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Research Article

Dynamics of a Stochastic Delayed Competitive Model with Impulsive Toxicant Input in Polluted Environments

Meiling Deng, Meng Liu, and Chuanzhi Bai

School of Mathematical Science, Huaiyin Normal University, Huaian 223300, China

Correspondence should be addressed to Chuanzhi Bai; czbai@hytc.edu.cn

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In the world today, with the rapid development of modern agriculture and industry, a large quantity of pollutants enter into ecosystems one by one, which is a threat to the persistence of the exposed populations. This paper investigates a stochastic delayed competitive system with impulsive toxicant input in a polluted environment. Under a simple condition, sufficient and necessary conditions for stability in the mean and extinction of each species are established. Some recent works are improved and extended greatly. Some numerical simulations are also included to illustrate and support the findings.

1. Introduction

In this paper, we consider the following stochastic delay competitive model in polluted environments with impulsive toxicant input:

$$\begin{aligned} dN_1(t) &= N_1(t)\left[r_{10} - r_{11}C_0(t) - a_{11}N_1(t) - a_{12}N_2(t-\tau_1)\right]dt + \lambda_1 N_1(t)\,dW_1(t), \\ dN_2(t) &= N_2(t)\left[r_{20} - r_{21}C_0(t) - a_{22}N_2(t) - a_{21}N_1(t-\tau_2)\right]dt + \lambda_2 N_2(t)\,dW_2(t), \\ \frac{dC_0(t)}{dt} &= kC_e(t) - (g+m)C_0(t), \\ \frac{dC_e(t)}{dt} &= -hC_e(t), \\ & \qquad t \neq n\gamma, \quad n \in Z^+, \\ \Delta N_i(t) &= 0, \qquad \Delta C_0(t) = 0, \qquad \Delta C_e(t) = b, \\ & \qquad t = n\gamma, \quad n \in Z^+, \quad i = 1,2, \end{aligned} \tag{1}$$

with initial data

$$\begin{gathered} N_i(t) = \psi_i(t) > 0, \\ t \in [-\tau, 0]; \quad \psi_i(0) > 0, \quad i = 1,2, \end{gathered} \tag{2}$$

where $\tau_i \geq 0$, $\tau = \max\{\tau_1, \tau_2\}$, and $\psi_i(t)$ is a continuous function on $[-\tau, 0]$. All coefficients in model (1) are positive. $\Delta f(t) = f(t^+) - f(t)$ and $Z^+ = \{1, 2, \ldots\}$; $N_i(t)$ is the size of the ith population, $i = 1, 2$; r_{i0} is the growth rate of the ith population; r_{i1} is the response to the pollutant present in the organism of the ith population; $C_0(t)$ is the toxicant concentration in the organism; $C_e(t)$ is the toxicant concentration in the environment; $kC_e(t)$ is the organism's net uptake of toxicant from the environment; $gC_0(t)+mC_0(t)$ is the egestion and depuration rates of the toxicant in the organism; $hC_e(t)$ is the toxicant loss from the environment itself; γ is the period of the impulsive effect about the exogenous input of toxicant; b is the amount of toxicant input at every time. $W_i(t)$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, \mathscr{P})$; λ_i^2 is the intensity of the environmental noise.

Recently, population models with toxicant effect have received great attention; see, for example, [1–18]. Liu and

Zhang [15] considered the following competitive model in polluted environments with impulsive toxicant input:

$$\begin{aligned}
&\frac{dN_1(t)}{dt} = N_1(t)\left[r_{10} - r_{11}C_0(t) - a_{11}N_1(t) - a_{12}N_2(t)\right],\\
&\frac{dN_2(t)}{dt} = N_2(t)\left[r_{20} - r_{21}C_0(t) - a_{21}N_1(t) - a_{22}N_2(t)\right],\\
&\frac{dC_0(t)}{dt} = kC_e(t) - (g+m)C_0(t),\\
&\frac{dC_e(t)}{dt} = -hC_e(t),\\
&\qquad t \neq n\gamma, \quad n \in Z^+.\\
&\Delta N_i(t) = 0, \quad \Delta C_0(t) = 0, \quad \Delta C_e(t) = b,\\
&\qquad t = n\gamma, \quad n \in Z^+, \quad i = 1,2,
\end{aligned} \tag{3}$$

For model (3), the authors [15] proved the following.

Lemma 1 (see [15]). *Define*

$$\begin{aligned}
&\Gamma = a_{11}a_{22} - a_{12}a_{21}, \qquad \Gamma_1 = r_{10}a_{22} - a_{12}r_{20},\\
&\Gamma_2 = r_{20}a_{11} - r_{10}a_{21}, \qquad \overline{\Gamma}_1 := a_{22}r_{11} - a_{12}r_{21},\\
&\overline{\Gamma}_2 := a_{11}r_{21} - a_{21}r_{11}, \qquad K = \frac{kb}{\gamma h(g+m)}.
\end{aligned} \tag{4}$$

Suppose that $\Gamma > 0$, $\Gamma_1 > 0$, $\Gamma_2 > 0$.

(A) *If* $\Gamma_1 > \overline{\Gamma}_1 K$ *and* $\Gamma_2 < \overline{\Gamma}_2 K$, *then* $\lim_{t\to+\infty} N_2(t) = 0$ *and* $\limsup_{t\to+\infty} t^{-1}\int_0^t N_1(s)ds > 0$.

(B) *If* $\Gamma_1 < \overline{\Gamma}_1 K$ *and* $\Gamma_2 > \overline{\Gamma}_2 K$, *then* $\lim_{t\to+\infty} N_1(t) = 0$ *and* $\limsup_{t\to+\infty} t^{-1}\int_0^t N_2(s)ds > 0$.

(C) *If* $\Gamma_1 > \overline{\Gamma}_1 K$ *and* $\Gamma_2 > \overline{\Gamma}_2 K$, *then* $\limsup_{t\to+\infty} t^{-1}\int_0^t N_i(s)ds > 0$, $i = 1, 2$.

From the work of Liu and Zhang [15], some important and interesting questions arise naturally.

(Q1) In the real world, the growth of population is inevitably affected by random environmental fluctuations. May [19] have claimed that population systems should be stochastic. Therefore, what happens if system (3) is affected by environmental fluctuations?

(Q2) Gopalsamy [20] have pointed out that, in order to be reality, time delays should not be ignored. Hence, what happens if system (3) incorporates with time delays?

(Q3) Can we improve the results obtained in Lemma 1?

The aims of this paper are to investigate the above questions. Recall that r_{i0} stands for the growth rate. In practice, we often estimate it by an average value plus an error term. Generally, by the famous central limit theorem, the error term follows a normal distribution. Hence, for short correlation time, we can replace r_{i0} with $r_{i0} + \lambda_i \dot{W}_i(t)$ (see, e.g., [21–29]), where $\dot{W}_i(t)$ is white noise and λ_i^2 is the intensity of the noise. At the same time, incorporating with time delays, we get model (1). For model (1), we will show the following.

Theorem 2. *Suppose that* $\Gamma > 0$. *Define*

$$\begin{aligned}
&\eta_i = r_{i0} - r_{i1}K - 0.5\lambda_i^2, \quad i = 1,2,\\
&\widetilde{\Gamma}_1 = 0.5a_{22}\lambda_1^2 - 0.5a_{12}\lambda_2^2, \qquad \widetilde{\Gamma}_2 = 0.5a_{11}\lambda_2^2 - 0.5a_{21}\lambda_1^2.
\end{aligned} \tag{5}$$

(I) *If* $\eta_1 < 0$ *and* $\eta_2 < 0$, *then both* N_1 *and* N_2 *go to extinction almost surely; that is,*

$$\lim_{t\to+\infty} N_i(t) = 0 \quad a.s., \ i = 1,2; \tag{6}$$

(II) *if* $\eta_1 > 0$ *and* $\eta_2 < 0$, *then* N_2 *goes to extinction a.s. and* N_1 *is stable in the mean a.s.; that is,*

$$\lim_{t\to+\infty} t^{-1}\int_0^t N_1(s)\,ds = \frac{\eta_1}{a_{11}}, \quad a.s.; \tag{7}$$

(III) *if* $\eta_1 < 0$ *and* $\eta_2 > 0$, *then* N_1 *goes to extinction a.s. and* N_2 *is stable in the mean a.s.; that is,*

$$\lim_{t\to+\infty} t^{-1}\int_0^t N_2(s)\,ds = \frac{\eta_2}{a_{22}}, \quad a.s.; \tag{8}$$

(IV) *if* $\eta_1 > 0$, $\eta_2 > 0$,

(A) *if* $\Gamma_1 > \overline{\Gamma}_1 K + \widetilde{\Gamma}_1$ *and* $\Gamma_2 < \overline{\Gamma}_2 K + \widetilde{\Gamma}_2$, *then* N_2 *goes to extinction a.s. and* N_1 *is stable in the mean a.s.*

$$\lim_{t\to+\infty} t^{-1}\int_0^t N_1(s)\,ds = \frac{\eta_1}{a_{11}}, \quad a.s.; \tag{9}$$

(B) *if* $\Gamma_1 < \overline{\Gamma}_1 K + \widetilde{\Gamma}_1$ *and* $\Gamma_2 > \overline{\Gamma}_2 K + \widetilde{\Gamma}_2$, *then* N_1 *goes to extinction a.s. and* N_2 *is stable in the mean a.s.*

$$\lim_{t\to+\infty} t^{-1}\int_0^t N_2(s)\,ds = \frac{\eta_2}{a_{22}}, \quad a.s.; \tag{10}$$

(C) *if* $\Gamma_1 > \overline{\Gamma}_1 K + \widetilde{\Gamma}_1$ *and* $\Gamma_2 > \overline{\Gamma}_2 K + \widetilde{\Gamma}_2$, *then both* N_1 *and* N_2 *are stable in the mean a.s.*

$$\begin{aligned}
&\lim_{t\to+\infty} t^{-1}\int_0^t N_1(s)\,ds = \frac{\Gamma_1 - \overline{\Gamma}_1 K - \widetilde{\Gamma}_1}{\Gamma}, \quad a.s.,\\
&\lim_{t\to+\infty} t^{-1}\int_0^t N_2(s)\,ds = \frac{\Gamma_2 - \overline{\Gamma}_1 K - \widetilde{\Gamma}_2}{\Gamma}, \quad a.s.
\end{aligned} \tag{11}$$

Remark 3. It is useful to point out that if $\eta_1 > 0$, $\eta_2 > 0$, and $\Gamma > 0$, then $\Gamma_1 < \overline{\Gamma}_1 + \widetilde{\Gamma}_1$ and $\Gamma_2 < \overline{\Gamma}_2 + \widetilde{\Gamma}_2$ will not hold simultaneously.

Remark 4. In comparison with most of the existing results, our key contributions in this paper are as follows.

(i) To the best of our knowledge, this paper is the first attempt to consider stochastic delay competitive model in polluted environments.

(ii) Our conditions are much weaker. For example, the authors [15] supposed $\Gamma_1 > 0$ and $\Gamma_2 > 0$ which are dropped in this paper.

(iii) Our results improve some recent works. For example, Lemma 1 indicates that the superior limit is positive while Theorem 2 proves that the limit exists and establishes the explicit form of the limit.

2. Proof

For simplicity, define

$$R_+^2 = \{a = (a_1, a_2) \in R^2 \mid a_i > 0,\ i = 1, 2\},$$
$$\langle g(t) \rangle = t^{-1} \int_0^t g(s)\, ds. \tag{12}$$

Lemma 5. *For any given initial data $\psi(\theta) = (\psi_1(\theta), \psi_2(\theta)) \in C([-\tau, 0], R_+^2)$, there is a unique global positive solution $N(t) = (N_1(t), N_2(t))$ to model (1) on $t \geq -\tau$ a.s. and*

$$\limsup_{t \to +\infty} \frac{\ln N_i(t)}{\ln t} \leq 1 \quad a.s.,\ i = 1, 2. \tag{13}$$

Proof. The proof is a special case of Theorems 5.1 and 5.2 in Liu and Wang [25] and hence is omitted. □

Lemma 6 (see [26]). *Suppose that $x(t) \in C[\Omega \times [0, +\infty), R_+]$.*

(I) *If there exist δ and positive constants δ_0, T such that*

$$\ln x(t) \leq \delta t - \delta_0 \int_0^t x(s)\, ds + \sum_{i=1}^n \beta_i W_i(t) \tag{14}$$

for $t \geq T$, where $W_i(t)$ are independent standard Brownian motions and β_i are constants, $1 \leq i \leq n$, then we have the following: if $\delta \geq 0$, then $\limsup_{t \to +\infty} \langle x(t) \rangle \leq \delta/\delta_0$ a.s.; if $\delta < 0$, then $\lim_{t \to +\infty} x(t) = 0$ a.s.

(II) *If there exist positive constants δ_0, T, and δ such that*

$$\ln x(t) \geq \delta t - \delta_0 \int_0^t x(s)\, ds + \sum_{i=1}^n \beta_i W_i(t) \tag{15}$$

for $t \geq T$, then $\liminf_{t \to +\infty} \langle x(t) \rangle \geq \delta/\delta_0$ a.s.

To begin with, let us consider the following subsystem of (1):

$$\begin{aligned} \frac{dC_0(t)}{dt} &= kC_e(t) - (g + m) C_0(t), \\ \frac{dC_e(t)}{dt} &= -hC_e(t), \\ & \qquad t \neq n\gamma,\ n \in Z^+, \\ \Delta C_0(t) &= 0, \quad \Delta C_e(t) = b, \quad t = n\gamma,\ n \in Z^+ \\ & 0 \leq C_0(0) \leq 1,\ 0 \leq C_e(0) \leq 1. \end{aligned} \tag{16}$$

Lemma 7 (see [12, 13]). *System (16) has a unique positive γ-periodic solution $(\widetilde{C}_0(t), \widetilde{C}_e(t))^T$ and for each solution $(C_0(t), C_e(t))^T$ of (16), $C_0(t) \to \widetilde{C}_0(t)$ and $C_e(t) \to \widetilde{C}_e(t)$ as $t \to \infty$. Moreover, $C_0(t) > \widetilde{C}_0(t)$ and $C_e(t) > \widetilde{C}_e(t)$ for all $t \geq 0$ if $C_0(0) > \widetilde{C}_0(0)$ and $C_e(0) > \widetilde{C}_e(0)$, $i = 1, 2$, where*

$$\begin{aligned} \widetilde{C}_0(t) &= \widetilde{C}_0(0) e^{-(g+m)(t-n\gamma)} + \frac{kb\left(e^{-(g+m)(t-n\gamma)} - e^{-h(t-n\gamma)}\right)}{(h - g - m)(1 - e^{-h\gamma})}, \\ \widetilde{C}_e(t) &= \frac{be^{-h(t-n\gamma)}}{1 - e^{-h\gamma}}, \\ \widetilde{C}_0(0) &= \frac{kb\left(e^{-(g+m)\gamma} - e^{-h\gamma}\right)}{(h - g - m)\left(1 - e^{-(g+m)\gamma}\right)\left(1 - e^{-h\gamma}\right)}, \\ \widetilde{C}_e(0) &= \frac{b}{1 - e^{-h\gamma}} \end{aligned} \tag{17}$$

for $t \in (n\gamma, (n+1)\gamma]$ and $n \in Z^+$. In addition,

$$\lim_{t \to +\infty} t^{-1} \int_0^t \widetilde{C}_0(s)\, ds = \frac{kb}{h(g+m)\gamma} = K. \tag{18}$$

Proof of Theorem 2. It follows from Lemma 7 that

$$\lim_{t \to +\infty} t^{-1} \int_0^t C_0(s)\, ds = \lim_{t \to +\infty} t^{-1} \int_0^t \widetilde{C}_0(s)\, ds = K. \tag{19}$$

That is to say, for all $\varepsilon > 0$, there is $T > 0$ such that

$$K - \varepsilon \leq \langle C_0(t) \rangle \leq K + \varepsilon, \quad t > T. \tag{20}$$

Applying Itô's formula to (1) leads to

$$\begin{aligned} d\ln N_1(t) &= \frac{dN_1(t)}{N_1(t)} - \frac{(dN_1(t))^2}{2N_1^2(t)} \\ &= \left[r_{10} - 0.5\lambda_1^2 - r_{11}C_0(t) - a_{11}N_1(t) - a_{12}N_2(t - \tau_1)\right] dt \\ &\quad + \lambda_1 dW_1(t), \\ d\ln N_2(t) &= \frac{dN_2(t)}{N_2(t)} - \frac{(dN_2(t))^2}{2N_2^2(t)} \\ &= \left[r_{20} - 0.5\lambda_2^2 - r_{21}C_0(t) - a_{21}N_1(t - \tau_2) - a_{22}N_2(t)\right] dt \\ &\quad + \lambda_2 dW_2(t). \end{aligned} \tag{21}$$

In other words, we have shown that

$$\begin{aligned}
&\ln N_1(t) - \ln N_1(0) \\
&\quad = \left(r_{10} - 0.5\lambda_1^2\right)t - r_{11}\int_0^t C_0(s)\,ds - a_{11}\int_0^t N_1(s)\,ds \\
&\qquad - a_{12}\int_0^t N_2(s-\tau_1)\,ds + \lambda_1 W_1(t) \\
&\quad = \left(r_{10} - 0.5\lambda_1^2\right)t - a_{12}\int_0^t N_2(s)\,ds \\
&\qquad + a_{12}\left[\int_{t-\tau_1}^t N_2(s)\,ds - \int_{-\tau_1}^0 N_2(s)\,ds\right] \\
&\qquad - r_{11}\int_0^t C_0(s)\,ds - a_{11}\int_0^t N_1(s)\,ds + \lambda_1 W_1(t), \\
&\ln N_2(t) - \ln N_2(0) \\
&\quad = \left(r_{20} - 0.5\lambda_2^2\right)t - r_{21}\int_0^t C_0(s)\,ds \\
&\qquad - a_{21}\int_0^t N_1(s-\tau_2)\,ds - a_{22}\int_0^t N_2(s)\,ds + \lambda_2 W_2(t) \\
&\quad = \left(r_{20} - 0.5\lambda_2^2\right)t - a_{21}\int_0^t N_1(s)\,ds \\
&\qquad + a_{21}\left[\int_{t-\tau_2}^t N_1(s)\,ds - \int_{-\tau_2}^0 N_1(s)\,ds\right] \\
&\qquad - r_{21}\int_0^t C_0(s)\,ds - a_{22}\int_0^t N_2(s)\,ds + \lambda_2 W_2(t).
\end{aligned} \tag{22}$$

(I) Assume that $\eta_1 = r_{10} - 0.5\lambda_1^2 - r_{11}K < 0$ and $\eta_2 = r_{20} - 0.5\lambda_2^2 - r_{21}K < 0$. In view of (22),

$$t^{-1}\ln\frac{N_1(t)}{N_1(0)} \le r_{10} - r_{11}\int_0^t C_0(s)\,ds - 0.5\lambda_1^2 + t^{-1}\lambda_1 W_1(t). \tag{23}$$

By $\lim_{t\to+\infty} W_i(t)/t = 0$ a.s., $i = 1, 2$ (19) and $r_{10} < 0.5\lambda_1^2 + r_{11}K$ we have that

$$\limsup_{t\to+\infty} t^{-1}\ln N_1(t) \le r_{10} - 0.5\lambda_1^2 - r_{11}K = \eta_1 < 0. \tag{24}$$

Therefore, $\lim_{t\to+\infty} N_1(t) = 0$, a.s. Similarly, it follows from (9) that if $\eta_2 < 0$, then $\lim_{t\to+\infty} N_2(t) = 0$, a.s.

(II) Assume that $\eta_1 > 0$ and $\eta_2 < 0$. Since $\eta_2 < 0$, then, by (I), $\lim_{t\to+\infty} N_2(t) = 0$, a.s. Hence, for arbitrary $\varepsilon > 0$, there exists $T > 0$ such that for $t \ge T$

$$\begin{aligned}
&-\frac{\varepsilon}{2} \le a_{12}t^{-1}\int_0^t N_2(s-\tau_1)\,ds \le \frac{\varepsilon}{2}, \\
&-\frac{\varepsilon}{2} \le t^{-1}\ln N_1(0) \le \frac{\varepsilon}{2}.
\end{aligned} \tag{25}$$

When the above inequalities and (20) are used in (22), we can obtain that, for $t \ge T$,

$$\ln N_1(t) \le \left(\eta_1 + 2\varepsilon\right)t - a_{11}\int_0^t N_1(s)\,ds + \lambda_1 W_1(t), \tag{26}$$

$$\ln N_1(t) \ge \left(\eta_1 - 2\varepsilon\right)t - a_{11}\int_0^t N_1(s)\,ds + \lambda_1 W_1(t). \tag{27}$$

Note that $\eta_1 > 0$; we can let ε be sufficiently small such that $\eta_1 - 2\varepsilon > 0$. Applying (i) and (ii) in Lemma 6 to (26) and (27), respectively, one can see that

$$\begin{aligned}
\frac{\eta_1 - 2\varepsilon}{a_{11}} &\le \liminf_{t\to+\infty}\langle N_1(t)\rangle \le \limsup_{t\to+\infty}\langle N_1(t)\rangle \\
&\le \frac{\eta_1 + 2\varepsilon}{a_{11}}, \quad \text{a.s.}
\end{aligned} \tag{28}$$

It therefore follows from the arbitrariness of ε that

$$\lim_{t\to+\infty}\langle N_1(t)\rangle = \frac{\eta_1}{a_{11}}, \quad \text{a.s.} \tag{29}$$

The proof of (III) can be obtained similarly and hence is omitted.

Now, we are in the position to prove (IV). Assume that $\eta_1 > 0$ and $\eta_2 > 0$. For $i = 1, 2$, consider the following stochastic equation:

$$\begin{aligned}
dy_i(t) &= y_i(t)\left[r_{i0} - r_{i1}C_0(t) - a_{ii}y_i(t)\right]dt + \lambda_i y_i(t)\,dW_i(t), \\
y_i(\theta) &= \psi(\theta), \quad \theta \in [-\tau, 0].
\end{aligned} \tag{30}$$

By the classical stochastic comparison theorem [30], one can see that

$$N_1(t) \le y_1(t), \qquad N_2(t) \le y_2(t). \tag{31}$$

Note that $r_{i0} > 0.5\lambda_i^2 + r_{i1}K$, $i = 1, 2$; an argument, identical to the argument used in the proof of (II), shows that

$$\lim_{t\to+\infty}\langle y_i(t)\rangle = \lim_{t\to+\infty} t^{-1}\int_0^t y_i(s)\,ds = \frac{\eta_i}{a_{ii}} \quad \text{a.s.}, \ i = 1, 2. \tag{32}$$

Consequently,

$$\begin{aligned}
&\lim_{t\to+\infty} t^{-1}\int_{t-\tau_1}^t y_2(s)\,ds \\
&\quad = \lim_{t\to+\infty}\left(t^{-1}\int_0^t y_2(s)\,ds - t^{-1}\int_0^{t-\tau_1} y_2(s)\,ds\right) = 0, \\
&\lim_{t\to+\infty} t^{-1}\int_{t-\tau_2}^t y_1(s)\,ds = 0, \quad \text{a.s.}
\end{aligned} \tag{33}$$

This, together with (31), implies that

$$\begin{aligned}
&\lim_{t\to+\infty} t^{-1}\int_{t-\tau_1}^t N_2(s)\,ds = 0, \\
&\lim_{t\to+\infty} t^{-1}\int_{t-\tau_2}^t N_1(s)\,ds = 0, \quad \text{a.s.}
\end{aligned} \tag{34}$$

On the other hand, computing (9) $\times a_{11}-$ (22) $\times a_{21}$ gives

$$\begin{aligned} a_{11}\ln\frac{N_2(t)}{N_2(0)} &= a_{11}a_{21}\left[\int_{t-\tau_2}^{t}N_1(s)\,ds-\int_{-\tau_2}^{0}N_1(s)\,ds\right] \\ &\quad -a_{21}a_{12}\left[\int_{t-\tau_1}^{t}N_2(s)\,ds-\int_{-\tau_1}^{0}N_2(s)\,ds\right] \\ &\quad +a_{21}\ln\frac{N_1(t)}{N_1(0)}+\left(\Gamma_2-\widetilde{\Gamma}_2\right)t-\overline{\Gamma}_2\int_0^t C_0(s)\,ds \\ &\quad -\Gamma\int_0^t N_2(s)\,ds-a_{21}\lambda_1W_1(t)+a_{11}\lambda_2W_2(t). \end{aligned}\tag{35}$$

In view of (13) and (34), for arbitrary $\varepsilon>0$, there exits $T>0$ such that, for $t\geq T$,

$$\begin{gathered} t^{-1}a_{21}\ln\frac{N_1(t)}{N_1(0)}<\frac{\varepsilon}{3},\qquad t^{-1}a_{11}\ln N_2(0)<\frac{\varepsilon}{3}, \\ t^{-1}a_{11}a_{21}\left[\int_{t-\tau_2}^{t}N_1(s)\,ds-\int_{-\tau_2}^{0}N_1(s)\,ds\right] \\ -t^{-1}a_{21}a_{12}\left[\int_{t-\tau_1}^{t}N_2(s)\,ds-\int_{-\tau_1}^{0}N_2(s)\,ds\right]\leq\frac{\varepsilon}{3}. \end{gathered}\tag{36}$$

When the above inequalities and (20) are used in (35), we can see that for $t>T$

$$\begin{aligned} a_{11}\ln N_2(t) &\leq\left(\Gamma_2-\overline{\Gamma}_2K-\widetilde{\Gamma}_2+2\varepsilon\right)t \\ &\quad -\Gamma\int_0^t N_2(s)\,ds-a_{21}\lambda_1W_1(t)+a_{11}\lambda_2W_2(t). \end{aligned}\tag{37}$$

Similarly, computing (22) $\times a_{22}-$ (9) $\times a_{12}$ gives

$$\begin{aligned} a_{22}\ln\frac{N_1(t)}{N_1(0)} &= a_{22}a_{12}\left[\int_{t-\tau_1}^{t}N_2(s)\,ds-\int_{-\tau_1}^{0}N_2(s)\,ds\right] \\ &\quad -a_{12}a_{21}\left[\int_{t-\tau_2}^{t}N_1(s)\,ds-\int_{-\tau_2}^{0}N_1(s)\,ds\right] \\ &\quad +a_{12}\ln\frac{N_2(t)}{N_2(0)}+\left(\Gamma_1-\widetilde{\Gamma}_1\right)t-\overline{\Gamma}_1\int_0^t C_0(s)\,ds \\ &\quad -\Gamma\int_0^t N_1(s)\,ds+a_{22}\lambda_1W_1(t)-a_{12}\lambda_2W_2(t). \end{aligned}\tag{38}$$

Hence, for sufficiently large t,

$$\begin{aligned} a_{22}\ln N_1(t) &\leq\left(\Gamma_1-\overline{\Gamma}_1K-\widetilde{\Gamma}_1+2\varepsilon\right)t \\ &\quad -\Gamma\int_0^t N_1(s)\,ds+a_{22}\lambda_1W_1(t)-a_{12}\lambda_2W_2(t). \end{aligned}\tag{39}$$

(A) Assume that $\Gamma_1>\overline{\Gamma}_1K+\widetilde{\Gamma}_1$ and $\Gamma_2<\overline{\Gamma}_2K+\widetilde{\Gamma}_2$. Since $\Gamma_2<\overline{\Gamma}_2K+\widetilde{\Gamma}_2$, then we can let ε be sufficiently small such that $\Gamma_2-\overline{\Gamma}_2K-\widetilde{\Gamma}_2+2\varepsilon<0$. Applying (i) in Lemma 6 to (37) results in $\lim_{t\to+\infty}N_2(t)=0$, a.s. The proof of $\lim_{t\to+\infty}\langle N_1(t)\rangle=\eta_1/a_{11}$ a.s. is the same as that in (II) and hence is omitted. The proof of (B) is similar to that of (A) and hence is omitted.

(C) Assume that $\Gamma_1>\overline{\Gamma}_1K+\widetilde{\Gamma}_1$ and $\Gamma_2>\overline{\Gamma}_2K+\widetilde{\Gamma}_2$. Notice that $\Gamma_2>\widetilde{\Gamma}_2$; then, by (37) and Lemma 6,

$$\limsup_{t\to+\infty}\langle N_2(t)\rangle\leq\frac{\Gamma_2-\overline{\Gamma}_2K-\widetilde{\Gamma}_2+2\varepsilon}{\Gamma},\quad\text{a.s.}\tag{40}$$

It then follows from the arbitrariness of ε that

$$\limsup_{t\to+\infty}\langle N_2(t)\rangle\leq\frac{\Gamma_2\overline{\Gamma}_2K-\widetilde{\Gamma}_2}{\Gamma},\quad\text{a.s.}\tag{41}$$

Similarly, by (39), Lemma 6, and the arbitrariness of ε, we have

$$\limsup_{t\to+\infty}\langle N_1(t)\rangle\leq\frac{\Gamma_1-\overline{\Gamma}_1K-\widetilde{\Gamma}_1}{\Gamma},\quad\text{a.s.}\tag{42}$$

Let ε be sufficiently small satisfying $a_{11}((\Gamma_1-\overline{\Gamma}_1K-\widetilde{\Gamma}_1)/\Gamma)-\varepsilon>0$. Substituting (34), (41), and (20) into (22) yields

$$\begin{aligned} t^{-1}\ln N_1(t) &= t^{-1}\ln N_1(0)+r_{10}-r_{11}\langle C_0(t)\rangle-0.5\lambda_1^2 \\ &\quad -a_{11}\langle N_1(t)\rangle-a_{12}\langle N_2(t)\rangle+\lambda_1W_1\frac{(t)}{t} \\ &\quad +a_{12}t^{-1}\left[\int_{t-\tau_1}^{t}N_2(s)\,ds-\int_{-\tau_1}^{0}N_2(s)\,ds\right] \\ &\geq\eta_1-2\varepsilon-a_{11}\langle N_1(t)\rangle-a_{12}\limsup_{t\to+\infty}\langle N_2(t)\rangle \\ &\quad +\lambda_1W_1\frac{(t)}{t} \\ &\geq\eta_1-2\varepsilon-a_{11}\langle N_1(t)\rangle-a_{12}\frac{\Gamma_2-\widetilde{\Gamma}_2}{\Gamma}+\lambda_1W_1\frac{(t)}{t} \\ &= a_{11}\frac{\Gamma_1-\overline{\Gamma}_1K-\widetilde{\Gamma}_1}{\Gamma}-2\varepsilon-a_{11}\langle N_1(t)\rangle+\lambda_1W_1\frac{(t)}{t}, \end{aligned}\tag{43}$$

for sufficiently large t. By (ii) in Lemma 6 and the arbitrariness of ε, one can observe that

$$\liminf_{t\to+\infty}\langle N_1(t)\rangle\geq\frac{\Gamma_1-\overline{\Gamma}_1K-\widetilde{\Gamma}_1}{\Gamma},\quad\text{a.s.}\tag{44}$$

Similarly, when (34) and (42) and (20) are used in (9), we can see that $\liminf_{t\to+\infty}\langle N_2(t)\rangle\geq(\Gamma_2-\overline{\Gamma}_2K-\widetilde{\Gamma}_2)/\Gamma$, a.s. This, together with (41), (42), and (44), indicates that

$$\begin{aligned} \lim_{t\to+\infty}\langle N_1(t)\rangle &= \frac{\Gamma_1-\overline{\Gamma}_1K-\widetilde{\Gamma}_1}{\Gamma}, \\ \lim_{t\to+\infty}\langle N_2(t)\rangle &= \frac{\Gamma_2-\overline{\Gamma}_2K-\widetilde{\Gamma}_2}{\Gamma},\quad\text{a.s.} \end{aligned}\tag{45}$$

□

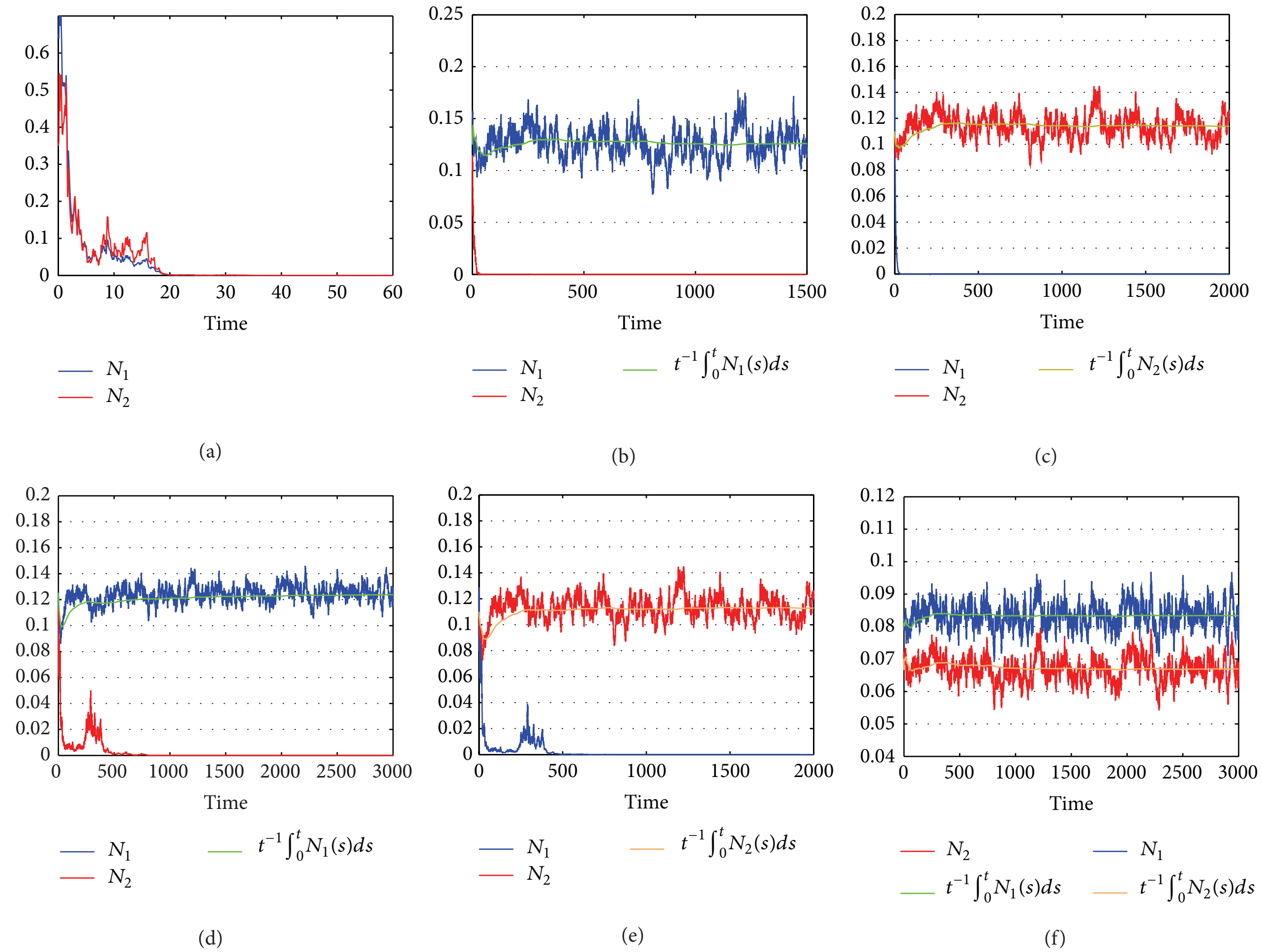

FIGURE 1: Solutions of system (1) for $r_{10} = 0.7$, $r_{20} = 0.5$, $a_{11} = 0.8$, $a_{12} = 0.5$, $a_{21} = 0.4$, $a_{22} = 0.7$, $\tau_1 = 10$, $\tau_2 = 12$, $k = g = m = 0.2$, $h = 0.5$, $b = 0.1$, $\gamma = 1$, $C_0(0) = C_e(0) = 0.1$, and step size $\Delta t = 0.001$. (a) is with $\lambda_1 = 1.1$, $\lambda_2 = 1$; (b) is with $\lambda_1 = 1$, $\lambda_2 = 1$; (c) is with $\lambda_1 = 1.2$, $\lambda_2 = 0.8$; (d) is with $\lambda_1 = 1$, $\lambda_2 = 0.8484$; (e) is with $\lambda_1 = 1.058$, $\lambda_2 = 0.8$; (f) is with $\lambda_1 = 1$, $\lambda_2 = 0.8$.

3. Numerical Simulations

In this section, using the classical Milstein method (see, e.g., [31]), we work out some numerical figures to support the analytical results. In Figure 1, we choose $r_{10} = 0.7$, $r_{20} = 0.5$, $a_{11} = 0.8$, $a_{12} = 0.5$, $a_{21} = 0.4$, $a_{22} = 0.7$, $\tau_1 = 10$, $\tau_2 = 12$, $k = g = m = 0.2$, $h = 0.5$, $b = 0.1$, and $\gamma = 1$. Then, $\Gamma = 0.36$, $\Gamma_1 = 0.24$, $\Gamma_2 = 0.12$, $\overline{\Gamma_1} = 0.2$, $\overline{\Gamma_2} = 0.4$, and $K = kb/\gamma h(g + m) = 0.1$. The only difference between conditions of Figures 1(a), 1(b), 1(c), 1(d), 1(e), and 1(f) is that the values of λ_1 and λ_2 are different.

(a) In Figure 1(a), we choose $\lambda_1 = 1.1$, $\lambda_2 = 1$. Then, $\eta_1 = r_{10} - r_{11}K - 0.5\lambda_1^2 = -0.12$, $\eta_2 = r_{20} - r_{21}K - 0.5\lambda_2^2 = -0.1$. By virtue of (I) in Theorem 2, both N_1 and N_2 are extinct; see Figure 1(a).

(b) In Figure 1(b), we set $\lambda_1 = 1$, $\lambda_2 = 1$. Then, $\eta_1 = 0.1$, $\eta_2 = -0.1$. In view of (II) in Theorem 2, N_2 is extinct and $\lim_{t\to+\infty} t^{-1}\int_0^t N_1(s)ds = \eta_1/a_{11} = 0.125$. Figure 1(b) confirms this.

(c) In Figure 1(c), we let $\lambda_1 = 1.2$, $\lambda_2 = 0.8$. Then, $\eta_1 = -0.12$, $\eta_2 = 0.08$. It follows from (III) in Theorem 2 that N_1 is extinct and

$$\lim_{t\to+\infty} t^{-1}\int_0^t N_2(s)\,ds = \frac{\eta_2}{a_{22}} = 0.1143. \tag{46}$$

See Figure 1(c).

(d) In Figure 1(d), we set $\lambda_1 = 1$, $\lambda_2 = 0.8484$. Then, $\eta_1 = 0.1$, $\eta_2 = 0.04$, $\Gamma_1 = 0.24 > \overline{\Gamma}_1 K + \widetilde{\Gamma}_1 = 0.19$, and $\Gamma_2 = 0.12 < \overline{\Gamma}_2 K + \widetilde{\Gamma}_2 = 0.128$. According to (A) in Theorem 2, N_2 is extinct and

$$\lim_{t\to+\infty} t^{-1}\int_0^t N_1(s)\,ds = \frac{\eta_1}{a_{11}} = \frac{0.1}{0.8} = 0.125. \tag{47}$$

Figure 1(d) confirms this.

(e) In Figure 1(e), we choose $\lambda_1 = 1.058$, $\lambda_2 = 0.8$. Then, $\eta_1 = 0.04$, $\eta_2 = 0.08$, $\Gamma_1 = 0.24 < \overline{\Gamma}_1 K + \widetilde{\Gamma}_1 = 0.252$, and

$\Gamma_2 = 0.12 > \overline{\Gamma}_2 K + \widetilde{\Gamma}_2 = 0.072$. By (B) in Theorem 2, N_1 is extinct and

$$\lim_{t\to+\infty} t^{-1}\int_0^t N_2(s)\,ds = \frac{\eta_2}{a_{22}} = \frac{0.08}{0.7} = 0.1143. \quad (48)$$

See Figure 1(e).

(f) In Figure 1(f), we let $\lambda_1 = 1$, $\lambda_2 = 0.8$. Then, $\Gamma_1 = 0.24 > \overline{\Gamma}_1 K + \widetilde{\Gamma}_1 = 0.21$ and $\Gamma_2 = 0.12 > \overline{\Gamma}_2 K + \widetilde{\Gamma}_2 = 0.096$. It follows from (C) in Theorem 2 that

$$\lim_{t\to+\infty} t^{-1}\int_0^t N_1(s)\,ds = \frac{\Gamma_1 - \overline{\Gamma}_1 K - \widetilde{\Gamma}_1}{\Gamma} = \frac{0.03}{0.36} = 0.0833,$$
$$\lim_{t\to+\infty} t^{-1}\int_0^t N_2(s)\,ds = \frac{\Gamma_2 - \overline{\Gamma}_2 K - \widetilde{\Gamma}_2}{\Gamma} = 0.0667. \quad (49)$$

See Figure 1(f).

4. Conclusions and Future Directions

This paper investigates a stochastic delay competitive model in a polluted environment with impulsive toxicant input. For each population, the critical value between stability in the mean and extinction is obtained. Some recent works are extended and improved. Our Theorem 2 has some important and interesting interpretation.

(1) Time delay is harmless for stability in the mean and extinction of the stochastic model (1).

(2) White noises can change the dynamics of the population model greatly.

Some interesting problems deserve further study. One can consider some more realistic systems, for example, stochastic delayed population model with the Markov switching (see, e.g., [22, 23, 29]). It is also interesting to extend Theorem 2 to n-species case.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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