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Research Article

New Existence Results for Fixed Point Problem and Minimization Problem in Compact Metric Spaces

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We first present some new existence theorems for fixed point problem and minimization problem in compact metric spaces without assuming that mappings possess convexity property. Some applications of our results to new fixed point theorems for nonself mappings in the setting of strictly convex normed linear spaces and usual metric spaces are also given.

1. Introduction and Preliminaries

Let (X,d) be a metric space. Denote by $\mathcal{N}(X)$ the family of all nonempty subsets of X. The symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ are used to denote the sets of positive integers, real numbers, and complex numbers, respectively. Let K be a nonempty subset of X, let $T : K \to X$ be a single-valued mapping, and let $S : K \to \mathcal{N}(X)$ be a multivalued mapping. A point v in K is said to be a *fixed point* of T (resp. S) if $Tv = v$ (resp. $v \in Sv$). The set of fixed points of T (resp. S) is denoted by $\mathcal{F}(T)$ (resp. $\mathcal{F}(S)$). An extended real valued function $f : X \to (-\infty, +\infty]$ is said to be *lower semicontinuous* at $\hat{x} \in X$ if, for any sequence $\{x_n\}$ in X with $x_n \to \hat{x}$, we have $f(\hat{x}) \leq \liminf_{n\to\infty} f(x_n)$. The function f is called to be lower semicontinuous on X if f is lower semicontinuous at every point of X. The function f is said to be *proper* if $f \not\equiv \infty$.

Let $(E, \|\cdot\|)$ be a normed linear space over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. E is said to be *strictly convex* if $x = y$ whenever $\|(x+y)/2\| = \|x\| = \|y\|$; in other words, the unit sphere of E does not contain nontrivial segments. It is worth mentioning that the strict convexity of a normed linear space E can be characterized by the properties: for any nonzero vectors $x, y \in E$, if $\|x+y\| = \|x\| + \|y\|$, then $y = cx$ for some real $c > 0$. The following four types of line segments between two distinct points a and b of E are defined as the sets:

$$\begin{aligned} \operatorname{Seg}(a,b) &= \{\lambda a + (1-\lambda) b : \lambda \in (0,1)\}, \\ \operatorname{Seg}[a,b) &= \{\lambda a + (1-\lambda) b : \lambda \in (0,1]\}, \\ \operatorname{Seg}(a,b] &= \{\lambda a + (1-\lambda) b : \lambda \in [0,1)\}, \\ \operatorname{Seg}[a,b] &= \{\lambda a + (1-\lambda) b : \lambda \in [0,1]\}. \end{aligned} \tag{1}$$

Clearly, $\operatorname{Seg}[a,b]$ is a closed subset of E.

The celebrated Banach contraction principle [1] plays an important role in various fields of nonlinear analysis and applied mathematical analysis.

Theorem 1 (Banach [1]). *Let (X,d) be a complete metric space and let $T : X \to X$ be a selfmap. Assume that there exists a nonnegative number $\gamma < 1$ such that*

$$d(T(x), T(y)) \leq \gamma d(x,y) \quad \forall x, y \in X. \tag{2}$$

Then T has a unique fixed point in X. Moreover, for each $x \in X$, the iterative sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to the unique fixed point of T.

Let K be a nonempty subset of a metric space (X, d) and let $T : K \to K$ be a mapping. Recall that T is said to be *contractive* [2] if

$$d(Tx, Ty) < d(x, y) \quad \forall x, y \in K \text{ with } x \neq y. \tag{3}$$

The following interesting fixed point theorem in the setting of compact metric spaces is due to Edelstein in [2].

Theorem 2 (Edelstein [2]). *Let (K, d) be a nonempty compact metric space and let $T : K \to K$ be contractive. Then T has a unique fixed point in K.*

In 1976, Caristi proved the following famous fixed point theorem to extend Banach contraction principle.

Theorem 3 (Caristi [30]). *Let (X, d) be a complete metric space and let $f : X \to \mathbb{R}$ be a lower semicontinuous and bounded below function. Suppose that T is a Caristi-type map on X dominated by f; that is, T satisfies*

$$d(x, Tx) \leq f(x) - f(Tx) \quad \text{for each } x \in X. \tag{4}$$

Then T has a fixed point in X.

It is well-known that Caristi's fixed point theorem is equivalent to Ekeland's variational principle, to Takahashi's nonconvex minimization theorem, to Daneš' drop theorem, to petal theorem, and to Oettli-Théra's theorem; see, for example, [3, 4] and references therein for more details. In view of the important contribution of Caristi's fixed point theorem on nonlinear analysis, a great deal of generalizations in various different directions of the Caristi's fixed point theorem has been investigated by several authors. For more details on these generalizations, one can refer to [3–19] and references therein.

During the last few decades, an interesting and important direction of research in metric fixed point theory is to study the existence and uniqueness of fixed points for single-valued nonself mappings or multivalued nonself mappings satisfying certain nonlinear conditions. A mass of such research has been investigated by many authors; see, for example, [20–29] and the references therein.

In this work, we first present some new existence theorems for fixed point problem and minimization problem in compact metric spaces without assuming that mappings possess convexity property. Some applications of our results to new fixed point theorems for nonself mappings in the setting of strictly convex normed linear spaces and usual metric spaces are also given.

2. Existence Results for Fixed Point Problem and Minimization Problem without Convexity

We start with the following crucial and useful existence result for fixed point problem and minimization problem which is one of the main results of this paper.

Theorem 4. *Let (K, d) be a nonempty compact metric space, let $f : K \to (-\infty, +\infty]$ be a proper lower semicontinuous function bounded from below, and let $T : K \to \mathcal{N}(K)$ be a multivalued mapping. Suppose that*

($H1$) *for any $x \in K$ with $x \notin Tx$, there exists $y \in Tx$ such that*

$$f(y) < f(x). \tag{5}$$

Then, there exists $v \in K$ such that

(a) $v \in Tv$,

(b) $f(v) = \inf\{f(x) : x \in Tv\} = \inf\{f(x) : x \in K\} < +\infty$.

Proof. Since f is bounded from below,

$$c := \inf\{f(x) : x \in K\} > -\infty. \tag{6}$$

Since f is proper, there exists $w \in K$ such that $f(w) < +\infty$. It follows that

$$c \leq f(w) < +\infty. \tag{7}$$

Hence, by (6) and (7), we know $c \in \mathbb{R}$. One can find a sequences $\{v_n\}_{n\in\mathbb{N}}$ in K such that

$$\lim_{n\to\infty} f(v_n) = c. \tag{8}$$

By the compactness of K, there exists subsequences $\{v_{n_j}\} \subset \{v_n\}$ and $v \in K$ such that

$$v_{n_j} \longrightarrow v \quad \text{as } j \longrightarrow \infty. \tag{9}$$

By the lower semicontinuity of f and (8), we have

$$c \leq f(v) \leq \liminf_{j\to\infty} f(v_{n_j}) = c, \tag{10}$$

which implies

$$f(v) = c = \inf\{f(x) : x \in K\}. \tag{11}$$

Next, we claim that $v \in Tv$. On the contrary, assume that $v \notin Tv$. Then, by our hypothesis ($H1$), there exists $y_v \in Tv$ such that

$$f(v) > f(y_v) \geq f(v), \tag{12}$$

which is a contradiction. Therefore $v \in Tv$ and the conclusion (a) is proved. Due to

$$f(v) = \inf\{f(x) : x \in K\} \leq \inf\{f(x) : x \in Tv\} \leq f(v), \tag{13}$$

we show the conclusion (b). The proof is completed. □

The following existence theorem is obviously an immediate result from Theorem 4.

Theorem 5. *Let (K, d) be a nonempty compact metric space, let $f : K \to (-\infty, +\infty]$ be a proper lower semicontinuous function bounded from below, and let $T : K \to K$ be a single-valued selfmapping. Suppose that*

$(H2)$ *$f(x) - f(Tx) > 0$ for any $x \in K$ with $x \neq Tx$.*

Then there exists $v \in K$ such that

(a) $Tv = v$,

(b) $f(v) = \inf\{f(x) : x \in K\} < +\infty$.

In fact, we have the following important fact.

Theorem 6. *Theorems 4 and 5 are equivalent.*

Proof. It suffices to show that Theorem 5 implies Theorem 4. Under the assumption $(H1)$ of Theorem 4, for any $x \in K$ with $x \notin Tx$, there exists $y_x \in Tx$ such that

$$f(y_x) < f(x). \tag{14}$$

So, we can define a single-valued selfmap $G : K \to K$ by

$$Gx := \begin{cases} x, & \text{if } x \in Tx, \\ y_x, & \text{if } x \notin Tx. \end{cases} \tag{15}$$

It is easy to see that G satisfies $f(x) - f(Gx) > 0$ for any $x \in K$ with $x \neq Gx$. So, all the hypotheses of Theorem 5 are fulfilled. It is therefore possible to apply Theorem 5 to get $v \in K$ such that

(a) $Gv = v$,

(b) $f(v) = \inf\{f(x) : x \in K\} < +\infty$.

By (a) and the definition of G, we have $v \in Tv$. From (b) and $v \in Tv$, we get

$$f(v) = \inf\{f(x) : x \in Tv\} = \inf\{f(x) : x \in K\} < +\infty. \tag{16}$$

Therefore Theorem 5 implies Theorem 4 and hence the proof is completed. □

Applying Theorem 4, we establish the following compactness version of Caristi's type fixed point theorem for multivalued mappings.

Theorem 7. *Let (K, d) be a nonempty compact metric space, let $f : K \to (-\infty, +\infty]$ be a proper lower semicontinuous function bounded from below, and let $T : K \to \mathcal{N}(K)$ be a multivalued mapping. Suppose that, for any $x \in K$, there exists $y \in Tx$ such that*

$$d(x, y) \leq f(x) - f(y). \tag{17}$$

Then there exists $v \in K$ such that

(a) $v \in Tv$,

(b) $f(v) = \inf\{f(x) : x \in Tv\} = \inf\{f(x) : x \in K\} < +\infty$.

Proof. For any $x \in K$ with $x \notin Tx$, by our hypothesis, there exists $y \in Tx$ such that

$$0 < d(x, y) \leq f(x) - f(y) \tag{18}$$

which implies

$$f(y) < f(x). \tag{19}$$

So $(H1)$ as in Theorem 4 is satisfied. Therefore the conclusion follows from Theorem 4. □

As a direct consequence of Theorem 7 we obtain the following result which is a compactness version of Caristi's fixed point theorem.

Theorem 8. *Let (K, d) be a nonempty compact metric space, let $f : K \to (-\infty, +\infty]$ be a proper lower semicontinuous function bounded from below, and let $T : K \to K$ be a single-valued selfmapping. Suppose that T is a Caristi-type map on K dominated by f; that is, T satisfies*

$$d(x, Tx) \leq f(x) - f(Tx) \quad \text{for each } x \in K. \tag{20}$$

Then there exists $v \in K$ such that

(a) $Tv = v$,

(b) $f(v) = \inf\{f(x) : x \in K\} < +\infty$.

Theorem 9. *Theorems 7 and 8 are equivalent.*

By applying Theorem 5 (or Theorem 4), we obtain the following new fixed point theorem for nonself mappings in metric spaces.

Theorem 10. *Let K be a nonempty compact subset of a metric space (X, d) and let $S : K \to X$ be a continuous mapping. Suppose that*

$(H3)$ *for any $x \in K$ with $x \neq Sx$ there exists $y \in K \setminus \{x\}$ such that*

$$d(y, Sy) < d(x, Sx). \tag{21}$$

Then S admits a fixed point in K.

Proof. Define $f : K \to \mathbb{R}$ by

$$f(x) = d(x, Sx) \quad \text{for } x \in K. \tag{22}$$

By the continuity of S, f is continuous and bounded below by 0. By the assumption $(H3)$, for any $x \in K$ with $x \neq Sx$, there exists $y_x \in K \setminus \{x\}$ such that

$$f(y_x) = d(y_x, Sy_x) < d(x, Sx) = f(x), \tag{23}$$

so we can define a single-valued selfmap $T : K \to K$ by

$$Tx := \begin{cases} x, & \text{if } x = Sx, \\ y_x, & \text{if } x \neq Sx. \end{cases} \tag{24}$$

For any $x \in K$ with $x \neq Sx$, by (23) and the definition of T, we obtain

$$f(x) - f(Tx) > 0. \tag{25}$$

Hence we prove that $(H3)$ implies $(H2)$ in Theorem 5. Applying Theorem 5, there exists $v \in K$ such that $Tv = v$, which deduce $Sv = v$. The proof is completed. □

Remark 11. Edelstein's fixed point theorem [2] (i.e., Theorem 2) is a special case of Theorem 10. Indeed, since T is contractive, it is easy to see that T is continuous on K. For any $x \in K$ with $x \neq Tx$, let $y = Tx$. Then $y \in K \setminus \{x\}$ and

$$\begin{aligned} d(x,Tx) - d(y,Ty) &= d(x,Tx) - d(Tx,T^2x) \\ &> d(x,Tx) - d(x,Tx) = 0. \end{aligned} \tag{26}$$

Hence $(H3)$ as in Theorem 10 is satisfied. Therefore the conclusion follows from Theorem 10.

3. Some Applications of Theorem 10

In this section, we study some applications of Theorem 10 to fixed point theory. We first establish a new fixed point theorem without assuming that nonself mappings possess convexity property in the setting of strictly convex normed linear spaces by exploiting Theorem 10.

Theorem 12. *Let $(E, \|\cdot\|)$ be a strictly convex normed linear space, let K be a nonempty compact subset of E, and let $T : K \to E$ be a continuous mapping. Suppose that*

- $(H4)$ *for any $x \in K$ with $x \neq Tx$ there exists $y \in K \setminus \{x\}$ and $y \neq Tx$ such that*

$$\begin{aligned} \|x - Tx\| &= \|x - y\| + \|y - Tx\|, \\ \|Tx - Ty\| &\leq \|x - y\|. \end{aligned} \tag{27}$$

Then T admits a fixed point in K.

Proof. We first claim that the condition (P) holds, where

- (P) for any $x \in K$ with $x \neq Tx$ there exists $y \in K \setminus \{x\}$ such that $\|y - Ty\| < \|x - Tx\|$.

Indeed, let $x \in K$ with $x \neq Tx$ be given. By $(H4)$, there exists $y_x \in K \setminus \{x\}$ and $y_x \neq Tx$ such that

$$\begin{aligned} \|x - Tx\| &= \|x - y_x\| + \|y_x - Tx\|, \\ \|Tx - Ty_x\| &\leq \|x - y_x\|. \end{aligned} \tag{28}$$

It follows that

$$\begin{aligned} \|y_x - Ty_x\| &\leq \|y_x - Tx\| + \|Tx - Ty_x\| \\ &\leq \|y_x - Tx\| + \|x - y_x\| \\ &= \|x - Tx\|. \end{aligned} \tag{29}$$

If $\|y_x - Ty_x\| < \|x - Tx\|$, then our claim (P) is finished.

Suppose $\|y_x - Ty_x\| = \|x - Tx\|$. Since E is strictly convex, $x \neq y_x$, $y_x \neq Tx$, and $\|x - Tx\| = \|x - y_x\| + \|y_x - Tx\|$, there exists $\lambda > 0$ such that

$$y_x - Tx = \lambda (x - y_x). \tag{30}$$

Let $c = \lambda/(1+\lambda)$. Then $c \in (0,1)$. By (30), we have

$$y_x = cx + (1-c)Tx \in \text{Seg}(x, Tx). \tag{31}$$

Hence $y_x \in \text{Seg}(x, Tx) \cap K$. Put

$$\mathscr{A} = \{w \in \text{Seg}(x,Tx) \cap K : \|w - Tw\| = \|x - Tx\|\}. \tag{32}$$

Since $y_x \in \mathscr{A}$, $\mathscr{A} \neq \emptyset$. Let $\rho := \sup_{w \in \mathscr{A}} \|w - x\|$. Then

$$0 < \|y_x - x\| \leq \rho < \sup_{w \in K} \|w - x\| < \infty. \tag{33}$$

We can choose a sequence $\{v_n\} \subset \mathscr{A}$, such that

$$\lim_{n \to \infty} \|v_n - x\| = \rho. \tag{34}$$

Since $\{v_n\} \subset \text{Seg}(x,Tx) \cap K \subset \text{Seg}[x,Tx] \cap K$ and $\text{Seg}[x,Tx] \cap K$ is a nonempty compact subset of E, there exist a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ and a vector $v \in \text{Seg}[x,Tx] \cap K$ such that

$$v_{n_i} \longrightarrow v \quad \text{as } i \longrightarrow \infty. \tag{35}$$

By taking into account (34) and (35), we get

$$\|v - x\| = \lim_{i \to \infty} \|v_{n_i} - x\| = \rho > 0, \tag{36}$$

which implies $v \neq x$. So $v \in \text{Seg}(x,Tx] \cap K$ and hence there exists $\kappa \in [0,1)$ such that

$$v = \kappa x + (1-\kappa)Tx. \tag{37}$$

On the other hand, by the continuity of T, we obtain

$$Tv_{n_i} \longrightarrow Tv \quad \text{as } i \longrightarrow \infty. \tag{38}$$

For any $i \in \mathbb{N}$, since $v_{n_i} \in \mathscr{A}$, we have

$$\|v_{n_i} - Tv_{n_i}\| = \|x - Tx\|. \tag{39}$$

Thus, by (37), we obtain

$$\begin{aligned} \|v - Tv\| &\leq \|v - v_{n_i}\| + \|v_{n_i} - Tv_{n_i}\| + \|Tv_{n_i} - Tv\| \\ &= \|v - v_{n_i}\| + \|x - Tx\| + \|Tv_{n_i} - Tv\| \quad \forall i \in \mathbb{N}. \end{aligned} \tag{40}$$

By taking the limit from both sides of the last inequality, we get

$$\|v - Tv\| \leq \|x - Tx\|. \tag{41}$$

If $\|v - Tv\| < \|x - Tx\|$, then our claim (P) is proved when we take $y = v$. Suppose $\|v - Tv\| = \|x - Tx\|$. Let

$$\mathscr{B} = \{w \in \text{Seg}[v,Tx] \cap K : \|w - Tw\| = \|v - Tv\|\}. \tag{42}$$

Then $\emptyset \neq \mathscr{B} \subset \mathscr{A}$. Let $\zeta := \sup_{w\in\mathscr{B}}\|w - x\|$. Then $0 < \zeta \leq \rho < \infty$. We can find a sequence $\{z_n\} \subset \mathscr{B} \subset \text{Seg}[v, Tx] \cap K$, such that

$$\lim_{n\to\infty} \|z_n - x\| = \zeta. \tag{43}$$

By the compactness of $\text{Seg}[v, Tx] \cap K$, there exist a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ and a vector $z \in \text{Seg}[v, Tx] \cap K$ such that

$$z_{n_j} \longrightarrow z \quad \text{as } j \longrightarrow \infty. \tag{44}$$

From (43) and (44), we get

$$\|z - x\| = \lim_{j\to\infty} \left\|z_{n_j} - x\right\| = \zeta. \tag{45}$$

By (44) and the continuity of T, we have

$$Tz_{n_j} \longrightarrow Tz \quad \text{as } j \longrightarrow \infty. \tag{46}$$

For any $j \in \mathbb{N}$,

$$\left\|z_{n_j} - Tz_{n_j}\right\| = \|v - Tv\| = \|x - Tx\| \quad \text{due to } v_{n_j} \in \mathscr{B}. \tag{47}$$

Since

$$\|z - Tz\| \leq \left\|z - z_{n_j}\right\| + \left\|z_{n_j} - Tz_{n_j}\right\| + \left\|Tz_{n_j} - Tz\right\| \quad \forall j \in \mathbb{N}, \tag{48}$$

taking into account (44), (46), and (47), we get

$$\|z - Tz\| \leq \|x - Tx\|. \tag{49}$$

We will verify $\|z - Tz\| < \|x - Tx\|$. Assume $\|z - Tz\| = \|x - Tx\| = \|v - Tv\|$. Then $z \in \mathscr{B}$. So $z = \eta v + (1 - \eta)Tx$ for some $\eta \in [0, 1]$. Thus, by (37), we have

$$\begin{aligned} v &= \kappa x + (1 - \kappa) Tx \\ &= \kappa x + (1 - \kappa)\left(\frac{1}{1-\eta}z - \frac{\eta}{1-\eta}v\right) \end{aligned} \tag{50}$$

which deduces

$$v = \frac{\kappa(1-\eta)}{1-\eta\kappa}x + \frac{1-\kappa}{1-\eta\kappa}z. \tag{51}$$

Since $\kappa \in [0, 1)$, we have $1 - \eta\kappa > 0$ and $(1 - \kappa)/(1 - \eta\kappa) > 0$. Because $\kappa(1 - \eta)/(1 - \eta\kappa) > 0$, $(1 - \kappa)/(1 - \eta\kappa) > 0$, and $\kappa(1-\eta)/(1-\eta\kappa)+(1-\kappa)/(1-\eta\kappa) = 1$, we know $v \in \text{Seg}(x, z)$ and hence

$$\|x - z\| = \|x - v\| + \|v - z\|. \tag{52}$$

Since $v \neq x$ and $v \neq z$, by (36), (45), and (52), we get

$$\zeta = \|z - x\| > \|v - x\| = \rho, \tag{53}$$

which leads a contradiction. Hence it must be $\|z - Tz\| < \|x - Tx\|$. So our claim (P) is proved when we take $y = z$. Now, all the hypotheses of Theorem 10 are fulfilled, so it is therefore possible to apply Theorem 10 to get the thesis. □

As another interesting application of Theorem 10, we give the following new fixed point result for nonself mappings in usual metric spaces. It is worth mentioning that condition $(H5)$ as in Theorem 13 is different from condition $(H4)$ as in Theorem 12.

Theorem 13. *Let K be a nonempty compact subset of a metric space (X, d) and let $S : K \to X$ be a continuous mapping. Suppose that*

$(H5)$ *for any $x \in K$ with $x \neq Sx$, there exists $y \in K \setminus \{x\}$ such that*

$$\begin{aligned} d(x, y) + d(y, Sx) &= d(x, Sx), \\ d(Sx, Sy) &< d(x, y). \end{aligned} \tag{54}$$

Then S admits a fixed point in K.

Proof. Let $x \in K$ with $x \neq Tx$ be given. Then, by $(H5)$, there exists $y \in K \setminus \{x\}$ such that

$$\begin{aligned} d(x, y) + d(y, Sx) &= d(x, Sx), \\ d(Sx, Sy) &< d(x, y). \end{aligned} \tag{55}$$

It follows from the last inequalities that

$$\begin{aligned} d(x, Sx) - d(y, Sy) &= d(x, y) + d(y, Sx) - d(y, Sy) \\ &\geq d(x, y) - d(Sx, Sy) \\ &> d(x, y) - d(x, y) = 0. \end{aligned} \tag{56}$$

So $(H3)$ as in Theorem 10 is satisfied. Hence the conclusion follows from Theorem 10. □

Let K be a nonempty subset of a metric space (X, d). A mapping $T : K \to K$ is said to be *metrically inward* [30] if, for each $x \in K$, there exists $u \in K$ such that

$$d(x, u) + d(u, Tx) = d(x, Tx), \tag{57}$$

where $u = x$ if and only if $x = Tx$.

Theorem 14. *Let K be a nonempty compact subset of a metric space (X, d) and let $S : K \to X$ be a metrically inward contractive mapping. Then S admits a unique fixed point in K.*

Proof. Applying Theorem 13, S has a fixed point in K. To see the uniqueness of fixed points of S, let $u, v \in \mathscr{F}(S)$. If $u \neq v$, since S is contractive, we have

$$d(u, v) = d(Su, Sv) < d(u, v), \tag{58}$$

a contradiction. Hence $u = v$ and $\mathscr{F}(S)$ is a singleton set. The proof is completed. □

Finally, the following example is given to illustrate Theorem 14.

Example 15 (see [26, Example 3.1]). Let $X = \mathbb{R}^2$. Define a norm on X by

$$\|x\| = \max\{|x_1|, |x_2|\} \quad \text{for } x = (x_1, x_2) \in X. \tag{59}$$

Then $(X, \|\cdot\|)$ is a Banach space and the norm is equivalent to the Euclidean norm on X. Let

$$K = \left\{ (x_1, x_2) \in X : 0 \leq x_2 \leq x_1, \; 0 \leq x_1 \leq \frac{1}{2} \right\} \cup \left\{ \left(1 - \sqrt{2}, 0\right) \right\}. \tag{60}$$

So K is a nonempty compact subset of $(X, \|\cdot\|)$. Define a mapping $S : K \to X$ by

$$Sx = \left(\frac{x_1^2 - 1}{2}, \frac{x_2^2}{2} \right) \quad \text{for } x = (x_1, x_2) \in K. \tag{61}$$

Hence $S : K \to X$ is a metrically inward contractive mapping (see [26, Example 3.1]). By applying Theorem 14, we know that S has a unique fixed point in K. In fact, precisely speaking, $(1 - \sqrt{2}, 0)$ is the unique fixed point of S.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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