparison principle for the operator $L = -\Delta + |\cdot|^{p-2}$. defined in Lemma 7

Lemma 11. *Let* $u, v \in C^{1,\alpha}(\overline{\Omega})$ *satisfy, for some* $k \geqslant 0$,

$$\begin{gathered} Lu + ku \geqslant Lv + kv \quad \text{in } \Omega \\ u \geqslant v \quad \text{on } \partial\Omega; \end{gathered} \tag{30}$$

then, $u \geqslant v$ *in* Ω.

Proof. Indeed, setting

$$Lu - Lv = -\Delta u + |u|^{p-2} - \left(-\Delta v + |v|^{p-2} v\right) \tag{31}$$

and defining $w = u - v$, it is noted that the derivative expression $(|s|^{p-2}s)' = (p-1)|s|^{p-2} \geqslant 0$. Then, by the mean value theorem, there is $\xi = \theta u + (1-\theta)v (0 < \theta < 1)$ satisfying

$$\begin{gathered} -\Delta w + (p-2)|\xi|^{p-2} w + kw \geqslant 0 \quad \text{in } \Omega \\ w \geqslant 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{32}$$

Applying the weak maximum principle, Theorem 8.1 P179 [3] by choosing $c(x) = (p-2)|\xi|^{p-2}$, we know that $w \geqslant 0$ and complete the proof. □

Since $u(x), h(x) \in C^{1,\alpha}(\overline{\Omega_j \setminus B})$ in Lemma 8, then, by the virtue of Lemma 11, $u \geqslant h_j$ in $\Omega_j \setminus B$. As $j \to \infty$, we find

$$u \geqslant h \quad \text{in } \Omega \setminus B, \tag{33}$$

when h solves

$$\begin{gathered} -\Delta h + |h|^{p-2}h + kh = 0, \quad h \geqslant 0 \quad \text{in } \Omega \setminus B \\ h = \delta \quad \text{on } \partial B \\ h = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{34}$$

By the Hopf lemma 3.4 P34 [3] with $c(x) = |h|^{p-2} + k$, one obtains for some $\epsilon > 0$

$$h(x) \geqslant \epsilon \operatorname{dist}(x, \partial\Omega) \quad \text{in } \Omega \setminus B. \tag{35}$$

The conclusion of Theorem 10 then follows directly.

Case 2. Consider $\mu \not\equiv 0$. Let $\zeta \in C_0^{\infty}(\Omega)$ be a cutoff function, $0 \leqslant \zeta \leqslant 1$, such that $\zeta\mu \not\equiv 0$. Let v be the solution of

$$\begin{gathered} (L + k) v = \zeta\mu \quad \text{in } \Omega \\ v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{36}$$

Since v is smooth outside a compact set $K \subset \Omega$, it follows and applies to the Hopf lemma as above for some $\epsilon > 0$,

$$v(x) \geqslant \epsilon \operatorname{dist}(x, \partial\Omega) \quad \text{in } \Omega \setminus B. \tag{37}$$

The conclusion of Theorem 10 is a direct consequence of the following.

Claim 2. One has $u \geqslant v$ in Ω.

Proof of Claim 2. Given any $\alpha > 0$, we will prove that

$$\overline{u} = u + \alpha \geqslant v \quad \text{in } \Omega. \tag{38}$$

The claim then follows.

Note that $w = \overline{u} - v$ satisfies

$$\begin{aligned} &(-\Delta + k) w + |\overline{u}|^{p-2}\overline{u} - |v|^{p-2}v \\ &\quad = (1-\zeta)\mu + |\overline{u}|^{p-2}\overline{u} - |u|^{p-2}u + k\alpha \geqslant 0 \quad \text{in } \Omega \end{aligned} \tag{39}$$

$$w \geqslant 0 \quad \text{in } N_\eta = \{x \in \Omega; \operatorname{dist}(x, \partial) < \eta\} \tag{40}$$

provided η is sufficiently small (depending on α). The property (39) follows from the inequality $|\overline{u}|^{p-2}\overline{u} - |u|^{p-2}u \geqslant 0$, a.e. $x \in \Omega$, when $\overline{u} = u + \alpha > u$. The last property (40) follows from the fact that v is smooth near $\partial\Omega$ and $v = 0$ on $\partial\Omega$.

Let ρ_j be a sequence of mollifiers with supp $\rho_j \subset B(0, 1/j)$ and set $w_j(x) = \int_\Omega \rho_j(x-y)w(y)$.

Clearly w_j is smooth, and, by (39) and the mean value theorem with $\xi = \theta\overline{u} + (1-\theta)v$, we have

$$\left(-\Delta + k + (p-1)|\xi|^{p-2}\right) w_j \geqslant 0 \quad \text{in } \Omega \setminus \overline{N}_{1/j}. \tag{41}$$

On the other hand, we deduce from (40) that

$$w_j \geqslant 0 \quad \text{on } \partial\left(\Omega \setminus \overline{N}_{1/j}\right) \tag{42}$$

provided $\eta > 2/j$. The maximum principle (Corollary 3.2 P33 [3]) of choosing $c(x) = (p-1)|\xi|^{p-2} \geqslant 0$ implies that

$$w_j \geqslant 0 \quad \text{in } \Omega \setminus \overline{N}_{1/j} \tag{43}$$

when $\eta > 2/j$. Passing to the limit as $j \to \infty$, we see that

$$w \geqslant 0 \quad \text{in } \Omega \tag{44}$$

which is the desired conclusion. The similar argument is also true as k is replaced by the nonnegative continuous function $c(x) \in C(\overline{\Omega})$ and the proof of Theorem 10 is completely finished. □

Now we are in a position to prove Theorem 9.

Proof of Theorem 9. On the basis of our above results, we can prove Theorem 9 by the similar argument as [7] and rewrite it here for the reader's convenience. We introduce an auxiliary function. Set

$$\tilde{f}(x,s) = \begin{cases} f(x,\underline{u}(x)) & \text{if } s < \underline{u}(x) \\ f(x,s) & \text{if } \underline{u}(x) \leqslant s \leqslant \overline{u}(x) \\ f(x,\overline{u}(x)) & \text{if } s > \overline{u}(x); \end{cases} \tag{45}$$

it is continuous in s. Then set

$$\begin{aligned} \widetilde{F}(x,u) &= \int_0^u \tilde{f}(x,s)\,ds, \\ \widetilde{\Phi}(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{p}|u|^p - \int_\Omega \widetilde{F}(x,u). \end{aligned} \tag{46}$$

By the similar argument as Claim 1, there is a minimum $u_0 \in X$ satisfying

$$-\Delta u_0 + |u_0|^{p-2}u_0 = \tilde{f}(x,u_0) \quad \text{in } \Omega. \tag{47}$$

And, with Lemma 8, we can get $u_0 \in W^{2,p}$, for all $p < \infty$. We claim that $\underline{u} \leqslant u_0 \leqslant \overline{u}$; we will just prove the first inequality. Indeed, we have

$$L(\underline{u}) - L(u_0) \leqslant f(x,\underline{u}) - \tilde{f}(x,u_0) \tag{48}$$

and in particular

$$L(\underline{u}) - L(u_0) \leqslant 0 \quad \text{in } A = \left\{x \in \Omega;\ u_0(x) < \underline{u}(x)\right\}. \tag{49}$$

Since $\underline{u} - u_0 \leqslant 0$ on ∂A, it follows from the comparison principle (i.e., Lemma 11) that $\underline{u} - u_0 \leqslant 0$ in A. Therefore, $A = \emptyset$ and the claim is proved.

Returning to (48), we have

$$\begin{aligned} &L(\underline{u}) - L(u_0) + k(\underline{u} - u_0) \\ &\quad \leqslant (f(x,\underline{u}) + \underline{u}) - (f(x,u_0) + ku_0) \leqslant 0. \end{aligned} \tag{50}$$

Since $\underline{u}$ is not a solution, it follows from Theorem 10 that there is some $\epsilon > 0$, such that

$$\underline{u}(x) - u_0(x) \leqslant -\epsilon\,\mathrm{dist}(x,\partial\Omega), \quad \forall x \in \Omega. \tag{51}$$

Similarly, for $\overline{u}$, we have

$$\underline{u}(x) + \epsilon\ \mathrm{dist}\ (x,\partial\Omega) \leqslant u_0(x) \leqslant \overline{u}(x) - \epsilon\,\mathrm{dist}(x,\partial\Omega), \quad \forall x \in \Omega. \tag{52}$$

It follows that, if $u \in C_0^1(\overline{\Omega})$ and $\| u - u_0\|_{C^1} \leqslant \epsilon$, then

$$\underline{u} \leqslant u \leqslant \overline{u} \quad \text{in } \Omega. \tag{53}$$

Finally, we apply the fact that $\widetilde{F}(x,u) - F(x,u)$ is a function of x alone for $u \in [\underline{u}(x), \overline{u}(x)]$. In particular, $\Phi(u) - \widetilde{\Phi}(u)$ is constant for $\|u - u_0\|_{C^1} \leqslant \epsilon$. Hence, u_0 is a local minimum of Φ in the C^1 topology (since it is a global minimum for $\widetilde{\Phi}$). So, from Theorem 1, we know that u_0 is also a local minimum of Φ in the X topology and the proof of Theorem 9 is finished. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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