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Research Article

Initial Coefficients of Biunivalent Functions

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An analytic function f defined on the open unit disk is biunivalent if the function f and its inverse f^{-1} are univalent in $\mathbb{D}$. Estimates for the initial coefficients of biunivalent functions f are investigated when f and f^{-1}, respectively, belong to some subclasses of univalent functions. Some earlier results are shown to be special cases of our results.

1. Introduction

Let $\mathcal{S}$ be the class of all univalent analytic functions f in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. For $f \in \mathcal{S}$, it is well known that the nth coefficient is bounded by n. The bounds for the coefficients give information about the geometric properties of these functions. Indeed, the bound for the second coefficient of functions in the class $\mathcal{S}$ gives rise to the growth, distortion and covering theorems for univalent functions. In view of the influence of the second coefficient in the geometric properties of univalent functions, it is important to know the bounds for the (initial) coefficients of functions belonging to various subclasses of univalent functions. In this paper, we investigate this coefficient problem for certain subclasses of biunivalent functions.

Recall that the Koebe one-quarter theorem [1] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius 1/4. Thus, every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$, $(z \in \mathbb{D})$, and

$$f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right). \tag{1}$$

A function $f \in \mathcal{S}$ is *biunivalent* in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Let σ denote the class of biunivalent functions defined in the unit disk $\mathbb{D}$. Lewin [2] investigated this class σ and obtained the bound for the second coefficient of the biunivalent functions. Several authors subsequently studied similar problems in this direction (see [3, 4]). A function $f \in \sigma$ is bistarlike or strongly bistarlike or biconvex of order α if f and f^{-1} are both starlike, strongly starlike, or convex of order α, respectively. Brannan and Taha [5] obtained estimates for the initial coefficients of bistarlike, strongly bistarlike, and biconvex functions. Bounds for the initial coefficients of several classes of functions were also investigated in [6–24].

An analytic function f is *subordinate* to an analytic function g, written $f(z) \prec g(z)$, if there is an analytic function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ satisfying $f(z) = g(w(z))$. Ma and Minda [25] unified various subclasses of starlike ($\mathcal{S}^*$) and convex functions ($\mathcal{C}$) by requiring that either the quantity $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a more general superordinate function φ with positive real part in the unit disk $\mathbb{D}$, $\varphi(0) = 1$, $\varphi'(0) > 0$, φ maps $\mathbb{D}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class $\mathcal{S}^*(\varphi)$ of Ma-Minda starlike functions with respect to φ consists of functions $f \in \mathcal{S}$ satisfying the subordination $zf'(z)/f(z) \prec \varphi(z)$. Similarly, the class $\mathcal{C}(\varphi)$ of Ma-Minda convex functions consists of functions $f \in \mathcal{S}$ satisfying the subordination $1 + zf''(z)/f'(z) \prec \varphi(z)$. Ma and Minda investigated growth and distortion properties of functions in $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ as well as Fekete-Szegö inequalities for $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$. Their proof of Fekete-Szegö inequalities requires the univalence of φ. Ali et al. [7] investigated Fekete-Szegö problems for various other classes and their proof does not require the univalence or starlikeness of φ. In particular, their results are valid even if one just assumes the function φ to have a series expansion of the form $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, $B_1 > 0$. So, in this paper, we assume that φ has series expansion $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, B_1, B_2 are real, and $B_1 > 0$. A function f is Ma-Minda bistarlike or Ma-Minda biconvex if both f and f^{-1} are, respectively, Ma-Minda starlike or convex. Motivated by the Fekete-Szegö problem for the classes of Ma-Minda starlike and Ma-Minda convex functions [25], Ali et al. [26] recently obtained estimates of the initial coefficients for biunivalent Ma-Minda starlike and Ma-Minda convex functions.

The present work is motivated by the results of Kędzierawski [27] who considered functions f belonging to certain subclasses of univalent functions while their inverses f^{-1} belong to some other subclasses of univalent functions. Among other results, he obtained the following coefficient estimates.

Theorem 1 (see [27]). *Let $f \in \sigma$ with Taylor series $f(z) = z + a_2 z^2 + \cdots$ and $g = f^{-1}$. Then,*

$$|a_2| \le \begin{cases} 1.5894 & \text{if } f \in \mathcal{S},\ g \in \mathcal{S}, \\ \sqrt{2} & \text{if } f \in \mathcal{S}^*,\ g \in \mathcal{S}^*, \\ 1.507 & \text{if } f \in \mathcal{S}^*,\ g \in \mathcal{S}, \\ 1.224 & \text{if } f \in \mathcal{C},\ g \in \mathcal{S}. \end{cases} \tag{2}$$

We need the following classes investigated in [6, 7, 26].

Definition 2. Let $\varphi : \mathbb{D} \to \mathbb{C}$ be analytic and $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$ and $B_2 \in \mathbb{R}$. For $\alpha \ge 0$, let

$$\begin{aligned} \mathcal{M}(\alpha, \varphi) &:= \left\{ f \in \mathcal{S} : (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\}, \\ \mathcal{L}(\alpha, \varphi) &:= \left\{ f \in \mathcal{S} : \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z) \right\}, \\ \mathcal{P}(\alpha, \varphi) &:= \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec \varphi(z) \right\}. \end{aligned} \tag{3}$$

In this paper, we obtain the estimates for the second and third coefficients of functions f when

(i) $f \in \mathcal{P}(\alpha, \varphi)$ and $g := f^{-1} \in \mathcal{P}(\beta, \psi)$, or $g \in \mathcal{M}(\beta, \psi)$, or $g \in \mathcal{L}(\beta, \psi)$,

(ii) $f \in \mathcal{M}(\alpha, \varphi)$ and $g \in \mathcal{M}(\beta, \psi)$, or $g \in \mathcal{L}(\beta, \psi)$,

(iii) $f \in \mathcal{L}(\alpha, \varphi)$ and $g \in \mathcal{L}(\beta, \psi)$.

2. Coefficient Estimates

In the sequel, it is assumed that φ and ψ are analytic functions of the form

$$\begin{aligned} \varphi(z) &= 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0), \\ \psi(z) &= 1 + D_1 z + D_2 z^2 + D_3 z^3 + \cdots \quad (D_1 > 0). \end{aligned} \tag{4}$$

Theorem 3. *Let $f \in \sigma$ and $g = f^{-1}$. If $f \in \mathcal{P}(\alpha, \varphi)$, $g \in \mathcal{P}(\beta, \psi)$ and f is of the form*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{5}$$

then

$$\begin{aligned} |a_2| \le{}& \left(B_1 D_1 \sqrt{B_1(1+3\beta) + D_1(1+3\alpha)}\right) \\ &\times \Big(\big|\sigma B_1^2 D_1^2 - (1+2\alpha)^2(1+3\beta)(B_2 - B_1)D_1^2 \\ &\quad -(1+2\beta)^2(1+3\alpha)(D_2 - D_1)B_1^2\big|\Big)^{-1/2}, \end{aligned} \tag{6}$$

$$\begin{aligned} 2\sigma|a_3| \le{}& B_1(3+10\beta) + D_1(1+2\alpha) + (3+10\beta)|B_2 - B_1| \\ &+ \frac{(1+2\beta)^2 B_1^2 |D_2 - D_1|}{D_1^2(1+2\alpha)}, \end{aligned} \tag{7}$$

where $\sigma := 2 + 7\alpha + 7\beta + 24\alpha\beta$.

Proof. Since $f \in \mathcal{P}(\alpha, \varphi)$ and $g \in \mathcal{P}(\beta, \psi)$, $g = f^{-1}$, then there exist analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$\begin{aligned} \frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} &= \varphi(u(z)), \\ \frac{wg'(w)}{g(w)} + \frac{\beta w^2 g''(w)}{g(w)} &= \psi(v(w)). \end{aligned} \tag{8}$$

Define the functions p_1 and p_2 by

$$\begin{aligned} p_1(z) &:= \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + \cdots, \\ p_2(z) &:= \frac{1+v(z)}{1-v(z)} = 1 + b_1 z + b_2 z^2 + \cdots, \end{aligned} \tag{9}$$

or, equivalently,

$$\begin{aligned} u(z) &= \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right), \\ v(z) &= \frac{p_2(z) - 1}{p_2(z) + 1} = \frac{1}{2}\left(b_1 z + \left(b_2 - \frac{b_1^2}{2}\right) z^2 + \cdots\right). \end{aligned} \tag{10}$$

Then, p_1 and p_2 are analytic in $\mathbb{D}$ with $p_1(0) = 1 = p_2(0)$. Since $u, v : \mathbb{D} \to \mathbb{D}$, the functions p_1 and p_2 have positive real part in $\mathbb{D}$, and $|b_i| \le 2$ and $|c_i| \le 2$. In view of (8) and (10), it is clear that

$$\begin{aligned} \frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} &= \varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right), \\ \frac{wg'(w)}{g(w)} + \frac{\beta w^2 g''(w)}{g(w)} &= \psi\left(\frac{p_2(w) - 1}{p_2(w) + 1}\right). \end{aligned} \tag{11}$$

Using (10) together with (4), it is evident that

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \cdots,$$
$$\psi\left(\frac{p_2(w)-1}{p_2(w)+1}\right) = 1 + \frac{1}{2}D_1b_1w + \left(\frac{1}{2}D_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2b_1^2\right)w^2 + \cdots. \quad (12)$$

Since f has the Maclaurin series given by (5), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 + \cdots. \quad (13)$$

Since

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = 1 + a_2(1+2\alpha)z + \left(2(1+3\alpha)a_3 - (1+2\alpha)a_2^2\right)z^2 + \cdots,$$
$$\frac{wg'(w)}{g(w)} + \frac{\beta w^2 g''(w)}{g(w)} = 1 - (1+2\beta)a_2w + \left((3+10\beta)a_2^2 - 2(1+3\beta)a_3\right)w^2 + \cdots, \quad (14)$$

it follows from (11) and (12) that

$$a_2(1+2\alpha) = \frac{1}{2}B_1c_1, \quad (15)$$

$$2(1+3\alpha)a_3 - (1+2\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2, \quad (16)$$

$$-(1+2\beta)a_2 = \frac{1}{2}D_1b_1, \quad (17)$$

$$(3+10\beta)a_2^2 - 2(1+3\beta)a_3 = \frac{1}{2}D_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}D_2b_1^2. \quad (18)$$

It follows from (15) and (17) that

$$b_1 = -\frac{B_1(1+2\beta)}{D_1(1+2\alpha)}c_1. \quad (19)$$

Equations (15), (16), (18), and (19) lead to

$$a_2^2 = B_1^2D_1^2\left[B_1(1+3\beta)c_2 + D_1(1+3\alpha)b_2\right] \times \left(2\left[\sigma B_1^2D_1^2 - (1+2\alpha)^2(1+3\beta)(B_2-B_1)D_1^2 - (1+2\beta)^2(1+3\alpha)(D_2-D_1)B_1^2\right]\right)^{-1}, \quad (20)$$

where $\sigma := 2 + 7\alpha + 7\beta + 24\alpha\beta$, which, in view of $|b_2| \le 2$ and $|c_2| \le 2$, gives us the desired estimate on $|a_2|$ as asserted in (6).

By using (16), (18), and (19), we get

$$2\sigma a_3 = \frac{1}{2}\left[B_1(3+10\beta)c_2 + D_1(1+2\alpha)b_2\right] + \frac{c_1^2}{4}\left[(3+10\beta)(B_2-B_1) + \frac{(1+2\beta)^2B_1^2(D_2-D_1)}{D_1^2(1+2\alpha)}\right], \quad (21)$$

and this yields the estimate given in (7). □

Remark 4. When $\alpha = \beta = 0$ and $B_1 = B_2 = 2$, $D_1 = D_2 = 2$, then (6) reduces to Theorem 1. When $\beta = \alpha$ and $\psi = \varphi$, Theorem 3 reduces to [26, Theorem 2.2].

Theorem 5. *Let $f \in \sigma$ and $g = f^{-1}$. If $f \in \mathscr{P}(\alpha,\varphi)$ and $g \in \mathscr{M}(\beta,\psi)$, then*

$$|a_2| \le \left(B_1D_1\sqrt{B_1(1+2\beta) + D_1(1+3\alpha)}\right) \times \left(\left|\sigma B_1^2D_1^2 - (1+2\alpha)^2(1+2\beta)(B_2-B_1)D_1^2 - (1+\beta)^2(1+3\alpha)(D_2-D_1)B_1^2\right|\right)^{-1/2}, \quad (22)$$

$$2\sigma|a_3| \le B_1(3+5\beta) + D_1(1+2\alpha) + (3+5\beta)|B_2-B_1| + \frac{(1+\beta)^2B_1^2|D_2-D_1|}{D_1^2(1+2\alpha)}, \quad (23)$$

where $\sigma := 2 + 7\alpha + 3\beta + 11\alpha\beta$.

Proof. Let $f \in \mathscr{P}(\alpha,\varphi)$ and $g \in \mathscr{M}(\beta,\psi)$, $g = f^{-1}$. Then, there exist analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, such that

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = \varphi(u(z)),$$
$$(1-\beta)\frac{wg'(w)}{g(w)} + \beta\left(1 + \frac{wg''(w)}{g'(w)}\right) = \psi(v(w)). \quad (24)$$

Since

$$\begin{aligned}
&\frac{zf'(z)}{f(z)}+\frac{\alpha z^2 f''(z)}{f(z)}\\
&\quad=1+a_2(1+2\alpha)z\\
&\qquad+\left(2(1+3\alpha)a_3-(1+2\alpha)a_2^2\right)z^2+\cdots,\\
&(1-\beta)\frac{wg'(w)}{g(w)}+\beta\left(1+\frac{wg''(w)}{g'(w)}\right)\\
&\quad=1-(1+\beta)a_2w\\
&\qquad+\left((3+5\beta)a_2^2-2(1+2\beta)a_3\right)w^2+\cdots,
\end{aligned}\tag{25}$$

(12) and (24) yield

$$a_2(1+2\alpha)=\frac{1}{2}B_1c_1,\tag{26}$$

$$2(1+3\alpha)a_3-(1+2\alpha)a_2^2=\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2,\tag{27}$$

$$-(1+\beta)a_2=\frac{1}{2}D_1b_1,\tag{28}$$

$$(3+5\beta)a_2^2-2(1+2\beta)a_3=\frac{1}{2}D_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}D_2b_1^2.\tag{29}$$

It follows from (26) and (28) that

$$b_1=-\frac{B_1(1+\beta)}{D_1(1+2\alpha)}c_1.\tag{30}$$

Hence, (26), (27), (29), and (30) lead to

$$\begin{aligned}
a_2^2=&\left(B_1^2D_1^2\left[B_1(1+2\beta)c_2+D_1(1+3\alpha)b_2\right]\right)\\
&\times\left(2\left[\sigma B_1^2D_1^2-(1+2\alpha)^2(1+2\beta)(B_2-B_1)D_1^2\right.\right.\\
&\left.\left.-(1+2\beta)^2(1+3\alpha)(D_2-D_1)B_1^2\right]\right)^{-1},
\end{aligned}\tag{31}$$

which gives us the desired estimate on $|a_2|$ as asserted in (22) when $|b_2|\le 2$ and $|c_2|\le 2$.

Further, (27), (29), and (30) give

$$\begin{aligned}
2\sigma a_3=&\frac{1}{2}\left[B_1(3+5\beta)c_2+D_1(1+2\alpha)b_2\right]\\
&+\frac{c_1^2}{4}\left[(3+5\beta)(B_2-B_1)+\frac{(1+\beta)^2B_1^2(D_2-D_1)}{D_1^2(1+2\alpha)}\right],
\end{aligned}\tag{32}$$

and this yields the estimate given in (23). □

Theorem 6. *Let $f\in\sigma$ and $g=f^{-1}$. If $f\in\mathscr{P}(\alpha,\varphi)$ and $g\in\mathscr{L}(\beta,\psi)$, then*

$$\begin{aligned}
|a_2|\le&\left(B_1D_1\sqrt{2\left[B_1(3-2\beta)+D_1(1+3\alpha)\right]}\right)\\
&\times\left(\left|\sigma B_1^2D_1^2-2(1+2\alpha)^2(3-2\beta)(B_2-B_1)D_1^2\right.\right.\\
&\left.\left.-2(2-\beta)^2(1+3\alpha)(D_2-D_1)B_1^2\right|\right)^{-1/2},\\
|\sigma a_3|\le&\frac{1}{2}B_1\left(\beta^2-11\beta+16\right)+D_1(1+2\alpha)\\
&+\frac{1}{2}\left(\beta^2-11\beta+16\right)|B_2-B_1|\\
&+\frac{(2-\beta)^2B_1^2|D_2-D_1|}{D_1^2(1+2\alpha)},
\end{aligned}\tag{33}$$

where $\sigma:=10+36\alpha-7\beta-25\alpha\beta+\beta^2+3\alpha\beta^2$.

Proof. Let $f\in\mathscr{P}(\alpha,\varphi)$ and $g\in\mathscr{L}(\beta,\psi)$, $g=f^{-1}$. Then, there are analytic functions $u,v:\mathbb{D}\to\mathbb{D}$, with $u(0)=v(0)=0$, satisfying

$$\begin{aligned}
&\frac{zf'(z)}{f(z)}+\frac{\alpha z^2f''(z)}{f(z)}=\varphi(u(z)),\\
&\left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\beta}=\psi(v(w)).
\end{aligned}\tag{34}$$

Using

$$\begin{aligned}
&\frac{zf'(z)}{f(z)}+\frac{\alpha z^2f''(z)}{f(z)}\\
&\quad=1+a_2(1+2\alpha)z\\
&\qquad+\left(2(1+3\alpha)a_3-(1+2\alpha)a_2^2\right)z^2+\cdots,\\
&\left(\frac{wg'(w)}{g(w)}\right)^{\beta}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\beta}\\
&\quad=1-(2-\beta)a_2w\\
&\qquad+\left(\left(8(1-\beta)+\frac{1}{2}\beta(\beta+5)\right)a_2^2-2(3-2\beta)a_3\right)w^2\\
&\qquad+\cdots,
\end{aligned}\tag{35}$$

and (12) and (34) will yield

$$a_2(1+2\alpha)=\frac{1}{2}B_1c_1,$$

$$2(1+3\alpha)a_3-(1+2\alpha)a_2^2=\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2,$$

$$-(2-\beta)a_2=\frac{1}{2}D_1b_1,$$

$$\left[8(1-\beta)+\frac{\beta}{2}(\beta+5)\right]a_2^2-2(3-2\beta)a_3$$
$$=\frac{1}{2}D_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}D_2b_1^2. \tag{36}$$

Further implication of (36) and applying the fact that $|b_2| \le 2$ and $|c_2| \le 2$ give the estimates in (33). □

Theorem 7. *Let $f \in \sigma$ and $g = f^{-1}$. If $f \in \mathscr{M}(\alpha,\varphi)$, $g \in \mathscr{M}(\beta,\psi)$, then*

$$\begin{aligned}|a_2| \le{}& \left(B_1D_1\sqrt{B_1(1+2\beta)+D_1(1+2\alpha)}\right)\\ &\times\Big(\big|\sigma B_1^2D_1^2-(1+\alpha)^2(1+2\beta)(B_2-B_1)D_1^2\\ &\quad-(1+\beta)^2(1+2\alpha)(D_2-D_1)B_1^2\big|\Big)^{-1/2},\\ 2\sigma|a_3| \le{}& B_1(3+5\beta)+D_1(1+3\alpha)+(3+5\beta)|B_2-B_1|\\ &+\frac{(1+\beta)^2(1+3\alpha)B_1^2|D_2-D_1|}{D_1^2(1+\alpha)^2},\end{aligned} \tag{37}$$

where $\sigma := 2+3\alpha+3\beta+4\alpha\beta$.

Proof. For $f \in \mathscr{M}(\alpha,\varphi)$ and $g \in \mathscr{M}(\beta,\psi)$, $g = f^{-1}$, there exist analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$\begin{aligned}(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)&=\varphi(u(z)),\\ (1-\beta)\frac{wg'(w)}{g(w)}+\beta\left(1+\frac{wg''(w)}{g'(w)}\right)&=\psi(v(w)).\end{aligned} \tag{38}$$

Since

$$\begin{aligned}&(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\\ &\quad=1+(1+\alpha)a_2z\\ &\qquad+\left(2(1+2\alpha)a_3-(1+3\alpha)a_2^2\right)z^2+\cdots,\\ &(1-\beta)\frac{wg'(w)}{g(w)}+\beta\left(1+\frac{wg''(w)}{g'(w)}\right)\\ &\quad=1-(1+\beta)a_2w\\ &\qquad+\left((3+5\beta)a_2^2-2(1+2\beta)a_3\right)w^2+\cdots,\end{aligned} \tag{39}$$

then (12) and (38) yield

$$\begin{aligned}a_2(1+\alpha)&=\frac{1}{2}B_1c_1,\\ 2(1+2\alpha)a_3-(1+3\alpha)a_2^2&\\ =\frac{1}{2}B_1&\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2,\\ -(1+\beta)a_2&=\frac{1}{2}D_1b_1,\\ (3+5\beta)a_2^2-2(1+2\beta)a_3&\\ =\frac{1}{2}D_1&\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}D_2b_1^2.\end{aligned} \tag{40}$$

Further implication of (40) and applying the fact that $|b_2| \le 2$ and $|c_2| \le 2$ give the estimates in (37). □

Remark 8. When $\beta = \alpha$ and $\psi = \varphi$, Theorem 7 reduces to [26, Theorem 2.3].

The following theorems give the estimates for the second and third coefficients of functions f when (i) $f \in \mathscr{M}(\alpha,\varphi)$ and $g \in \mathscr{L}(\beta,\psi)$ and (ii) $f \in \mathscr{L}(\alpha,\varphi)$ and $g \in \mathscr{L}(\beta,\psi)$. The proofs are similar as for the theorems above; hence, they are omitted here.

Theorem 9. *Let $f \in \sigma$ and $g = f^{-1}$. If $f \in \mathscr{M}(\alpha,\varphi)$ and $g \in \mathscr{L}(\beta,\psi)$, then*

$$\begin{aligned}|a_2| \le{}& \left(B_1D_1\sqrt{2[B_1(3-2\beta)+D_1(1+2\alpha)]}\right)\\ &\times\Big(\big|\sigma B_1^2D_1^2-2(1+\alpha)^2(3-2\beta)(B_2-B_1)D_1^2\\ &\quad-2(2-\beta)^2(1+2\alpha)(D_2-D_1)B_1^2\big|\Big)^{-1/2},\\ |\sigma a_3| \le{}& \frac{B_1}{2}\left(\beta^2-11\beta+16\right)+D_1(1+3\alpha)\\ &+\frac{1}{2}\left(\beta^2-11\beta+16\right)|B_2-B_1|\\ &+\frac{(2-\beta)^2(1+3\alpha)B_1^2|D_2-D_1|}{D_1^2(1+\alpha)^2},\end{aligned} \tag{41}$$

where $\sigma := 10+14\alpha-7\beta+\beta^2+2\alpha\beta^2-10\alpha\beta$.

Theorem 10. *Let $f \in \sigma$ and $g = f^{-1}$. If $f \in \mathscr{L}(\alpha,\varphi)$ and $g \in \mathscr{L}(\beta,\psi)$, then*

$$\begin{aligned}|a_2| \le{}& \left(B_1D_1\sqrt{2[B_1(3-2\beta)+D_1(3-2\alpha)]}\right)\\ &\times\Big(\big|\sigma B_1^2D_1^2-2(2-\alpha)^2(3-2\beta)(B_2-B_1)D_1^2\\ &\quad-2(2-\beta)^2(3-2\alpha)(D_2-D_1)B_1^2\big|\Big)^{-1/2},\end{aligned} \tag{42}$$

$$\begin{aligned}2\left|\sigma a_3\right| \leq\, & B_1\left(\beta^2-11\beta+16\right)+D_1\left(8-5\alpha-\alpha^2\right)\\&+\left(\beta^2-11\beta+16\right)\left|B_2-B_1\right|\\&+\frac{(2-\beta)^2\left(\alpha^2+5\alpha-8\right)B_1^2\left|D_2-D_1\right|}{D_1^2(2-\alpha)^2},\end{aligned}\tag{43}$$

where $\sigma := 24+3\alpha^2+3\beta^2-17\alpha-17\beta-2\beta\alpha^2-2\alpha\beta^2+12\alpha\beta$.

Remark 11. When $\beta=\alpha$ and $\psi=\varphi$, Theorem 10 reduces to [26, Theorem 2.4].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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