

## Research Article

# More on Generalizations and Modifications of Iterative Methods for Solving Large Sparse Indefinite Linear Systems

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Continuing from the works of Li et al. (2014), Li (2007), and Kincaid et al. (2000), we present more generalizations and modifications of iterative methods for solving large sparse symmetric and nonsymmetric indefinite systems of linear equations. We discuss a variety of iterative methods such as GMRES, MGMRES, MINRES, LQ-MINRES, QR MINRES, MMINRES, MGRES, and others.

## 1. Introduction

When solving large sparse linear systems of the form

$$\mathbf{Ax} = \mathbf{b}, \quad (1)$$

in which the coefficient matrix  $\mathbf{A}$  is indefinite, there are basis methods and a variety of generalizations and modifications of them. For example, basic iterative methods for symmetric indefinite linear systems are the MINRES method and the SYMMLQ method, while a basic method for nonsymmetric linear systems is the GMRES method. (See, e.g., Lanczos [1], Golub and Van Loan [2], Paige and Saunders [3], Saad [4], and Saad and Schultz [5].)

In Section 2, we review the Arnoldi process and present background material. In Sections 3 and 4, we describe the LQ-MINRES and the QR-MINRES methods, respectively, as well as discussing their relationship in Section 5. In Section 6, we take a closer look at the QR-MINRES method and the SYMMQR method. In Sections 7 and 8, we describe the modified MINRES (MMINRES) method and the generalized QR-MINRES method, respectively. In Section 9, we review the GMRES method. Finally, we discuss the differences between the modified MINRES (MMINRES) method and the modified GMRES (MGMRES) method, in Section 10.

## 2. Arnoldi Process

First, we assume that matrix  $\mathbf{A}$  is symmetric. In [6, 7], we use a short term recurrence to generate orthonormal vectors  $\{\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(n-3)}, \mathbf{w}^{(n-2)}, \mathbf{w}^{(n-1)}\}$  as follows:

$$\mathbf{u}^{(0)} = \text{arbitrary},$$

$$\mathbf{r}^{(0)} = \mathbf{b} - \mathbf{A}\mathbf{u}^{(0)},$$

$$\widetilde{\mathbf{w}}^{(0)} = \mathbf{r}^{(0)},$$

$$\sigma_0 = \sqrt{\langle \widetilde{\mathbf{w}}^{(0)}, \widetilde{\mathbf{w}}^{(0)} \rangle},$$

$$\mathbf{w}^{(0)} = \left( \frac{1}{\sigma_0} \right) \widetilde{\mathbf{w}}^{(0)},$$

for  $i = 1, 2, \dots, n-1$ ,

$$\alpha_{i-1} = \langle \mathbf{A}\mathbf{w}^{(i-1)}, \mathbf{w}^{(i-1)} \rangle,$$

$$\beta_{i-1} = \langle \mathbf{A}\mathbf{w}^{(i-1)}, \mathbf{w}^{(i-2)} \rangle,$$

$$\widetilde{\mathbf{w}}^{(i)} = \mathbf{A}\mathbf{w}^{(i-1)} - \alpha_{i-1}\mathbf{w}^{(i-1)} - \beta_{i-1}\mathbf{w}^{(i-2)},$$

$$\sigma_i = \sqrt{\langle \widetilde{\mathbf{w}}^{(i)}, \widetilde{\mathbf{w}}^{(i)} \rangle},$$

$$\mathbf{w}^{(i)} = \left( \frac{1}{\sigma_i} \right) \widetilde{\mathbf{w}}^{(i)},$$

end for.

(2)

Here we assume that  $\mathbf{w}^{(-1)} = \mathbf{0}$  and  $\sigma_i \neq 0$ , for all  $i$ . If we let  $\widetilde{\mathbf{w}}^{(0)} = \mathbf{r}^{(0)} = \mathbf{b} - \mathbf{A}\mathbf{u}^{(0)}$ , then the subspace,

$$\text{Span} \{ \mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(n-3)}, \mathbf{w}^{(n-2)}, \mathbf{w}^{(n-1)} \}, \quad (3)$$

is equivalent to the Krylov subspace

$$\begin{aligned} \mathcal{K}_n(\mathbf{r}^{(0)}, \mathbf{A}) = \text{Span} \{ & \mathbf{r}^{(0)}, \mathbf{A}\mathbf{r}^{(0)}, \mathbf{A}^2\mathbf{r}^{(0)}, \mathbf{A}^3\mathbf{r}^{(0)}, \dots, \mathbf{A}^{n-3}\mathbf{r}^{(0)}, \\ & \mathbf{A}^{n-2}\mathbf{r}^{(0)}, \mathbf{A}^{n-1}\mathbf{r}^{(0)} \}. \end{aligned} \quad (4)$$

Consequently, we have the following properties, for  $(0 \leq i, j \leq n-1)$ :

$$\begin{aligned} \langle \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \rangle &= \delta_{ij}, \\ \beta_i &= \sigma_i, \end{aligned} \quad (5)$$

as well as these matrix equations

$$\mathbf{A}\mathbf{W}_{n-1} = \mathbf{W}_{n-1}\mathbf{T}_n + \sigma_n \mathbf{w}^{(n)} \mathbf{e}_n^T \quad (6)$$

$$= \mathbf{W}_n \widetilde{\mathbf{T}}_{n+1},$$

$$\mathbf{W}_{n-1}^T \mathbf{W}_{n-1} = \mathbf{I}, \quad (7)$$

where

$$\mathbf{W}_{n-1} = [\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(n-3)}, \mathbf{w}^{(n-2)}, \mathbf{w}^{(n-1)}]_{n \times n}, \quad (8)$$

$$\mathbf{e}_n^T = [0, 0, 0, \dots, 0, 0, 0, 1]_{1 \times n}, \quad (9)$$

$$\mathbf{T}_n = \begin{bmatrix} \alpha_0 & \beta_1 & & & & & & \\ \sigma_1 & \alpha_1 & \beta_2 & & & & & \\ & \sigma_2 & \alpha_2 & \beta_3 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \sigma_{n-3} & \alpha_{n-3} & \beta_{n-2} & & \\ & & & & \sigma_{n-2} & \alpha_{n-2} & \beta_{n-1} & \\ & & & & & \sigma_{n-1} & \alpha_{n-1} & \end{bmatrix}_{n \times n}, \quad (10)$$

$$\widetilde{\mathbf{T}}_{n+1} = \begin{bmatrix} \alpha_0 & \beta_1 & & & & & & \\ \sigma_1 & \alpha_1 & \beta_2 & & & & & \\ & \sigma_2 & \alpha_2 & \beta_3 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \sigma_{n-3} & \alpha_{n-3} & \beta_{n-2} & & \\ & & & & \sigma_{n-2} & \alpha_{n-2} & \beta_{n-1} & \\ & & & & & \sigma_{n-1} & \alpha_{n-1} & \\ & & & & & & & \sigma_n \end{bmatrix}_{(n+1) \times n} \quad (11)$$

$$= \mathbf{T}_n + \mathbf{T}_n \mathbf{e}_{n+1}^T.$$

*Example.* We illustrate (6), for the case  $n = 3$ .

From (2) and (7), we have

$$\begin{aligned} \sigma_1 \mathbf{w}^{(1)} &= \widetilde{\mathbf{w}}^{(1)} = \mathbf{A}\mathbf{w}^{(0)} - \alpha_0 \mathbf{w}^{(0)} - \beta_0 \mathbf{w}^{(-1)} \\ \sigma_2 \mathbf{w}^{(2)} &= \widetilde{\mathbf{w}}^{(2)} = \mathbf{A}\mathbf{w}^{(1)} - \alpha_1 \mathbf{w}^{(1)} - \beta_1 \mathbf{w}^{(0)} \\ \sigma_3 \mathbf{w}^{(3)} &= \widetilde{\mathbf{w}}^{(3)} = \mathbf{A}\mathbf{w}^{(2)} - \alpha_2 \mathbf{w}^{(2)} - \beta_2 \mathbf{w}^{(1)}. \end{aligned} \quad (12)$$

Since  $\mathbf{w}^{(-1)} = \mathbf{0}$ , we obtain

$$\begin{aligned} \mathbf{A}\mathbf{W}_2 &= \mathbf{A} [\mathbf{w}^{(0)} \quad \mathbf{w}^{(1)} \quad \mathbf{w}^{(2)}] = [\mathbf{A}\mathbf{w}^{(0)} \quad \mathbf{A}\mathbf{w}^{(1)} \quad \mathbf{A}\mathbf{w}^{(2)}] \\ &= \begin{bmatrix} \beta_0 \mathbf{w}^{(-1)} + \alpha_0 \mathbf{w}^{(0)} + \sigma_1 \mathbf{w}^{(1)} \\ \beta_1 \mathbf{w}^{(0)} + \alpha_1 \mathbf{w}^{(1)} + \sigma_2 \mathbf{w}^{(2)} \\ \beta_2 \mathbf{w}^{(1)} + \alpha_2 \mathbf{w}^{(2)} + \sigma_3 \mathbf{w}^{(3)} \end{bmatrix}^T \\ &= [\mathbf{w}^{(0)} \quad \mathbf{w}^{(1)} \quad \mathbf{w}^{(2)} \quad \mathbf{w}^{(3)}] \begin{bmatrix} \alpha_0 & \beta_1 & 0 \\ \sigma_1 & \alpha_1 & \beta_2 \\ 0 & \sigma_2 & \alpha_2 \\ 0 & 0 & \sigma_3 \end{bmatrix}_{4 \times 3} \\ &= \mathbf{W}_3 \widetilde{\mathbf{T}}_4 = [\mathbf{W}_2 \quad \mathbf{w}^{(3)}] \begin{bmatrix} \mathbf{T}_3 \\ 0 & 0 & \sigma_3 \end{bmatrix} \\ &= \mathbf{W}_2 \mathbf{T}_3 + \sigma_3 \mathbf{w}^{(3)} \mathbf{e}_3^T. \end{aligned} \quad (13)$$

So we obtain (6), with  $n = 3$ :

$$\mathbf{A}\mathbf{W}_2 = \mathbf{W}_2 \mathbf{T}_3 + \sigma_3 \mathbf{w}^{(3)} \mathbf{e}_3^T = \mathbf{W}_3 \widetilde{\mathbf{T}}_4. \quad (14)$$

### 3. LQ-MINRES Method

We choose  $\mathbf{u}^{(n)}$  such that  $\mathbf{u}^{(n)} - \mathbf{u}^{(0)} \in \mathcal{K}_n(\mathbf{r}^{(0)}, \mathbf{A})$ . Hence, we have

$$\begin{aligned} \mathbf{u}^{(n)} &= \mathbf{u}^{(0)} + \mathbf{W}_{n-1} \mathbf{y}^{(n)}, \\ \mathbf{r}^{(n)} &= \mathbf{r}^{(0)} - (\mathbf{A}\mathbf{W}_{n-1}) \mathbf{y}^{(n)}. \end{aligned} \quad (15)$$

For the MINRES method [3], we let

$$\begin{aligned} f(\mathbf{y}^{(n)}) &= (\mathbf{r}^{(n)})^T (\mathbf{r}^{(n)}), \\ &= (\mathbf{r}^{(0)} - \mathbf{A}\mathbf{W}_{n-1} \mathbf{y}^{(n)})^T (\mathbf{r}^{(0)} - \mathbf{A}\mathbf{W}_{n-1} \mathbf{y}^{(n)}); \end{aligned} \quad (16)$$

then

$$\begin{aligned} \nabla f(\mathbf{y}^{(n)}) &= -2\mathbf{W}_{n-1}^T \mathbf{A} (\mathbf{r}^{(0)} - \mathbf{A}\mathbf{W}_{n-1} \mathbf{y}^{(n)}) \\ &= -2\mathbf{W}_{n-1}^T \mathbf{A} \mathbf{r}^{(0)} + 2\mathbf{W}_{n-1}^T \mathbf{A}^2 \mathbf{W}_{n-1} \mathbf{y}^{(n)}. \end{aligned} \quad (17)$$

Letting  $\nabla f(\mathbf{y}^{(n)}) = \mathbf{0}$ , we can minimize  $\|\mathbf{r}^{(n)}\|$  by solving this linear system for  $\mathbf{y}^{(n)}$

$$(\mathbf{W}_{n-1}^T \mathbf{A}^2 \mathbf{W}_{n-1}) \mathbf{y}^{(n)} = \mathbf{W}_{n-1}^T \mathbf{A} \mathbf{r}^{(0)}. \quad (18)$$









For the LQ-MINRES method (73), we have

$$\begin{aligned}
 c_1 &= \frac{\alpha_0}{\gamma_0}, & c_k^L &= \frac{\bar{\gamma}_{k-1}^L}{\gamma_{k-1}^L}, \\
 s_1 &= \frac{\beta_1}{\gamma_0}, & s_k^L &= \frac{\beta_k}{\gamma_{k-1}^L}, \\
 \bar{\delta}_k^L &= \begin{cases} \beta_1, & k = 1, \\ -c_{k-1}^L \beta_k, & \end{cases} \\
 \delta_k^L &= c_k^L \bar{\delta}_k^L + s_k^L \alpha_k, \\
 \varepsilon_k^L &= s_{k-1}^L \beta_k, \\
 \bar{\gamma}_k^L &= \begin{cases} \alpha_0, & k = 0, \\ -s_k^L \bar{\delta}_k^L - c_k^L \alpha_k, & \end{cases} \\
 \gamma_k^L &= \sqrt{(\bar{\gamma}_k^L)^2 + \sigma_{k+1}^2}.
 \end{aligned} \tag{75}$$

In the QR-MINRES method (46), we have

$$\begin{aligned}
 c_1 &= \frac{\alpha_0}{\gamma_0}, & c_k^R &= \frac{\bar{\gamma}_{k-1}^R}{\gamma_{k-1}^R}, \\
 s_1 &= -\frac{\beta_1}{\gamma_0}, & s_k^R &= -\frac{\beta_k}{\gamma_{k-1}^R}, \\
 \bar{\delta}_k^R &= \begin{cases} \beta_1, & k = 1, \\ c_{k-1}^R \beta_k, & \end{cases} \\
 \delta_k^R &= c_k^R \bar{\delta}_k^R - s_k^R \alpha_k, \\
 \varepsilon_k^R &= -s_{k-1}^R \beta_k, \\
 \bar{\gamma}_k^R &= \begin{cases} \alpha_0, & k = 0, \\ s_k^R \bar{\delta}_k^R + c_k^R \alpha_k, & \end{cases} \\
 \gamma_k^R &= \sqrt{(\bar{\gamma}_k^R)^2 + \sigma_{k+1}^2}.
 \end{aligned} \tag{76}$$

From the computation and by induction, we have the following relations between the LQ-MINRES method and the QR-MINRES method:

$$\begin{aligned}
 c_k^L &= \begin{cases} c_k^R, & k: \text{ odd}, \\ -c_k^R, & k: \text{ even}, \end{cases} \\
 \bar{\gamma}_k^L &= \begin{cases} -\bar{\gamma}_k^R, & k: \text{ odd}, \\ \bar{\gamma}_k^R, & k: \text{ even}, \end{cases} \\
 \bar{\delta}_k^L &= \begin{cases} \delta_k^R, & k: \text{ odd}, \\ -\delta_k^R, & k: \text{ even}, \end{cases} \\
 s_k^L &= -s_k^R.
 \end{aligned} \tag{77}$$

By induction, we have

$$\delta_k^L = \delta_k^R, \quad \varepsilon_k^L = \varepsilon_k^R, \quad \gamma_k^L = \gamma_k^R. \tag{78}$$

Hence, we obtain

$$\mathbf{L}_n^T = \mathbf{R}_n. \tag{79}$$

Moreover, if we let

$$\begin{aligned}
 \sigma_0 \bar{\mathbf{D}}_n^L \mathbf{Q}_L^T \mathbf{e}_1 &= [\xi_0^L, \xi_1^L, \xi_2^L, \dots, \xi_{n-3}^L, \xi_{n-2}^L, \xi_{n-1}^L]_{1 \times n}^T, \\
 \sigma_0 \bar{\mathbf{D}}_n^R \mathbf{Q}_R \mathbf{e}_1 &= [\xi_0^R, \xi_1^R, \xi_2^R, \dots, \xi_{n-3}^R, \xi_{n-2}^R, \xi_{n-1}^R]_{1 \times n}^T,
 \end{aligned} \tag{80}$$

then

$$\begin{aligned}
 \xi_i^L &= c_{i+1}^L s_1^L s_2^L s_3^L \dots s_{i-2}^L s_{i-1}^L s_i^L \sigma_0 \\
 &= c_{i+1}^L (-1)^i s_1^R s_2^R s_3^R \dots s_{i-2}^R s_{i-1}^R s_i^R \sigma_0 \\
 &= c_{i+1}^R s_1^R s_2^R s_3^R \dots s_{i-2}^R s_{i-1}^R s_i^R \sigma_0 \\
 &= \xi_i^R.
 \end{aligned} \tag{81}$$

Thus, we obtain

$$\mathbf{u}_L^{(n)} = \mathbf{u}_R^{(n)}. \tag{82}$$

## 6. A Closer Look at QR-MINRES and SYMMQR

In the SYMMQR method [6, 7], we have two estimated solutions:  $\mathbf{u}_{ls}^{(n)}$  and  $\mathbf{u}_{Gal}^{(n)}$ .

The first estimated solution is

$$\mathbf{u}_{ls}^{(n)} = \mathbf{u}^{(0)} + \mathbf{W}_{n-1} \mathbf{y}_{ls}^{(n)}, \tag{83}$$

where  $\mathbf{y}_{ls}^{(n)}$  is the solution of this linear system

$$\mathbf{R}_n \mathbf{y}^{(n)} = \mathbf{z}^{(n)}, \tag{84}$$

with the right-hand side vector being

$$\mathbf{z}^{(n)} = [\xi_0, \xi_1, \xi_2, \dots, \xi_{n-3}, \xi_{n-2}, \xi_{n-1}]_{1 \times n}^T. \tag{85}$$

From these equations, we have

$$\mathbf{z}^{(n)} = \sigma_0 \bar{\mathbf{D}}_n^R \mathbf{Q}_R \mathbf{e}_1. \tag{86}$$

Thus, we have

$$\mathbf{u}_R^{(n)} = \mathbf{u}_{ls}^{(n)}, \tag{87}$$

where  $\mathbf{u}_{ls}^{(n)}$  is the solution of this least square problem

$$\min_{\mathbf{y}^{(n)} \in \mathbb{R}^n} \|\bar{\mathbf{T}}_{n+1} \mathbf{y}^{(n)} - \sigma_0 \mathbf{e}_1\|. \tag{88}$$

The second estimated solution is

$$\mathbf{u}_{Gal}^{(n)} = \mathbf{u}^{(0)} + \mathbf{W}_{n-1} \mathbf{y}_{Gal}^{(n)}, \tag{89}$$

where  $\mathbf{y}_{Gal}^{(n)}$  is the solution of this linear system

$$\bar{\mathbf{R}}_n \mathbf{y}^{(n)} = \bar{\mathbf{z}}^{(n)}, \tag{90}$$

$$\mathbf{u}_{Gal}^{(n)} - \mathbf{u}^{(0)} = \bar{\mathbf{D}}_n^{-2} (\mathbf{u}_R^{(n)} - \mathbf{u}^{(0)}). \tag{91}$$

## 7. Modified MINRES (MMINRES) Method

Next, we assume that matrix  $\mathbf{A}$  is nonsymmetric. In [6, 7], we use a long term recurrence to generate orthonormal vectors  $\{\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(n-3)}, \mathbf{w}^{(n-2)}, \mathbf{w}^{(n-1)}\}$  as follows:

$$\begin{aligned} \mathbf{u}^{(0)} &= \text{arbitrary,} \\ \mathbf{r}^{(0)} &= \mathbf{b} - \mathbf{A}\mathbf{u}^{(0)}, \\ \widetilde{\mathbf{w}}^{(0)} &= \mathbf{r}^{(0)}, \\ \sigma_0 &= \sqrt{\langle \widetilde{\mathbf{w}}^{(0)}, \widetilde{\mathbf{w}}^{(0)} \rangle}, \\ \mathbf{w}^{(0)} &= \left( \frac{1}{\sigma_0} \right) \widetilde{\mathbf{w}}^{(0)}, \\ \text{for } i &= 1, 2, \dots, n-1, \end{aligned} \quad (92)$$

$$\widetilde{\mathbf{w}}^{(i)} = \mathbf{A}\mathbf{w}^{(i-1)} - \sum_{j=0}^{i-1} \mathbf{b}_{i,j} \mathbf{w}^{(j)},$$

$$\text{where } \mathbf{b}_{i,j} = \langle \mathbf{A}\mathbf{w}^{(i-1)}, \mathbf{w}^{(j)} \rangle,$$

$$\sigma_i = \sqrt{\langle \widetilde{\mathbf{w}}^{(i)}, \widetilde{\mathbf{w}}^{(i)} \rangle},$$

$$\mathbf{w}^{(i)} = \left( \frac{1}{\sigma_i} \right) \widetilde{\mathbf{w}}^{(i)},$$

end for.

Consequently, we obtain the following matrix equations:

$$\begin{aligned} \mathbf{A}\mathbf{W}_{n-1} &= \mathbf{W}_{n-1}\mathbf{H}_n + \sigma_n \mathbf{w}^{(n)} \mathbf{e}_n^T \\ &= \mathbf{W}_n \widetilde{\mathbf{H}}_{n+1}, \\ \mathbf{W}_{n-1}^T \mathbf{W}_{n-1} &= \mathbf{I}, \end{aligned} \quad (93)$$

where

$$\mathbf{W}_{n-1} = [\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(n-3)}, \mathbf{w}^{(n-2)}, \mathbf{w}^{(n-1)}]_{n \times n},$$

$$\mathbf{e}_n^T = [0, 0, 0, \dots, 0, 0, 1]_{1 \times n},$$

$$\mathbf{H}_n = \begin{bmatrix} b_{1,0} & b_{2,0} & b_{3,0} & \cdots & b_{n-2,0} & b_{n-1,0} & b_{n,0} \\ \sigma_1 & b_{2,1} & b_{3,1} & \cdots & b_{n-2,1} & b_{n-1,1} & b_{n,1} \\ & \sigma_2 & b_{3,2} & \cdots & b_{n-2,2} & b_{n-1,2} & b_{n,2} \\ & & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & & \sigma_{n-3} & b_{n-2,n-3} & b_{n-1,n-3} & b_{n,n-3} \\ & & & & \sigma_{n-2} & b_{n-1,n-2} & b_{n,n-2} \\ & & & & & \sigma_{n-1} & b_{n,n-1} \end{bmatrix}_{n \times n},$$

$$\widetilde{\mathbf{H}}_{n+1} = \begin{bmatrix} b_{1,0} & b_{2,0} & b_{3,0} & \cdots & b_{n-2,0} & b_{n-1,0} & b_{n,0} \\ \sigma_1 & b_{2,1} & b_{3,1} & \cdots & b_{n-2,1} & b_{n-1,1} & b_{n,1} \\ & \sigma_2 & b_{3,2} & \cdots & b_{n-2,2} & b_{n-1,2} & b_{n,2} \\ & & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & & \sigma_{n-3} & b_{n-2,n-3} & b_{n-1,n-3} & b_{n,n-3} \\ & & & & \sigma_{n-2} & b_{n-1,n-2} & b_{n,n-2} \\ & & & & & \sigma_{n-1} & b_{n,n-1} \\ & & & & & & \sigma_n \end{bmatrix}_{(n+1) \times n} \quad (94)$$

Since the matrix  $\mathbf{H}_n$  is a full upper Hessenberg matrix, the LQ-MINRES method is *not* a practical procedure. Hence, we discuss only a generalization of the QR-MINRES method.

## 8. Generalized QR-MINRES Method

Since the matrix  $\mathbf{A}$  is nonsymmetric, to minimize an expression such as this

$$\min_{\mathbf{y}^{(n)} \in \mathbb{R}^n} f(\mathbf{y}^{(n)}) = (\mathbf{r}^{(n)})^T (\mathbf{r}^{(n)}), \quad (95)$$

we choose  $\mathbf{y}^{(n)}$  to satisfy

$$(\mathbf{W}_{n-1}^T \mathbf{A}^T \mathbf{A} \mathbf{W}_{n-1}) \mathbf{y}^{(n)} = \mathbf{W}_{n-1}^T \mathbf{A}^T \mathbf{r}^{(0)}. \quad (96)$$

First, from the Arnoldi process and from the left-hand side of linear system (96), we can write the coefficient matrix of this linear system as

$$\begin{aligned} \mathbf{W}_{n-1}^T \mathbf{A}^T \mathbf{A} \mathbf{W}_{n-1} &= (\mathbf{A}\mathbf{W}_{n-1})^T \mathbf{A}\mathbf{W}_{n-1} \\ &= (\mathbf{W}_{n-1}\mathbf{H}_n + \sigma_n \mathbf{w}^{(n)} \mathbf{e}_n^T)^T (\mathbf{W}_{n-1}\mathbf{H}_n + \sigma_n \mathbf{w}^{(n)} \mathbf{e}_n^T) \\ &= \mathbf{H}_n^T \mathbf{H}_n + \sigma_n^2 \mathbf{e}_n \mathbf{e}_n^T. \end{aligned} \quad (97)$$

Second, from the right-hand side vector of the linear system (96), we have

$$\begin{aligned} (\mathbf{W}_{n-1}^T \mathbf{A}^T) \mathbf{r}^{(0)} &= (\mathbf{A}\mathbf{W}_{n-1})^T \sigma_0 \mathbf{w}^{(0)} \\ &= (\mathbf{W}_{n-1}\mathbf{H}_n + \sigma_n \mathbf{w}^{(n)} \mathbf{e}_n^T)^T \sigma_0 \mathbf{w}^{(0)} \\ &= (\mathbf{H}_n^T \mathbf{W}_{n-1}^T) \sigma_0 \mathbf{w}^{(0)} \\ &= \sigma_0 \mathbf{H}_n^T \mathbf{e}_1. \end{aligned} \quad (98)$$



Consequently, instead of solving (96), we solve this linear system

$$\left(\mathbf{H}_n^T \mathbf{H}_n + \sigma_n^2 \mathbf{e}_n \mathbf{e}_n^T\right) \mathbf{y}^{(n)} = \sigma_0 \mathbf{H}_n^T \mathbf{e}_1. \quad (99)$$

First, we repeatedly apply Givens rotations to the left-hand side of  $\mathbf{H}_n$

$$\mathbf{Q}_{n-1,n} \mathbf{Q}_{n-2,n-1} \mathbf{Q}_{n-3,n-2} \cdots \mathbf{Q}_{3,4} \mathbf{Q}_{2,3} \mathbf{Q}_{1,2} \mathbf{H}_n \equiv \bar{\mathbf{R}}_n. \quad (100)$$

By defining

$$\mathbf{Q}_{n-1} \equiv \mathbf{Q}_{n-1,n} \mathbf{Q}_{n-2,n-1} \mathbf{Q}_{n-3,n-2} \cdots \mathbf{Q}_{3,4} \mathbf{Q}_{2,3} \mathbf{Q}_{1,2}, \quad (101)$$

we obtain

$$\mathbf{Q}_{n-1} \mathbf{H}_n = \bar{\mathbf{R}}_n, \quad (102)$$

$$\mathbf{H}_n = \mathbf{Q}_{n-1}^T \bar{\mathbf{R}}_n, \quad (103)$$

so that

$$\mathbf{H}_n^T = \bar{\mathbf{R}}_n^T \mathbf{Q}_{n-1}. \quad (104)$$

Thus, the coefficient matrix in linear system (99) has the following form by using (103) and (104):

$$\begin{aligned} \mathbf{H}_n^T \mathbf{H}_n + \sigma_n^2 \mathbf{e}_n \mathbf{e}_n^T &= \bar{\mathbf{R}}_n^T \bar{\mathbf{R}}_n + \sigma_n^2 \mathbf{e}_n \mathbf{e}_n^T \\ &= \mathbf{R}_n^T \mathbf{R}_n + \sigma_n^2 \mathbf{e}_n \mathbf{e}_n^T. \end{aligned} \quad (105)$$

Moreover, by using (102) and (104), the right-hand side vector in linear system (99) is

$$\begin{aligned} \sigma_0 \mathbf{H}_n^T \mathbf{e}_1 &= \sigma_0 \bar{\mathbf{R}}_n^T \mathbf{Q}_{n-1} \mathbf{e}_1 \\ &= \sigma_0 \mathbf{R}_n^T \bar{\mathbf{D}}_n \mathbf{Q}_{n-1} \mathbf{e}_1, \end{aligned} \quad (106)$$

where

$$\mathbf{R}_n = \begin{bmatrix} \gamma_0 & \times & \times & \cdots & \cdots & \cdots & \times \\ & \gamma_1 & \times & \times & \cdots & \cdots & \times \\ & & \gamma_2 & \times & \times & \cdots & \times \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & \gamma_{n-3} & \times & \times \\ & & & & & \gamma_{n-2} & \times \\ & & & & & & \gamma_{n-1} \end{bmatrix}_{n \times n}, \quad (107)$$

$$\bar{\mathbf{D}}_n = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & c_n \end{bmatrix}_{n \times n}, \quad (108)$$

where  $c_n = \bar{\gamma}_{n-1}/\gamma_{n-1}$  and  $\gamma_{n-1} = \sqrt{\bar{\gamma}_{n-1}^2 + \sigma_n^2}$ . Thus, we obtain  $\bar{\gamma}_{n-1} = c_n \gamma_{n-1}$ . Then, we have

$$\bar{\mathbf{R}}_n = \bar{\mathbf{D}}_n \mathbf{R}_n. \quad (109)$$

Let

$$\mathbf{B} \equiv \begin{bmatrix} \gamma_0 & \times & \times & \times & \times & \times & \times \\ & \gamma_1 & \ddots & \ddots & & & \vdots \\ & & \gamma_2 & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & \gamma_{n-3} & \ddots & \times \\ & & & & & \gamma_{n-2} & \times \end{bmatrix}_{(n-1) \times n}. \quad (110)$$

Then, we have

$$\mathbf{R}_n = \mathbf{B} + \gamma_{n-1} \mathbf{e}_n, \quad (111)$$

and we obtain

$$\begin{aligned} \mathbf{R}_n^T \mathbf{R}_n &= (\mathbf{B}^T + \gamma_{n-1} \mathbf{e}_n^T) (\mathbf{B} + \gamma_{n-1} \mathbf{e}_n) \\ &= \mathbf{B}^T \mathbf{B} + \gamma_{n-1}^2 \mathbf{e}_n \mathbf{e}_n^T. \end{aligned} \quad (112)$$

We have

$$\begin{aligned} \bar{\mathbf{R}}_n &= \bar{\mathbf{D}}_n \mathbf{R}_n \\ &= \mathbf{B} + c_n \gamma_{n-1} \mathbf{e}_n \\ &= \mathbf{B} + \bar{\gamma}_{n-1} \mathbf{e}_n, \end{aligned} \quad (113)$$

$$\begin{aligned} \bar{\mathbf{R}}_n^T \bar{\mathbf{R}}_n &= (\mathbf{B}^T + \bar{\gamma}_{n-1} \mathbf{e}_n^T) (\mathbf{B} + \bar{\gamma}_{n-1} \mathbf{e}_n) \\ &= \mathbf{B}^T \mathbf{B} + \bar{\gamma}_{n-1}^2 \mathbf{e}_n \mathbf{e}_n^T. \end{aligned} \quad (114)$$

Using  $\bar{\gamma}_{n-1}^2 + \sigma_n^2 = \gamma_{n-1}^2$ , (112), and (114) we find that the coefficient matrix in linear system (104) is

$$\bar{\mathbf{R}}_n^T \bar{\mathbf{R}}_n + \sigma_n^2 \mathbf{e}_n \mathbf{e}_n^T = \mathbf{R}_n^T \mathbf{R}_n, \quad (115)$$

and the right-hand side vector is

$$\begin{aligned} \sigma_0 \mathbf{H}_n^T \mathbf{e}_1 &= \sigma_0 \bar{\mathbf{R}}_n^T \mathbf{Q}_{n-1} \mathbf{e}_1 \\ &= \sigma_0 \mathbf{R}_n^T \bar{\mathbf{D}}_n \mathbf{Q}_{n-1} \mathbf{e}_1. \end{aligned} \quad (116)$$

Since  $\mathbf{R}_n$  is nonsingular, we obtain this linear system

$$\left(\mathbf{R}_n^T \mathbf{R}_n\right) \mathbf{y}^{(n)} = \sigma_0 \mathbf{R}_n^T \bar{\mathbf{D}}_n \mathbf{Q}_{n-1} \mathbf{e}_1, \quad (117)$$

which reduces to this linear system

$$\mathbf{R}_n \mathbf{y}^{(n)} = \sigma_0 \bar{\mathbf{D}}_n \mathbf{Q}_{n-1} \mathbf{e}_1. \quad (118)$$

Thus, we obtain the  $n$ th iteration of the generalized QR-MINRES method

$$\begin{aligned} \mathbf{u}_R^{(n)} &= \mathbf{u}^{(0)} + \mathbf{W}_{n-1} \mathbf{y}^{(n)} \\ &= \mathbf{u}^{(0)} + \mathbf{W}_{n-1} \mathbf{R}_n^{-1} (\sigma_0 \widetilde{\mathbf{D}}_n \mathbf{Q}_{n-1} \mathbf{e}_1). \end{aligned} \tag{119}$$

### 9. GMRES Method

In the GMRES method, we let

$$\mathbf{u}^{(n)} = \mathbf{u}^{(0)} + \mathbf{W}_{n-1} \mathbf{y}^{(n)}. \tag{120}$$

Multiplying by  $\mathbf{A}$ , we have

$$\begin{aligned} \mathbf{A} \mathbf{u}^{(n)} &= \mathbf{A} \mathbf{u}^{(0)} + (\mathbf{A} \mathbf{W}_{n-1}) \mathbf{y}^{(n)}, \\ \mathbf{b} - \mathbf{A} \mathbf{u}^{(n)} &= \mathbf{b} - \mathbf{A} \mathbf{u}^{(0)} - (\mathbf{A} \mathbf{W}_{n-1}) \mathbf{y}^{(n)}. \end{aligned} \tag{121}$$

Then

$$\mathbf{r}^{(n)} = \mathbf{r}^{(0)} - (\mathbf{A} \mathbf{W}_{n-1}) \mathbf{y}^{(n)}. \tag{122}$$

For minimizing  $\|\mathbf{r}^{(n)}\|$ , we need to solve

$$\min_{\mathbf{y} \in \mathbb{R}^n} \|\widetilde{\mathbf{H}}_{n+1} \mathbf{y} - \sigma_0 \mathbf{e}_1\|, \tag{123}$$

by using Givens rotation, which is the GMRES method.

In Saad's book [4], there is a relation between the FOM method and the GMRES method. For the FOM method, we impose the Galerkin condition  $\mathbf{r}^{(n)} \perp \mathbf{W}_{n-1}$  and we solve this linear system for  $\mathbf{y}$

$$\mathbf{H}_n \mathbf{y} = \sigma_0 \mathbf{e}_1. \tag{124}$$

For the  $(n + 1)$ st iteration, we solve this linear system for  $\mathbf{y}$

$$\mathbf{H}_{n+1} \mathbf{y} = \sigma_0 \mathbf{e}_1. \tag{125}$$

Between the  $n$ th iteration and the  $(n + 1)$ st iteration of the FOM method, we obtain the solution of the this least squares problem

$$\min_{\mathbf{y} \in \mathbb{R}^n} \|\widetilde{\mathbf{H}}_{n+1} \mathbf{y} - \sigma_0 \mathbf{e}_1\|, \tag{126}$$

which is the same as in the GMRES method.

### 10. Differences between MMINRES and MGMRES

In this section, we assume that matrices  $\mathbf{E}$  and  $\mathbf{EA}$  are nonsingular symmetric matrices (but *not* necessarily positive definite). In [6, 7], we use a short term recurrence to generate

orthonormal vectors  $\{\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(n-3)}, \mathbf{w}^{(n-2)}, \mathbf{w}^{(n-1)}\}$  as follows:

$$\begin{aligned} \mathbf{u}^{(0)} &= \text{arbitrary}, \\ \mathbf{r}^{(0)} &= \mathbf{b} - \mathbf{A} \mathbf{u}^{(0)}, \\ \widetilde{\mathbf{w}}^{(0)} &= \mathbf{r}^{(0)}, \\ \sigma_0 &= \sqrt{\langle \mathbf{E} \widetilde{\mathbf{w}}^{(0)}, \widetilde{\mathbf{w}}^{(0)} \rangle}, \\ \mathbf{w}^{(0)} &= \left( \frac{1}{\sigma_0} \right) \widetilde{\mathbf{w}}^{(0)}, \end{aligned} \tag{127}$$

**for**  $i = 1, 2, \dots, n - 1$ ,

$$\alpha_{i-1} = \frac{\langle \mathbf{E} \mathbf{A} \mathbf{w}^{(i-1)}, \mathbf{w}^{(i-1)} \rangle}{\langle \mathbf{E} \mathbf{w}^{(i-1)}, \mathbf{w}^{(i-1)} \rangle}, \tag{128}$$

$$\beta_{i-1} = \frac{\langle \mathbf{E} \mathbf{A} \mathbf{w}^{(i-1)}, \mathbf{w}^{(i-2)} \rangle}{\langle \mathbf{E} \mathbf{w}^{(i-2)}, \mathbf{w}^{(i-2)} \rangle},$$

$$\widetilde{\mathbf{w}}^{(i)} = \mathbf{A} \mathbf{w}^{(i-1)} - \alpha_{i-1} \mathbf{w}^{(i-1)} - \beta_{i-1} \mathbf{w}^{(i-2)}, \tag{129}$$

$$\sigma_i = \sqrt{|\langle \mathbf{E} \widetilde{\mathbf{w}}^{(i)}, \widetilde{\mathbf{w}}^{(i)} \rangle|},$$

$$\mathbf{w}^{(i)} = \left( \frac{1}{\sigma_i} \right) \widetilde{\mathbf{w}}^{(i)}, \tag{130}$$

**end for.**

We obtain these properties, for  $(0 \leq i, j \leq n - 1)$ ,

$$\langle \mathbf{E} \mathbf{w}^{(i)}, \mathbf{w}^{(j)} \rangle = 0, \quad \text{if } i \neq j, \tag{131}$$

$$\beta_i = \sigma_i,$$

$$(\mathbf{E} \mathbf{W}_{n-1})^T \mathbf{W}_{n-1}$$

$$= \begin{bmatrix} d_0 & & & & & & & & & \\ & d_1 & & & & & & & & \\ & & d_2 & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & d_{n-3} & & & & & \\ & & & & & d_{n-2} & & & & \\ & & & & & & d_{n-1} & & & \end{bmatrix}_{n \times n} \tag{132}$$

$$\equiv \mathbf{D}_n,$$

where

$$d_i = \langle \mathbf{E} \mathbf{w}^{(i)}, \mathbf{w}^{(i)} \rangle = \begin{cases} 1, & \text{if } \langle \mathbf{E} \widetilde{\mathbf{w}}^{(i)}, \widetilde{\mathbf{w}}^{(i)} \rangle > 0, \\ -1, & \text{if } \langle \mathbf{E} \widetilde{\mathbf{w}}^{(i)}, \widetilde{\mathbf{w}}^{(i)} \rangle < 0. \end{cases} \tag{133}$$

Consequently, we obtain these matrix equations

$$\begin{aligned} \mathbf{A}\mathbf{W}_{n-1} &= \mathbf{W}_{n-1}\mathbf{T}_n + \sigma_n \mathbf{w}^{(n)} \mathbf{e}_n^T \\ &= \mathbf{W}_n \widehat{\mathbf{T}}_{n+1}, \end{aligned} \quad (134)$$

$$\mathbf{W}_{n-1} = [\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(n-3)}, \mathbf{w}^{(n-2)}, \mathbf{w}^{(n-1)}]_{n \times n}, \quad (135)$$

$$\mathbf{e}_n^T = [0, 0, 0, \dots, 0, 0, 0, 1]_{1 \times n}, \quad (136)$$

$$\mathbf{T}_n = \begin{bmatrix} \alpha_0 & \beta_1 & & & & & & & \\ \sigma_1 & \alpha_1 & \beta_2 & & & & & & \\ & \sigma_2 & \alpha_2 & \beta_3 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & \sigma_{n-3} & \alpha_{n-3} & \beta_{n-2} & & & \\ & & & & \sigma_{n-2} & \alpha_{n-2} & \beta_{n-1} & & \\ & & & & & \sigma_{n-1} & \alpha_{n-1} & & \end{bmatrix}_{n \times n}, \quad (137)$$

$$\widehat{\mathbf{T}}_{n+1} = \begin{bmatrix} \alpha_0 & \beta_1 & & & & & & & \\ \sigma_1 & \alpha_1 & \beta_2 & & & & & & \\ & \sigma_2 & \alpha_2 & \beta_3 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & \sigma_{n-3} & \alpha_{n-3} & \beta_{n-2} & & & \\ & & & & \sigma_{n-2} & \alpha_{n-2} & \beta_{n-1} & & \\ & & & & & \sigma_{n-1} & \alpha_{n-1} & & \\ & & & & & & \sigma_n & & \end{bmatrix}_{(n+1) \times n}. \quad (138)$$

From (134) and (132), we obtain

$$\begin{aligned} &(\mathbf{E}\mathbf{W}_{n-1})^T (\mathbf{A}\mathbf{W}_{n-1}) \\ &= (\mathbf{E}\mathbf{W}_{n-1})^T (\mathbf{W}_{n-1}\mathbf{T}_n + \sigma_n \mathbf{w}^{(n)} \mathbf{e}_n^T) \\ &= (\mathbf{E}\mathbf{W}_{n-1})^T \mathbf{W}_{n-1}\mathbf{T}_n + (\mathbf{E}\mathbf{W}_{n-1})^T \sigma_n \mathbf{w}^{(n)} \mathbf{e}_n^T \\ &= (\mathbf{E}\mathbf{W}_{n-1})^T \mathbf{W}_{n-1}\mathbf{T}_n + \sigma_n \langle \mathbf{E}\mathbf{W}_{n-1}, \mathbf{w}^{(n)} \rangle \mathbf{e}_n^T \\ &= \mathbf{D}_n \mathbf{T}_n + \mathbf{0} = \mathbf{D}_n \mathbf{T}_n. \end{aligned} \quad (139)$$

Assume that  $\mathbf{x}^{(n)} - \mathbf{x}^{(0)} \in \mathbf{W}_{n-1}$ ; then we have

$$\begin{aligned} \mathbf{x}^{(n)} &= \mathbf{x}^{(0)} + \mathbf{W}_{n-1} \mathbf{y}^{(n)} \\ \mathbf{r}^{(n)} &= \mathbf{r}^{(0)} - (\mathbf{A}\mathbf{W}_{n-1}) \mathbf{y}^{(n)}. \end{aligned} \quad (140)$$

From the Galerkin condition  $\mathbf{E}\mathbf{r}^{(n)} \perp \mathbf{W}_{n-1}$ , we have

$$(\mathbf{E}\mathbf{r}^{(n)})^T \mathbf{W}_{n-1} = \mathbf{0}. \quad (141)$$

From (141) and since  $\mathbf{E}$  is symmetric, we obtain

$$\begin{aligned} \mathbf{0} &= (\mathbf{E}\mathbf{r}^{(n)})^T \mathbf{W}_{n-1} = (\mathbf{r}^{(n)})^T \mathbf{E}^T \mathbf{W}_{n-1} \\ &= (\mathbf{r}^{(n)})^T \mathbf{E} \mathbf{W}_{n-1} = (\mathbf{E}\mathbf{W}_{n-1})^T \mathbf{r}^{(n)}. \end{aligned} \quad (142)$$

Then by (142), we have

$$(\mathbf{E}\mathbf{W}_{n-1})^T \mathbf{r}^{(n)} = \mathbf{0}. \quad (143)$$

From (139), (140), and (143), we obtain this linear system

$$\begin{aligned} (\mathbf{D}_n \mathbf{T}_n) \mathbf{y}^{(n)} &= (\mathbf{E}\mathbf{W}_{n-1})^T (\mathbf{A}\mathbf{W}_{n-1}) \mathbf{y}^{(n)} \\ &= (\mathbf{E}\mathbf{W}_{n-1})^T (\mathbf{r}^{(0)} - \mathbf{r}^{(n)}) \\ &= (\mathbf{E}\mathbf{W}_{n-1})^T \mathbf{r}^{(0)} \\ &= \sigma_0 (\mathbf{E}\mathbf{W}_{n-1})^T \mathbf{w}^{(0)} \\ &= \sigma_0 \langle \mathbf{E}\mathbf{W}_{n-1}, \mathbf{w}^{(0)} \rangle \\ &= \sigma_0 \mathbf{e}_n, \end{aligned} \quad (144)$$

using (127), where  $\sigma_0 = d_0 \sqrt{|\langle \mathbf{E}\mathbf{r}^{(0)}, \mathbf{r}^{(0)} \rangle|}$  and  $\sigma_0 \mathbf{w}^{(0)} = \mathbf{r}^{(0)}$ .  
By (132) and (137), we have

$$\mathbf{D}_n \mathbf{T}_n = \begin{bmatrix} d_0 \alpha_0 & d_0 \beta_1 & & & & & & & \\ d_1 \sigma_1 & d_1 \alpha_1 & d_1 \beta_2 & & & & & & \\ & d_2 \sigma_2 & d_2 \alpha_2 & d_2 \beta_3 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & d_{n-3} \sigma_{n-3} & d_{n-3} \alpha_{n-3} & d_{n-3} \beta_{n-2} & & & \\ & & & & d_{n-2} \sigma_{n-2} & d_{n-2} \alpha_{n-2} & d_{n-2} \beta_{n-1} & & \\ & & & & & d_{n-1} \sigma_{n-1} & d_{n-1} \alpha_{n-1} & & \end{bmatrix}_{n \times n}. \quad (145)$$

Notice that  $\mathbf{D}_n \mathbf{T}_n$  is symmetric since

$$\begin{aligned} d_{i-1} \beta_i &= \frac{d_{i-1} \langle \mathbf{E} \mathbf{A} \mathbf{w}^{(i)}, \mathbf{w}^{(i-1)} \rangle}{\langle \mathbf{E} \mathbf{w}^{(i-1)}, \mathbf{w}^{(i-1)} \rangle} \\ &= \frac{d_{i-1} \langle \mathbf{w}^{(i)}, \mathbf{E} \mathbf{A} \mathbf{w}^{(i-1)} \rangle}{\langle \mathbf{E} \mathbf{w}^{(i-1)}, \mathbf{w}^{(i-1)} \rangle} \\ &= \frac{d_{i-1} \langle \mathbf{w}^{(i)}, \mathbf{E} (\sigma_i \mathbf{w}^{(i)} + \alpha_{i-1} \mathbf{w}^{(i-1)} + \beta_{i-1} \mathbf{w}^{(i-2)}) \rangle}{\langle \mathbf{E} \mathbf{w}^{(i-1)}, \mathbf{w}^{(i-1)} \rangle} \\ &= \left( d_{i-1} \left[ \sigma_i \langle \mathbf{E} \mathbf{w}^{(i)}, \mathbf{w}^{(i)} \rangle + \alpha_{i-1} \langle \mathbf{E} \mathbf{w}^{(i)}, \mathbf{w}^{(i-1)} \rangle \right. \right. \\ &\quad \left. \left. + \beta_{i-1} \langle \mathbf{E} \mathbf{w}^{(i)}, \mathbf{w}^{(i-2)} \rangle \right] \right) \\ &\quad \times \left( \langle \mathbf{E} \mathbf{w}^{(i-1)}, \mathbf{w}^{(i-1)} \rangle \right)^{-1} \\ &= \frac{d_{i-1} \sigma_i \langle \mathbf{E} \mathbf{w}^{(i)}, \mathbf{w}^{(i)} \rangle}{\langle \mathbf{E} \mathbf{w}^{(i-1)}, \mathbf{w}^{(i-1)} \rangle} \\ &= \frac{d_{i-1} \sigma_i d_i}{d_{i-1}} = d_i \sigma_i. \end{aligned} \tag{146}$$

Here we use (128), EA is symmetric, (129), (130), E is symmetric, (131), and (133).

Let  $\bar{\mathbf{x}}_{\text{sym}}^{(n)}$  be the estimated solution corresponding to the Galerkin condition. Then

$$\bar{\mathbf{x}}_{\text{sym}}^{(n)} = \mathbf{x}^{(0)} + \mathbf{W}_{n-1} \bar{\mathbf{y}}^{(n)}, \tag{147}$$

where  $\bar{\mathbf{y}}^{(n)}$  satisfies this linear system

$$(\mathbf{D}_n \mathbf{T}_n) \bar{\mathbf{y}}^{(n)} = \sigma_0 \mathbf{e}_n. \tag{148}$$

Let  $\mathbf{x}_{\text{min}}^{(n)}$  be the estimated solution corresponding to the least squares problem. Then

$$\mathbf{x}_{\text{min}}^{(n)} = \mathbf{x}^{(0)} + \mathbf{W}_{n-1} \mathbf{y}^{(n)}, \tag{149}$$

where  $\mathbf{y}^{(n)}$  satisfies this expression

$$\min_{\mathbf{y}^{(n)} \in \mathbb{R}^n} \left\| (\mathbf{D}_{n+1} \hat{\mathbf{T}}_{n+1}) \mathbf{y}^{(n)} - \sigma_0 \mathbf{e}_{n+1} \right\|. \tag{150}$$

We call  $\mathbf{x}_{\text{min}}^{(n)}$  the  $n$ th iteration of the MMINRES method.

Notice that if to matrix  $\mathbf{D}_{n+1} \mathbf{T}_{n+1}$  we add this bottom row

$$[0, 0, 0, \dots, 0, 0, 0, d_n \sigma_n]_{1 \times n} \tag{151}$$

and add this far-right column

$$[0, 0, 0, \dots, 0, 0, 0, d_{n-1} \beta_n, d_n \alpha_n]^T_{1 \times n+1}, \tag{152}$$

see (132) and (138), then we obtain the matrix  $\mathbf{D}_{n+1} \hat{\mathbf{T}}_{n+1}$ .

We define these Given rotation matrices

$$\mathbf{Q}_{n-1} \equiv \mathbf{Q}_{n-1,n} \mathbf{Q}_{n-2,n-1} \mathbf{Q}_{n-3,n-2} \cdots \mathbf{Q}_{3,4} \mathbf{Q}_{2,3} \mathbf{Q}_{1,2}, \tag{153}$$

where  $\mathbf{Q}_{i,i+1}$  is given by (23). Multiplying matrix  $\mathbf{Q}_{n-1}$  times both sides of linear system (148), we obtain this linear system

$$(\mathbf{Q}_{n-1} \mathbf{D}_n \mathbf{T}_n) \bar{\mathbf{y}}^{(n)} = \mathbf{Q}_{n-1} \sigma_0 \mathbf{e}_n, \tag{154}$$

where, from (137), the coefficient matrix is

$$\begin{aligned} &\mathbf{Q}_{n-1} \mathbf{D}_n \mathbf{T}_n \\ &= \left[ \begin{array}{cccc|c} \gamma_0 & \delta_1 & \varepsilon_2 & & \\ & \gamma_1 & \delta_2 & \varepsilon_3 & \\ & & \gamma_2 & \delta_3 & \varepsilon_4 \\ & & & \ddots & \ddots \\ & & & & \gamma_{n-3} & \delta_{n-2} & \varepsilon_{n-1} \\ & & & & & \gamma_{n-2} & \delta_{n-1} \\ & & & & & & \bar{\gamma}_{n-1} \end{array} \right]_{n \times n} \\ &\equiv \bar{\mathbf{R}}_n, \end{aligned} \tag{155}$$

and the right-hand side vector is

$$\begin{aligned} \mathbf{Q}_{n-1} \sigma_0 \mathbf{e}_n &= [\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{n-3}, \zeta_{n-2}, \bar{\zeta}_{n-1}]_{1 \times n}^T \\ &\equiv \mathbf{z}^{(n)}. \end{aligned} \tag{156}$$

Note that  $\bar{\gamma}_{n-1}$  is *not* always nonzero, so that  $\bar{\mathbf{R}}_n$  might be singular, but here, we assume that  $\bar{\mathbf{R}}_n$  is nonsingular. Then we define this matrix factorization

$$\mathbf{W}_{n-1} = \bar{\mathbf{P}}_n \bar{\mathbf{R}}_n, \tag{157}$$

where

$$\bar{\mathbf{P}}_n \equiv [\mathbf{p}^{(0)}, \mathbf{p}^{(1)}, \mathbf{p}^{(2)}, \dots, \mathbf{p}^{(n-3)}, \mathbf{p}^{(n-2)}, \bar{\mathbf{p}}^{(n-1)}]_{n \times n}. \tag{158}$$

Moreover, we define the right-hand side vector in (156) as

$$\bar{\mathbf{z}}^{(n)} = \bar{\mathbf{R}}_n \bar{\mathbf{y}}^{(n)}. \tag{159}$$

From (147), we have, using (157) and (159),

$$\begin{aligned} \bar{\mathbf{x}}_{\text{sym}}^{(n)} &= \mathbf{x}^{(0)} + \mathbf{W}_{n-1} \bar{\mathbf{y}}^{(n)} \\ &= \mathbf{x}^{(0)} + \bar{\mathbf{P}}_n \bar{\mathbf{R}}_n \bar{\mathbf{y}}^{(n)} \\ &= \mathbf{x}^{(0)} + \bar{\mathbf{P}}_n \bar{\mathbf{z}}^{(n)}. \end{aligned} \tag{160}$$

For the next iterate  $\bar{\mathbf{x}}_{\text{sym}}^{(n+1)}$ , we need to solve this linear system

$$(\mathbf{D}_{n+1} \mathbf{T}_{n+1}) \bar{\mathbf{y}}^{(n+1)} = \sigma_0 \mathbf{e}_{n+1}, \tag{161}$$

where, by (132) and (137), the coefficient matrix is

$$\mathbf{D}_{n+1} \mathbf{T}_{n+1} = \left[ \begin{array}{cccccccc|c} d_0\alpha_0 & d_0\beta_1 & & & & & & & \\ d_1\sigma_1 & d_1\alpha_1 & d_1\beta_2 & & & & & & \\ & d_2\sigma_2 & d_2\alpha_2 & d_2\beta_3 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & d_{n-3}\sigma_{n-3} & d_{n-3}\alpha_{n-3} & d_{n-3}\beta_{n-2} & & & \\ & & & & d_{n-2}\sigma_{n-2} & d_{n-2}\alpha_{n-2} & d_{n-2}\beta_{n-1} & & \\ & & & & & d_{n-1}\sigma_{n-1} & d_{n-1}\alpha_{n-1} & d_{n-1}\beta_n & \\ & & & & & & d_n\sigma_n & d_n\alpha_n & \end{array} \right]_{(n+1) \times (n+1)} \quad (162)$$

Multiplying  $\mathbf{Q}_{n-1}$  times both sides of linear system (161), we obtain this coefficient matrix

$$\mathbf{Q}_{n-1} (\mathbf{D}_{n+1} \mathbf{T}_{n+1}) = \left[ \begin{array}{cccccccc|c} \gamma_0 & \delta_1 & \varepsilon_2 & & & & & & \\ & \gamma_1 & \delta_2 & \varepsilon_3 & & & & & \\ & & \gamma_2 & \delta_3 & \varepsilon_4 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & \gamma_{n-3} & \delta_{n-2} & \varepsilon_{n-1} & & \\ & & & & & \gamma_{n-2} & \delta_{n-1} & \varepsilon_n & \\ & & & & & & \bar{\gamma}_{n-1} & \psi & \\ & & & & & & d_n\sigma_n & d_n\alpha_n & \end{array} \right]_{(n+1) \times (n+1)} \quad (163)$$

The last columns in matrices (162) and (163) are related as follows:

$$\begin{aligned} & \mathbf{Q}_{n-1,n} [0, 0, 0, \dots, 0, 0, 0, d_{n-1}\beta_n, d_n\alpha_n]_{(n+1) \times 1}^T \\ & = [0, 0, 0, \dots, 0, 0, 0, \varepsilon_n, \psi, d_n\alpha_n]_{(n+1) \times 1}^T. \end{aligned} \quad (164)$$

To eliminate  $d_n\sigma_n$ , we compute the  $n$ th Givens rotation matrix  $\mathbf{Q}_{n,n+1}$  with

$$\gamma_{n-1} = \sqrt{\bar{\gamma}_{n-1}^2 + (d_n\sigma_n)^2}, \quad c_n = \frac{\bar{\gamma}_{n-1}}{\gamma_{n-1}}, \quad s_n = -\frac{d_n\sigma_n}{\gamma_{n-1}}. \quad (165)$$

Recall, by (23), that  $c_i^2 + s_i^2 = 1$ . Next, we multiply matrix  $\mathbf{Q}_{n,n+1}$  times the coefficient matrix  $\mathbf{Q}_{n-1}(\mathbf{D}_{n+1} \mathbf{T}_{n+1})$ , in (163),

$$\mathbf{Q}_{n,n+1} \mathbf{Q}_{n-1} (\mathbf{D}_{n+1} \mathbf{T}_{n+1}) = \left[ \begin{array}{cccccccc|c} \gamma_0 & \delta_1 & \varepsilon_2 & & & & & & \\ & \gamma_1 & \delta_2 & \varepsilon_3 & & & & & \\ & & \gamma_2 & \delta_3 & \varepsilon_4 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & \gamma_{n-3} & \delta_{n-2} & \varepsilon_{n-1} & & \\ & & & & & \gamma_{n-2} & \delta_{n-1} & \varepsilon_n & \\ & & & & & & \gamma_{n-1} & \delta_n & \\ & & & & & & & \bar{\gamma}_n & \end{array} \right]_{(n+1) \times (n+1)} \equiv \bar{\mathbf{R}}_{n+1}. \quad (166)$$

Then multiplying matrix  $\mathbf{Q}_{n,n+1}$  times the right-hand side vector  $\mathbf{Q}_{n-1}\sigma_0\mathbf{e}_{n+1}$ , we obtain

$$\begin{aligned} & \mathbf{Q}_{n,n+1} (\mathbf{Q}_{n-1}\sigma_0\mathbf{e}_{n+1}) = [\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{n-2}, \zeta_{n-1}, \bar{\zeta}_n]_{(n+1) \times 1}^T \\ & \equiv \bar{\mathbf{z}}_{n+1}. \end{aligned} \quad (167)$$

Looking at the last columns in (163) and (166), we see that

$$\begin{aligned} & \mathbf{Q}_{n,n+1} [0, 0, 0, \dots, 0, 0, 0, \varepsilon_n, \psi, d_n\alpha_n]_{(n+1) \times 1}^T \\ & = [0, 0, 0, \dots, 0, 0, 0, \varepsilon_n, \delta_n, \bar{\gamma}_n]_{(n+1) \times 1}^T, \end{aligned} \quad (168)$$

where

$$\zeta_{n-1} = c_n \bar{\zeta}_{n-1}, \quad \bar{\zeta}_n = s_n \bar{\zeta}_{n-1}. \quad (169)$$

Recall, by (23),  $c_n^2 + s_n^2 = 1$ .

Before forming the iterate  $\bar{\mathbf{x}}_{\text{sym}}^{(n+1)}$ , we consider this least square problem (150)

$$\min_{\mathbf{y}^{(n)} \in \mathbb{R}^n} \left\| (\mathbf{D}_{n+1} \hat{\mathbf{T}}_{n+1}) \mathbf{y}^{(n)} - \sigma_0 \mathbf{e}_{n+1} \right\|. \quad (170)$$

We have

$$\begin{aligned} & \left\| \mathbf{Q}_{n,n+1} \mathbf{Q}_{n-1} (\mathbf{D}_{n+1} \hat{\mathbf{T}}_{n+1} \mathbf{y}^{(n)} - \sigma_0 \mathbf{e}_{n+1}) \right\| \\ &= \left\| \begin{bmatrix} \mathbf{R}_n \\ 0, 0, 0, \dots, 0, 0, 0 \end{bmatrix}_{(n+1) \times n} \mathbf{y}^{(n)} - \begin{bmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{n-2} \\ \zeta_{n-1} \\ \bar{\zeta}_n \end{bmatrix}_{(n+1) \times 1} \right\|, \end{aligned} \quad (171)$$

where

$$\mathbf{R}_n \equiv \left[ \begin{array}{cccc|cccc} \gamma_0 & \delta_1 & \varepsilon_2 & & & & & \\ & \gamma_1 & \delta_2 & \varepsilon_3 & & & & \\ & & \gamma_2 & \delta_3 & \varepsilon_4 & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & \gamma_{n-4} & \delta_{n-3} & \varepsilon_{n-2} & \\ & & & & & \gamma_{n-3} & \delta_{n-2} & \varepsilon_{n-1} \\ & & & & & & \gamma_{n-2} & \delta_{n-1} \\ & & & & & & & \gamma_{n-1} \end{array} \right]_{n \times n}. \quad (172)$$

Next define

$$\mathbf{z}^{(n)} \equiv [\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{n-3}, \zeta_{n-2}, \zeta_{n-1}]_{1 \times n}^T. \quad (173)$$

Hence,  $\mathbf{y}^{(n)}$  is the solution of this linear system

$$\mathbf{R}_n \mathbf{y}^{(n)} = \mathbf{z}^{(n)}, \quad (174)$$

which minimizes the expression

$$\min_{\mathbf{y}^{(n)} \in \mathbb{R}^n} \left\| (\mathbf{D}_{n+1} \hat{\mathbf{T}}_{n+1}) \mathbf{y}^{(n)} - \sigma_0 \mathbf{e}_{n+1} \right\|, \quad (175)$$

and then

$$\bar{\zeta}_n = \left\| (\mathbf{D}_{n+1} \hat{\mathbf{T}}_{n+1}) \mathbf{y}^{(n)} - \sigma_0 \mathbf{e}_{n+1} \right\|. \quad (176)$$

If  $\sigma_n \neq 0$ , then  $\gamma_n \neq 0$  and  $\mathbf{R}_n$  is nonsingular, and we can solve linear system (174) for  $\mathbf{y}^{(n)}$ . We discuss the case  $\sigma_n = 0$  later.

By (149), let

$$\begin{aligned} \mathbf{x}_{\min}^{(n)} &= \mathbf{x}^{(0)} + \mathbf{W}_{n-1} \mathbf{y}^{(n)} \\ &= \mathbf{x}^{(0)} + (\mathbf{W}_{n-1} \mathbf{R}_n^{-1}) \mathbf{z}^{(n)}, \end{aligned} \quad (177)$$

using (174). Here

$$\mathbf{W}_{n-1} = \mathbf{P}_n \mathbf{R}_n, \quad (178)$$

$$\mathbf{P}_{n-1} = [\mathbf{p}^{(0)}, \mathbf{p}^{(1)}, \mathbf{p}^{(2)}, \dots, \mathbf{p}^{(n-3)}, \mathbf{p}^{(n-2)}, \mathbf{p}^{(n-1)}]_{n \times n}. \quad (179)$$

From (177) and (179), we have

$$\begin{aligned} \mathbf{x}_{\min}^{(n)} &= \mathbf{x}^{(0)} + \mathbf{P}_n \mathbf{z}^{(n)} \\ &= \mathbf{x}_{\min}^{(n-1)} + \zeta_{n-1} \mathbf{p}^{(n-1)}, \end{aligned} \quad (180)$$

where, by (172),

$$\mathbf{p}^{(n-1)} = \left( \frac{1}{\gamma_{n-1}} \right) [\mathbf{w}^{(n-1)} - \delta_{n-1} \mathbf{p}^{(n-2)} - \varepsilon_{n-1} \mathbf{p}^{(n-3)}]. \quad (181)$$

From (147), we obtain, by (159), (157), and (180),

$$\begin{aligned} \bar{\mathbf{x}}_{\text{sym}}^{(n+1)} &= \mathbf{x}^{(0)} + \mathbf{W}_n \bar{\mathbf{y}}^{(n+1)} \\ &= \mathbf{x}^{(0)} + \mathbf{W}_n (\mathbf{R}_{n+1}^{-1} \mathbf{z}^{(n+1)}) \\ &= \mathbf{x}^{(0)} + (\mathbf{W}_n \bar{\mathbf{R}}_{n+1}^{-1}) \bar{\mathbf{z}}^{(n+1)} \\ &= \mathbf{x}^{(0)} + \bar{\mathbf{P}}_{n+1} \bar{\mathbf{z}}^{(n+1)} \\ &= \mathbf{x}^{(0)} + \mathbf{P}_{n-1} \mathbf{z}^{(n)} + \bar{\zeta}_n \bar{\mathbf{p}}^{(n)} \\ &= \mathbf{x}_{\min}^{(n)} + \bar{\zeta}_n \bar{\mathbf{p}}^{(n)}. \end{aligned} \quad (182)$$

We note that  $\bar{\mathbf{x}}_{\text{sym}}^{(n)}$  is the estimated solution satisfying the Galerkin condition, while

$$\mathbf{x}_{\min}^{(n)} = \mathbf{x}^{(0)} + \mathbf{W}_{n-1} \mathbf{y}^{(n)}, \quad (183)$$

where  $\mathbf{y}^{(n)}$  minimizes this expression

$$\min_{\mathbf{y}^{(n)} \in \mathbb{R}^n} \left\| (\mathbf{D}_{n+1} \hat{\mathbf{T}}_{n+1}) \mathbf{y}^{(n)} - \sigma_0 \mathbf{e}_{n+1} \right\|. \quad (184)$$

**Proposition 1.** If  $\sigma_n = 0$  and  $\bar{\mathbf{w}}^{(n)} = \mathbf{0}$ , then  $\bar{\mathbf{x}}_{\text{sym}}^{(n)}$  is the exact solution of  $\mathbf{A}\mathbf{u} = \mathbf{b}$ . Moreover, one obtains  $\mathbf{x}_{\min}^{(n)} = \bar{\mathbf{x}}_{\text{sym}}^{(n)}$ .

*Proof.* We consider the residual vector  $\bar{\mathbf{r}}^{(n)}$  corresponding to the iterate  $\bar{\mathbf{x}}_{\text{sym}}^{(n)}$ . Then, by (147), we have

$$\begin{aligned} \bar{\mathbf{r}}^{(n)} &= \mathbf{b} - \mathbf{A}\bar{\mathbf{x}}_{\text{sym}}^{(n)} \\ &= \mathbf{b} - \mathbf{A}(\mathbf{x}^{(0)} + \mathbf{W}_{n-1}\bar{\mathbf{y}}^{(n)}) \\ &= (\mathbf{b} - \mathbf{A}\mathbf{x}^{(0)}) - (\mathbf{A}\mathbf{W}_{n-1})\bar{\mathbf{y}}^{(n)} \\ &= \mathbf{r}^{(0)} - (\mathbf{A}\mathbf{W}_{n-1})\bar{\mathbf{y}}^{(n)}. \end{aligned} \tag{185}$$

Here  $\bar{\mathbf{y}}^{(n)}$  satisfies (148) and we have

$$(\mathbf{D}_n\mathbf{T}_n)\bar{\mathbf{y}}^{(n)} = \sigma_0\mathbf{e}_n, \tag{186}$$

$$\mathbf{T}_n\bar{\mathbf{y}}^{(n)} = \mathbf{D}_n^{-1}\sigma_0\mathbf{e}_n. \tag{187}$$

Hence, starting from (185) and by using (134) and (187), we have

$$\begin{aligned} \bar{\mathbf{r}}^{(n)} &= \mathbf{r}^{(0)} - (\mathbf{A}\mathbf{W}_{n-1})\bar{\mathbf{y}}^{(n)} \\ &= \mathbf{r}^{(0)} - (\mathbf{W}_{n-1}\mathbf{T}_n + \sigma_n\mathbf{w}^{(n)}\mathbf{e}_n^T)\bar{\mathbf{y}}^{(n)} \\ &= \mathbf{r}^{(0)} - \mathbf{W}_{n-1}(\mathbf{T}_n\bar{\mathbf{y}}^{(n)}) - \sigma_n\mathbf{w}^{(n)}\mathbf{e}_n^T\bar{\mathbf{y}}^{(n)} \\ &= \mathbf{r}^{(0)} - \mathbf{W}_{n-1}(\mathbf{D}_n^{-1}\sigma_0\mathbf{e}_n) - \sigma_n\mathbf{w}^{(n)}\mathbf{e}_n^T\bar{\mathbf{y}}^{(n)} \\ &= \sigma_0\mathbf{w}^{(0)} - \sigma_0\mathbf{w}^{(0)} - \sigma_n\mathbf{w}^{(n)}\mathbf{e}_n^T\bar{\mathbf{y}}^{(n)} \\ &= \mathbf{0} - \sigma_n\mathbf{w}^{(n)}\mathbf{e}_n^T\bar{\mathbf{y}}^{(n)} \\ &= -\sigma_n\bar{\eta}_n\mathbf{w}^{(n)}, \end{aligned} \tag{188}$$

since  $\mathbf{r}^{(0)} = \sigma_0\mathbf{w}^{(0)}$ . Here  $\bar{\eta}_n$  is the  $n$ th component of the vector  $\bar{\mathbf{y}}^{(n)}$ .

If  $\sigma_n = 0$ , then  $\bar{\mathbf{r}}^{(n)} = \mathbf{0}$ . Thus, we obtain  $\bar{\mathbf{x}}_{\text{sym}}^{(n)}$ , which is the exact solution. It follows from (165) that if  $\sigma_n = 0$ , then

$$c_n = \frac{\bar{\gamma}_{n-1}}{|\bar{\gamma}_{n-1}|} = \begin{cases} 1, & \bar{\gamma}_{n-1} > 0, \\ -1, & \bar{\gamma}_{n-1} < 0, \end{cases} \tag{189}$$

$$s_n = 0.$$

Since

$$\bar{\mathbf{x}}_{\text{sym}}^{(n)} = \mathbf{x}^{(0)} + \mathbf{W}_{n-1}\bar{\mathbf{y}}^{(n)}, \quad \bar{\mathbf{R}}_n\bar{\mathbf{y}}^{(n)} = \bar{\mathbf{z}}^{(n)}, \tag{190}$$

$$\mathbf{x}_{\text{min}}^{(n)} = \mathbf{x}^{(0)} + \mathbf{W}_{n-1}\mathbf{y}^{(n)}, \quad \mathbf{R}_n\mathbf{y}^{(n)} = \mathbf{z}^{(n)},$$

the only difference between  $\bar{\mathbf{R}}_n$  and  $\mathbf{R}_n$  is the  $(k, k)$ -entries, which are  $\bar{\gamma}_{n-1}$  and  $\gamma_{n-1}$ , respectively. Let the  $n$ th components of  $\bar{\mathbf{y}}^{(n)}$  and  $\mathbf{y}^{(n)}$  be  $\bar{\eta}_n$  and  $\eta_n$ , respectively. It follows that both matrices  $\bar{\mathbf{R}}_n$  and  $\mathbf{R}_n$  are triangular, where  $\eta_n = \zeta_{n-1}/\gamma_{n-1}$  and  $\bar{\eta}_n = \bar{\zeta}_{n-1}/\bar{\gamma}_{n-1}$ . By (169), we have

$$\eta_n = \frac{c_n\bar{\zeta}_{n-1}}{|\bar{\gamma}_{n-1}|} = \frac{\bar{\zeta}_{n-1}}{\bar{\gamma}_{n-1}} = \bar{\eta}_n. \tag{191}$$

Hence, we have

$$\mathbf{y}^{(n)} = \bar{\mathbf{y}}^{(n)}, \quad \mathbf{x}_{\text{min}}^{(n)} = \bar{\mathbf{x}}_{\text{sym}}^{(n)}. \tag{192}$$

□

## 11. Final Notes

In the MGMRES method, we solve this linear system

$$(\tilde{\mathbf{T}}_{n+1}^T\mathbf{D}_{n+1}\mathbf{T}_{n+1})\mathbf{y}^{(n+1)} = \sigma_0\mathbf{e}_1, \tag{193}$$

while in the MMINRES method, we solve this linear system

$$(\tilde{\mathbf{T}}_{n+1}^T\tilde{\mathbf{T}}_{n+1})\mathbf{y}^{(n+1)} = \mathbf{D}_{n+1}\sigma_0\mathbf{e}_1. \tag{194}$$

Clearly, the MMINRES method and the MGMRES method are the same when the diagonal matrix  $\mathbf{D}_{n+1}$  is the identity matrix  $\mathbf{I}_{n+1}$ .

See also Kincaid et al. [8].

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## References

- [1] C. Lanczos, "An iteration method for the solution of the eigenvalue problem of linear differential and integral operators," *Journal of Research of the National Bureau of Standards*, vol. 45, pp. 255–282, 1950.
- [2] G. H. Golub and C. F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, 3rd edition, 1996.
- [3] C. C. Paige and M. A. Saunders, "Solutions of sparse indefinite systems of linear equations," *SIAM Journal on Numerical Analysis*, vol. 12, no. 4, pp. 617–629, 1975.
- [4] Y. Saad, *Iterative Methods for Sparse Linear Systems*, SIAM, 2nd edition, 2003.
- [5] Y. Saad and M. H. Schultz, "GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems," *Society for Industrial and Applied Mathematics*, vol. 7, no. 3, pp. 856–869, 1986.
- [6] Y. Li, J. Chen, and D. R. Kincaid, "Some generalizations and modifications of iterative methods for solving large sparse symmetric indefinite linear systems," *Abstract and Applied Analysis*, vol. 2014, Article ID 237808, 11 pages, 2014.
- [7] Y. Li, *The modified, MINRES methods for solving large sparse non-symmetric linear systems [M.S. thesis]*, Department of Mathematics, National Kaohsiung Normal University, Kaohsiung, Taiwan, 2007.
- [8] D. R. Kincaid, D. M. Young, and J.-Y. Chen, "Overview of MGMRES and LAN/MGMRES methods for solving non-symmetric linear systems," *Taiwanese Journal of Mathematics*, vol. 4, no. 3, pp. 385–396, 2000.