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Research Article

Travelling Waves of an n-Species Food Chain Model with Spatial Diffusion and Time Delays

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We investigate an n-species food chain model with spatial diffusion and time delays. By using Schauder's fixed point theorem, we obtain the result about the existence of the travelling wave solutions of the food chain model with reaction term satisfying the partial quasimonotonicity conditions.

1. Introduction

In the past few decades, the dynamic relationship between predators and preys has been investigated intensively due to its universal existence and importance in both ecology and mathematical ecology [1–3].

Recently, traveling waves for reaction-diffusion systems (see, e.g., [4–7]) have received considerable attention since they determine the long-term behavior of other solutions in many situations. By now, many powerful methods have been used to study the travelling wave solutions for reaction-diffusion systems, like phase plane techniques in [8], degree theory methods [9, 10], the shooting methods [11], the monotone iteration [1], and so on [12, 13].

Although the existence of travelling wave solutions to reaction-diffusion systems without delay has been widely studied [14–16], delayed reaction-diffusion systems which are more realistic in population dynamic and biological models are much more complicated than ordinary systems. Recently, a number of researchers have studied the existence of travelling wave solutions in delayed reaction-diffusion systems. In [17, 18], Wu and Zou considered delayed reaction-diffusion systems with reaction terms satisfying the so-called quasimonotonicity or exponential quasimonotonicity conditions. In [19], Huang and Zou employed the upper-lower solution technique and the monotone to study the existence of travelling wave solutions for a class of diffusion cooperative Lotka-Volterra systems with delays. In [20], Ma used Schauder's fixed point theorem to study the existence of travelling wave solutions to reaction-diffusion systems with quasimonotonicity reaction terms. In 2010, Gan et al. [21] investigated the existence of travelling wave solutions to the following three-species food chain models:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= D_1 \frac{\partial^2 u}{\partial x^2} + u(t,x)\left(r_1 - a_{11}u(t,x) - a_{12}v(t,x)\right), \\
\frac{\partial v}{\partial t} &= D_2 \frac{\partial^2 v}{\partial x^2} + v(t,x)\left(-r_2 + a_{21}v(t-\tau,x)\right. \\
&\qquad \left. - a_{22}v(t,x) - a_{23}w(t,x)\right), \\
\frac{\partial w}{\partial t} &= D_3 \frac{\partial^2 w}{\partial x^2} + w(t,x)\left(-r_3 + a_{32}v(t-\tau,x) - a_{33}w(t,x)\right).
\end{aligned} \tag{1}$$

They obtain the existence of traveling waves by using Schauder's fixed point theorem with the reaction term satisfying the partial quasimonotonicity conditions instead of the quasimonotonicity conditions.

Motivated by the above papers, in this paper, we investigate an n-species food chain model with spatial diffusion and time delays:

$$\frac{\partial u_i(t,x)}{\partial t} = D_i \frac{\partial^2 u_i(t,x)}{\partial x^2} - a_i u_i(t,x) + \sum_{j=1}^{n} w_{ij} f_j(u_j(t,x)) + \sum_{j=1}^{n} h_{ij} g_j(u_j(t-\tau_j(t),x)), \quad (2)$$

where D_i, a_i, w_{ij}, and h_{ij} are positive constants and $i = 1,2,\ldots,n$.

The main propose of this paper is to obtain the sufficient condition for the existence of the travelling wave solutions of system (2) by employing Schauder's fixed point theorem. This paper is organized as follows. In Section 2, some definition and lemmas are given. And the main results of the paper are established in the last section.

2. Preliminary

On substituting $u_i(t,x) = \phi_i(x+ct)$ and denoting the traveling wave coordinate $x+ct$ still by t, we derive from (2) that

$$D_i\phi_i''(t) - c\phi_i'(t) - a_i\phi_i(t) + \sum_{j=1}^{n} w_{ij} f_j(\phi_j(t)) + \sum_{j=1}^{n} h_{ij} g_j(\phi_j(t-\tau_j(t))) = 0. \quad (3)$$

If for some $c > 0$, system (2) has a solution defined on R^n satisfying

$$\lim_{t\to-\infty}\phi_i(t) = \phi_i^-, \qquad \lim_{t\to\infty}\phi_i(t) = \phi_i^+, \quad (4)$$

where $(\phi_1^-,\phi_2^-,\ldots,\phi_n^-)$ and $(\phi_1^+,\phi_2^+,\ldots,\phi_n^+)$ are steady states of (2), then $u_i(t,x) = \phi_i(x+ct)$ is called a traveling wave solution of (2) with speed c. Without loss of generality, we assume that $(\phi_1^-,\phi_2^-,\ldots,\phi_n^-) = (0,0,\ldots,0)$ and $(\phi_1^+,\phi_2^+,\ldots,\phi_n^+) = (k_1,k_2,\ldots,k_n)$.

Rewrite model (3) as

$$D_i\phi_i''(t) - c\phi_i'(t) - a_i\phi_i(t) + H_i(\phi_1,\phi_2,\ldots,\phi_n)(t) = 0, \quad (5)$$

where

$$H_i(\phi_1,\phi_2,\ldots,\phi_n)(t) = \sum_{j=1}^{n} w_{ij} f_j(\phi_j(t)) + \sum_{j=1}^{n} h_{ij} g_j(\phi_j(t-\tau_j(t))). \quad (6)$$

In the following parts, we assume that the nonlinear reaction terms $H_i(\phi_1,\phi_2,\ldots,\phi_n)(t) (i = 1,2,\ldots,n)$ satisfy the partial quasimonotonicity conditions (PQM):

(1) H_n is nondecreasing. That is, for $0 \le \underline{\phi}_j \le \phi_j \le \overline{\phi}_j \le k_j$ $(j = 1,2,\ldots,n)$,

$$H_n(\underline{\phi}_1,\underline{\phi}_2,\ldots,\underline{\phi}_n) \le H_n(\phi_1,\phi_2,\ldots,\phi_n) \le H_n(\overline{\phi}_1,\overline{\phi}_2,\ldots,\overline{\phi}_n). \quad (7)$$

(2) For $t \in R$, and $0 \le \underline{\phi}_j \le \phi_j \le \overline{\phi}_j \le k_j$ $(j = 1,2,\ldots,n)$,

$$\begin{aligned} &H_i(\underline{\phi}_1,\underline{\phi}_2,\ldots,\underline{\phi}_i,\overline{\phi}_{i+1},\underline{\phi}_{i+2},\ldots,\underline{\phi}_n)(t) \\ &\le H_i(\phi_1,\phi_2,\ldots,\phi_n)(t) \\ &\le H_i(\overline{\phi}_1,\overline{\phi}_2,\ldots,\overline{\phi}_i,\underline{\phi}_{i+1},\overline{\phi}_{i+2},\ldots,\overline{\phi}_n)(t), \\ &\qquad i = 1,2,\ldots,n-1. \end{aligned} \quad (8)$$

Let $C_k(R,R^n) = \{(\phi_1,\phi_2,\ldots,\phi_n) \in C(R,R^n) : 0 \le \phi_i \le k_i,\ i = 1,2,\ldots,n\}$, and for $(\phi_1,\phi_2,\ldots,\phi_n) \in C_k(R,R^n)$, define $F = (F_1,F_2,\ldots,F_n) : C_k(R,R^n) \to C_k(R,R^n)$ by

$$\begin{aligned} F_i(\phi_1,\phi_2,\ldots,\phi_n)(t) &= \frac{1}{D_i(\lambda_{i2}-\lambda_{i1})} \times \left[\int_{-\infty}^{t} e^{\lambda_{i1}(t-s)} H_i(\phi_1,\phi_2,\ldots,\phi_n)(s)\,ds \right. \\ &\quad \left. + \int_{t}^{\infty} e^{\lambda_{i2}(t-s)} H_i(\phi_1,\phi_2,\ldots,\phi_n)(s)\,ds\right], \end{aligned} \quad (9)$$

where

$$\lambda_{i1} = \frac{c-\sqrt{c^2+4D_ia_i}}{2D_i}, \qquad \lambda_{i2} = \frac{c+\sqrt{c^2+4D_ia_i}}{2D_i}. \quad (10)$$

It is easy to see that $F_i(\phi_1,\phi_2,\ldots,\phi_n)$ satisfy

$$D_iF_i''(\phi_1,\phi_2,\ldots,\phi_n) - cF_i'(\phi_1,\phi_2,\ldots,\phi_n) - a_iF_i(\phi_1,\phi_2,\ldots,\phi_n) + H_i(\phi_1,\phi_2,\ldots,\phi_n)(t) = 0. \quad (11)$$

Throughout this paper, we always assume that the following assumptions hold.

(H_1)

$$\sum_{j=1}^{n} w_{ij} f_j(0) + \sum_{j=1}^{n} h_{ij} g_j(0) = \sum_{j=1}^{n} w_{ij} f_j(k_j) + \sum_{j=1}^{n} h_{ij} g_j(k_j) = 0, \quad i = 1,2,\ldots,n. \quad (12)$$

(H_2) For any $0 \le \phi_i^1,\ \phi_i^2 \le k_i$ $(i = 1,2,\ldots,n)$, there exist some positive constants $L_i, K_i > 0$ such that

$$\begin{aligned} \left|f_i(\phi_i^1) - f_i(\phi_i^2)\right| &\le L_i\left|\phi_i^1 - \phi_i^2\right|, \\ \left|g_i(\phi_i^1) - g_i(\phi_i^2)\right| &\le K_i\left|\phi_i^1 - \phi_i^2\right|. \end{aligned} \quad (13)$$

Let $u > 0$ and equip $\Phi = (\phi_1, \phi_2, \dots, \phi_n) \in C(R, R^n)$ with the exponential decay norm defined by $|\Phi|_\mu = \sup_{t\in R} e^{-\mu|t|}|\Phi(t)|_{R^n}$. And define $B_\mu(R, R^n) = \{\Phi \in C(R, R^n) : |\Phi|_\mu < \infty\}$; then it is easy to see that $(B_\mu(R, R^n), |\cdot|_\mu)$ is a Banach space.

We will study traveling wave solution to system (2) in the following profile set:

$$\Gamma\left(\left(\underline{\phi}_1, \underline{\phi}_2, \dots, \underline{\phi}_n\right), \left(\overline{\phi}_1, \overline{\phi}_2, \dots, \overline{\phi}_n\right)\right) = \begin{cases} (1)\ \phi_n \text{ is nondecreasing in } R; \\ (2)\ \underline{\phi}_i \le \phi_i \le \overline{\phi}_i, \quad i = 1, 2, \dots, n. \end{cases} \tag{14}$$

It is easy to check that $\Gamma((\underline{\phi}_1, \underline{\phi}_2, \dots, \underline{\phi}_n), (\overline{\phi}_1, \overline{\phi}_2, \dots, \overline{\phi}_n))$ is nonempty, convex, closed, and bounded.

Definition 1. Function $\overline{\Phi} = (\overline{\phi}_1, \overline{\phi}_2, \dots, \overline{\phi}_n)$ is called upper solution of system (2) if $\overline{\Phi}$ is twice differentiable almost everywhere in R, and there hold

$$\begin{aligned} &D_i\overline{\phi}_i''(t) - c\overline{\phi}_i'(t) - a_i\overline{\phi}_i(t) \\ &\quad + H_i\left(\overline{\phi}_1, \overline{\phi}_2, \dots, \overline{\phi}_i, \underline{\phi}_{i+1}, \overline{\phi}_{i+2}, \dots, \overline{\phi}_n\right)(t) \\ &\le 0, \quad \text{a.e. in } R, \\ &D_n\overline{\phi}_n''(t) - c\overline{\phi}_n'(t) - a_i\overline{\phi}_n(t) \\ &\quad + H_n\left(\overline{\phi}_1, \overline{\phi}_2, \dots, \overline{\phi}_n\right) \le 0, \quad \text{a.e. in } R. \end{aligned} \tag{15}$$

Reversing the direction of above inequalities, we can get the lower solution.

In this paper, we assume that the upper-lower solutions of system (2) $\overline{\Phi} = (\overline{\phi}_1, \overline{\phi}_2, \dots, \overline{\phi}_n)$ and $\underline{\Phi} = (\underline{\phi}_1, \underline{\phi}_2, \dots, \underline{\phi}_n)$ satisfy

(P_1) $(0, 0, \dots, 0) \le (\underline{\phi}_1, \underline{\phi}_2, \dots, \underline{\phi}_n) \le (\overline{\phi}_1, \overline{\phi}_2, \dots, \overline{\phi}_n) \le (k_1, k_2, \dots, k_n)$, $t \in R$,

(P_2) $\lim_{t\to-\infty}(\underline{\phi}_1, \underline{\phi}_2, \dots, \underline{\phi}_n) = (0, 0, \dots, 0)$, and $\lim_{t\to\infty}(\overline{\phi}_1, \overline{\phi}_2, \dots, \overline{\phi}_n) = (k_1, k_2, \dots, k_n)$.

Lemma 2. *Assume that (PQM) holds; then one has*

(1) *$F_n(\phi_1, \phi_2, \dots, \phi_n)(t)$ is nondecreasing for $t \in R$.*

(2) *For $t \in R$, $i = 1, 2, \dots, n-1$ and $0 \le \underline{\phi}_j \le \overline{\phi}_j \le k_j$ $(j = 1, 2, \dots, n)$, there has*

$$\begin{aligned} &F_i\left(\underline{\phi}_i, \underline{\phi}_2, \dots, \underline{\phi}_i, \overline{\phi}_{i+1}, \underline{\phi}_{i+2}, \dots, \underline{\phi}_n\right)(t) \\ &\le F_i\left(\phi_1, \phi_2, \dots, \phi_n\right)(t) \\ &\le F_i\left(\overline{\phi}_i, \overline{\phi}_2, \dots, \overline{\phi}_i, \underline{\phi}_{i+1}, \overline{\phi}_{i+2}, \dots, \overline{\phi}_n\right)(t). \end{aligned} \tag{16}$$

Lemma 2 is easy to prove, so we omit it.

Lemma 3. *Assume that (H_2) holds; then $F_i(\phi_1, \phi_2, \dots, \phi_n)(t)$ is continuous with respect to the norm $|\cdot|$ in $B_\mu(R, R^n)$.*

Proof. For any fixed $\epsilon > 0$, choose $\delta < \epsilon/(\sum_{j=1}^n w_{ij}L_j + \sum_{j=1}^n h_{ij}K_j)$; a direct calculation shows that $|\phi_j^1 - \phi_j^2|_\mu < \delta$, $|\phi_j^1(t - \tau_j(t)) - \phi_j^2(t - \tau_j(t))| < \delta$; then there exists

$$\begin{aligned} &\left|H_i\left(\phi_1^1, \phi_2^1, \dots, \phi_n^1\right)(t) - H_i\left(\phi_1^2, \phi_2^2, \dots, \phi_n^2\right)(t)\right| e^{-\mu|t|} \\ &= \left|\sum_{j=1}^n w_{ij} f_j\left(\phi_j^1(t)\right) + \sum_{j=1}^n h_{ij} g_j\left(\phi_j\left(t - \tau_j(t)\right)\right) - \sum_{j=1}^n w_{ij} f_j\left(\phi_j^2(t)\right) - \sum_{j=1}^n h_{ij} g_j\left(\phi_j^2\left(t - \tau_j(t)\right)\right)\right| e^{-\mu|t|} \\ &\le \sum_{j=1}^n w_{ij}\left|f_j\left(\phi_j^1(t)\right) - f_j\left(\phi_j^2(t)\right)\right| e^{-\mu|t|} \\ &\quad + \sum_{j=1}^n h_{ij}\left|g_j\left(\phi_j^1\left(t - \tau_j(t)\right)\right) - g_j\left(\phi_j^2\left(t - \tau_j(t)\right)\right)\right| e^{-\mu|t|} \\ &\le \sum_{j=1}^n w_{ij}L_j\delta + \sum_{j=1}^n h_{ij}K_j\delta \le \epsilon. \end{aligned} \tag{17}$$

For $t > 0$, we can see that

$$\begin{aligned} &\left|F_i\left(\phi_1^1, \phi_2^1, \dots, \phi_n^1\right) - F_i\left(\phi_1^2, \phi_2^2, \dots, \phi_n^2\right)\right| e^{-\mu|t|} \\ &\le \frac{1}{D_i\left(\lambda_{i2} - \lambda_{i1}\right)} \\ &\quad \times \left[\int_{-\infty}^t e^{\lambda_{i1}(t-s)} \times \left|H_i\left(\phi_1^1, \phi_2^1, \dots, \phi_n^1\right)(s) - H_i\left(\phi_1^2, \phi_2^2, \dots, \phi_n^2\right)(s)\right| e^{-\mu|s|}e^{\mu|s|}ds \right. \\ &\quad \left. + \int_t^{+\infty} e^{\lambda_{i2}(t-s)} \times \left|H_i\left(\phi_1^1, \phi_2^1, \dots, \phi_n^1\right)(s) - H_i\left(\phi_1^2, \phi_2^2, \dots, \phi_n^2\right)(s)\right| \times e^{-\mu|s|}e^{\mu|s|}ds\right] e^{-\mu t} \\ &\le \frac{\epsilon}{D_i\left(\lambda_{i2} - \lambda_{i1}\right)}\left[\int_{-\infty}^t e^{\lambda_{i1}(t-s)}e^{\mu|s|}ds + \int_t^{+\infty} e^{\lambda_{i2}(t-s)}e^{\mu|s|}ds\right] e^{-\mu t} \end{aligned}$$

$$= \frac{\epsilon}{D_i(\lambda_{i2}-\lambda_{i1})}\left[\frac{2\mu}{\lambda_{i1}^2-\mu^2}e^{(\lambda_{i1}-\mu)t}+\frac{\lambda_{i2}-\lambda_{i1}}{(\mu-\lambda_{i1})(\lambda_{i2}-\mu)}\right] \le \frac{\epsilon}{D_i(\lambda_{i2}-\lambda_{i1})}\left[\frac{2\mu}{\lambda_{i1}^2-\mu^2}+\frac{\lambda_{i2}-\lambda_{i1}}{(\mu-\lambda_{i1})(\lambda_{i2}-\mu)}\right]. \quad (18)$$

For $t<0$, we have

$$\begin{aligned}&\left|F_i(\phi_1^1,\phi_2^1,\ldots,\phi_n^1)-F_i(\phi_1^2,\phi_2^2,\ldots,\phi_n^2)\right|e^{-\mu|t|}\\ &\le \frac{1}{D_i(\lambda_{i2}-\lambda_{i1})}\times\left[\int_{-\infty}^{t}e^{\lambda_{i1}(t-s)}\left|H_i(\phi_1^1,\phi_2^1,\ldots,\phi_n^1)(s)-H_i(\phi_1^2,\phi_2^2,\ldots,\phi_n^2)(s)\right|e^{-\mu|s|}e^{\mu|s|}ds\right.\\ &\quad\left.+\int_{t}^{+\infty}e^{\lambda_{i2}(t-s)}\left|H_i(\phi_1^1,\phi_2^1,\ldots,\phi_n^1)(s)-H_i(\phi_1^2,\phi_2^2,\ldots,\phi_n^2)(s)\right|\times e^{-\mu|s|}e^{\mu|s|}ds\right]e^{\mu t}\\ &\le \frac{\epsilon}{D_i(\lambda_{i2}-\lambda_{i1})}\left[\int_{-\infty}^{t}e^{\lambda_{i1}(t-s)}e^{\mu|s|}ds+\int_{t}^{+\infty}e^{\lambda_{i2}(t-s)}e^{\mu|s|}ds\right]e^{\mu t}\\ &\le \frac{\epsilon}{D_i(\lambda_{i2}-\lambda_{i1})}\left[\frac{2\mu}{\lambda_{i2}^2-\mu^2}-\frac{\lambda_{i2}-\lambda_{i1}}{(\mu+\lambda_{i1})(\lambda_{i2}+\mu)}\right].\end{aligned} \quad (19)$$

So, $F_i : B_\mu(R,R^n)\to B_\mu(R,R^n)$ is continuous with respect to the norm $|\cdot|$ in $B_\mu(R,R^n)$. □

Lemma 4. *Assume that* (H_1) *and* (PQM) *hold; then*

$$F\left(\Gamma\left((\underline{\phi}_1,\underline{\phi}_2,\ldots,\underline{\phi}_n),(\overline{\phi}_1,\overline{\phi}_2,\ldots,\overline{\phi}_n)\right)\right) \subset \Gamma\left((\underline{\phi}_1,\underline{\phi}_2,\ldots,\underline{\phi}_n),(\overline{\phi}_1,\overline{\phi}_2,\ldots,\overline{\phi}_n)\right). \quad (20)$$

Proof. According to Lemma 2, for any $(\phi_1,\phi_2,\ldots,\phi_n)$ with $(\underline{\phi}_1,\underline{\phi}_2,\ldots,\underline{\phi}_n)\le(\phi_1,\phi_2,\ldots,\phi_n)\le(\overline{\phi}_1,\overline{\phi}_2,\ldots,\overline{\phi}_n)$, there exists

$$\begin{aligned}&F_i\left(\underline{\phi}_i,\underline{\phi}_2,\ldots,\underline{\phi}_i,\overline{\phi}_{i+1},\underline{\phi}_{i+2},\ldots,\underline{\phi}_n\right)(t)\\ &\le F_i(\phi_1,\phi_2,\ldots,\phi_n)(t)\\ &\le F_i\left(\overline{\phi}_i,\overline{\phi}_2,\ldots,\overline{\phi}_i,\underline{\phi}_{i+1},\overline{\phi}_{i+2},\ldots,\overline{\phi}_n\right)(t);\end{aligned} \quad (21)$$

$$\begin{aligned}&F_n\left(\underline{\phi}_1,\underline{\phi}_2,\ldots,\underline{\phi}_n\right)\\ &\le F_n(\phi_1,\phi_2,\ldots,\phi_n)\le F_n\left(\overline{\phi}_1,\overline{\phi}_2,\ldots,\overline{\phi}_n\right).\end{aligned}$$

By the definition of upper-lower solutions, we have

$$\begin{aligned}&D_i\overline{\phi}_i''(t)-c\overline{\phi}_i'(t)-a_i\overline{\phi}_i(t)\\ &\quad+H_i\left(\overline{\phi}_i,\overline{\phi}_2,\ldots,\overline{\phi}_i,\underline{\phi}_{i+1},\overline{\phi}_{i+2},\ldots,\overline{\phi}_n\right)(t)\le 0.\end{aligned} \quad (22)$$

Choosing $(\phi_1,\phi_2,\ldots,\phi_n)=(\overline{\phi}_i,\overline{\phi}_2,\ldots,\overline{\phi}_i,\underline{\phi}_{i+1},\overline{\phi}_{i+2},\ldots,\overline{\phi}_n)$ in (9) and denoting $\widetilde{\phi}_i = F_i(\overline{\phi}_i,\overline{\phi}_2,\ldots,\overline{\phi}_i,\underline{\phi}_{i+1},\overline{\phi}_{i+2},\ldots,\overline{\phi}_n)$, we get

$$\begin{aligned}&D_i\widetilde{\phi}_i''(t)-c\widetilde{\phi}_i'(t)-a_i\widetilde{\phi}_i(t)\\ &\quad+H_i\left(\overline{\phi}_1,\overline{\phi}_2,\ldots,\overline{\phi}_i,\underline{\phi}_{i+1},\overline{\phi}_{i+2},\ldots,\overline{\phi}_n\right)(t)=0.\end{aligned} \quad (23)$$

Setting $w_i(t)=\widetilde{\phi}_i-\overline{\phi}_i$ and combining (22) and (23), we have

$$D_iw_i''-cw_i'-a_i\ge 0. \quad (24)$$

Repeating the proof of Lemma 3.3 in Wu and Zou [17], we obtain $w_i(t)\le 0$, which implies that $F_i(\phi_1,\phi_2,\ldots,\phi_n)\le\overline{\phi}_i$ $(i=1,2,\ldots,n-1)$.

Next, choosing $(\phi_1,\phi_2,\ldots,\phi_n)=(\overline{\phi}_i,\overline{\phi}_2,\ldots,\overline{\phi}_n)$ in (9), we can get $F_n(\phi_1,\phi_2,\ldots,\phi_n)\le\overline{\phi}_n$. Then, by a similar argument, we know that $F_i(\phi_1,\phi_2,\ldots,\phi_n)\ge\underline{\phi}_i$, $(i=1,2,\ldots,n-1)$, and $F_n(\phi_1,\phi_2,\ldots,\phi_n)\ge\underline{\phi}_n$; then

$$F\left(\Gamma\left((\underline{\phi}_1,\underline{\phi}_2,\ldots,\underline{\phi}_n),(\overline{\phi}_1,\overline{\phi}_2,\ldots,\overline{\phi}_n)\right)\right) \subset \Gamma\left((\underline{\phi}_1,\underline{\phi}_2,\ldots,\underline{\phi}_n),(\overline{\phi}_1,\overline{\phi}_2,\ldots,\overline{\phi}_n)\right). \quad (25)$$

This completes the proof. □

Lemma 5. *The above defined function F is compact.*

Proof. By the definition of F, we have

$$\begin{aligned}&F_i'(\phi_1,\phi_2,\ldots,\phi_n)(t)\\ &=\frac{\lambda_{i1}e^{\lambda_{i1}t}}{D_i(\lambda_{i2}-\lambda_{i1})}\int_{-\infty}^{t}e^{-\lambda_{i1}s}H_i(\phi_1,\phi_2,\ldots,\phi_n)(s)\,ds\\ &\quad+\frac{\lambda_{i2}e^{\lambda_{i2}t}}{D_i(\lambda_{i2}-\lambda_{i1})}\int_{t}^{\infty}e^{-\lambda_{i2}s}H_i(\phi_1,\phi_2,\ldots,\phi_n)(s)\,ds.\end{aligned} \quad (26)$$

Thus,

$$\begin{aligned}
&\left|F_i'(\phi_1,\phi_2,\dots,\phi_n)(t)\right|_\mu \\
&= \sup_{t\in R}\left|\frac{\lambda_{i1}e^{\lambda_{i1}t}}{D_i(\lambda_{i2}-\lambda_{i1})}\times\int_{-\infty}^{t}e^{-\lambda_{i1}s}H_i(\phi_1,\phi_2,\dots,\phi_n)(s)\,ds + \frac{\lambda_{i2}e^{\lambda_{i2}t}}{D_i(\lambda_{i2}-\lambda_{i1})}\times\int_t^{\infty}e^{-\lambda_{i2}s}H_i(\phi_1,\phi_2,\dots,\phi_n)(s)\,ds\right|e^{-\mu|t|} \\
&\le \frac{-\lambda_{i1}}{D_i(\lambda_{i2}-\lambda_{i1})}\times\sup_{t\in R}e^{\lambda_{i1}t-\mu|t|}\int_{-\infty}^{t}e^{-\lambda_{i1}s}\left|H_i(\phi_1,\phi_2,\dots,\phi_n)(s)\right|ds \\
&\quad+\frac{\lambda_{i2}}{D_i(\lambda_{i2}-\lambda_{i1})}\times\sup_{t\in R}e^{\lambda_{i2}t-\mu|t|}\int_t^{\infty}e^{-\lambda_{i2}s}\left|H_i(\phi_1^2,\phi_2^2,\dots,\phi_n^2)(s)\right|ds \\
&\le \frac{-\lambda_{i1}}{D_i(\lambda_{i2}-\lambda_{i1})}\left|H_i(\phi_1^2,\phi_2^2,\dots,\phi_n^2)(t)\right|_\mu\times\sup_{t\in R}e^{\lambda_{i1}t-\mu|t|}\int_{-\infty}^{t}e^{-\lambda_{i1}s}e^{\mu|s|}ds \\
&\quad+\frac{\lambda_{i2}}{D_i(\lambda_{i2}-\lambda_{i1})}\left|H_i(\phi_1^2,\phi_2^2,\dots,\phi_n^2)(t)\right|_\mu\times\sup_{t\in R}e^{\lambda_{i2}t-\mu|t|}\int_t^{\infty}e^{-\lambda_{i2}s}e^{\mu|s|}ds.
\end{aligned} \tag{27}$$

We will complete the proof by two cases as follows:

(i) case $t>0$

$$\begin{aligned}
&\left|F_i'(\phi_1,\phi_2,\dots,\phi_n)(t)\right|_\mu \\
&\le \frac{-\lambda_{i1}}{D_i(\lambda_{i2}-\lambda_{i1})}\times\sup_{t\in R}e^{(\lambda_{i1}-\mu)t}\left[\int_{-\infty}^{0}e^{-(\lambda_{i1}+\mu)s}ds+\int_0^t e^{(\mu-\lambda_{i1})s}ds\right]\times\left|H_i(\phi_1,\phi_2,\dots,\phi_n)(t)\right|_\mu \\
&\quad+\frac{\lambda_{i2}}{D_i(\lambda_{i2}-\lambda_{i1})}\times\sup_{t\in R}e^{(\lambda_{i2}-\mu)t}\int_t^{\infty}e^{(\mu-\lambda_{i2})s}ds\left|H_i(\phi_1,\phi_2,\dots,\phi_n)(t)\right|_\mu \\
&\le \frac{\lambda_{i1}}{D_i(\mu+\lambda_{i1})(\lambda_{i2}-\lambda_{i1})}\left|H_i(\phi_1,\phi_2,\dots,\phi_n)(t)\right|_\mu+\frac{\lambda_{i2}}{D_i(\lambda_{i2}-\mu)(\lambda_{i2}-\lambda_{i1})}\left|H_i(\phi_1,\phi_2,\dots,\phi_n)(t)\right|_\mu \\
&= \frac{1}{D_i(\lambda_{i2}-\lambda_{i1})}\left[\frac{\lambda_{i1}}{\mu+\lambda_{i1}}+\frac{\lambda_{i2}}{\lambda_{i2}-\mu}\right]\times\left|H_i(\phi_1,\phi_2,\dots,\phi_n)(t)\right|_\mu;
\end{aligned} \tag{28}$$

(ii) case $t<0$

$$\begin{aligned}
&\left|F_i'(\phi_1,\phi_2,\dots,\phi_n)(t)\right|_\mu \\
&\le \frac{-\lambda_{i1}}{D_i(\lambda_{i2}-\lambda_{i1})}\times\sup_{t\in R}e^{(\lambda_{i1}+\mu)t}\int_{-\infty}^{t}e^{-(\lambda_{i1}+\mu)s}ds\times\left|H_i(\phi_1,\phi_2,\dots,\phi_n)(t)\right|_\mu \\
&\quad+\frac{\lambda_{i2}}{D_i(\lambda_{i2}-\lambda_{i1})}\times\sup_{t\in R}e^{(\lambda_{i2}+\mu)t}\left[\int_t^{0}e^{(-\lambda_{i2}-\mu)s}ds+\int_0^{\infty}e^{(\mu-\lambda_{i2})s}ds\right]\times\left|H_i(\phi_1,\phi_2,\dots,\phi_n)(t)\right|_\mu \\
&\le \frac{\lambda_{i1}}{D_i(\mu+\lambda_{i1})(\lambda_{i2}-\lambda_{i1})}\left|H_i(\phi_1,\phi_2,\dots,\phi_n)(t)\right|_\mu+\frac{\lambda_{i2}}{D_i(\lambda_{i2}-\mu)(\lambda_{i2}-\lambda_{i1})}\left|H_i(\phi_1,\phi_2,\dots,\phi_n)(t)\right|_\mu \\
&= \frac{1}{D_i(\lambda_{i2}-\lambda_{i1})}\left[\frac{\lambda_{i1}}{\mu+\lambda_{i1}}+\frac{\lambda_{i2}}{\lambda_{i2}-\mu}\right]\times\left|H_i(\phi_1,\phi_2,\dots,\phi_n)(t)\right|_\mu.
\end{aligned} \tag{29}$$

According to the conditions in (PQM), we get that $|H_i(\phi_1,\phi_2,\dots,\phi_n)(t)|_\mu$ is bounded by a positive number. Therefore, $|F_i'(\phi_1,\phi_2,\dots,\phi_n)(t)|_\mu$ is bounded. The above estimate for F' shows that $F(\Gamma((\underline{\phi}_1,\underline{\phi}_2,\dots,\underline{\phi}_n),(\overline{\phi}_1,\overline{\phi}_2,\dots,\overline{\phi}_n)))$ is equicontinuous. It follows from Lemma 4 that $F(\Gamma((\underline{\phi}_1,\underline{\phi}_2,\dots,\underline{\phi}_n),(\overline{\phi}_1,\overline{\phi}_2,\dots,\overline{\phi}_n)))$ is uniformly bounded.

Next, we define

$$F^m(\phi_1,\phi_2,\dots,\phi_n)(t) = \begin{cases}(1)\ F(\phi_1,\phi_2,\dots,\phi_n)(t), & t\in[-m,m];\\ (2)\ F(\phi_1,\phi_2,\dots,\phi_n)(m), & t\in(m,\infty);\\ (3)\ F(\phi_1,\phi_2,\dots,\phi_n)(-m), & t\in(-\infty,-m).\end{cases} \tag{30}$$

Then, for each $m \geq 1$, $F^m(\phi_1, \phi_2, \ldots, \phi_n)(t)$ is also equicontinuous and uniformly bounded. In the interval $[-m, m]$, it follows from Ascoli-Arzela theorem that F^m is compact. On the other hand, $F^m \to F$ in $B_\mu(R, R^n)$ as $m \to \infty$, since

$$\begin{aligned}\sup_{t\in R} &\left|F^m(\phi_1, \phi_2, \ldots, \phi_n)(t) - F(\phi_1, \phi_2, \ldots, \phi_n)(t)\right| e^{-\mu|t|} \\ &= \sup_{t\in(-\infty,-m)\cup(m,\infty)} \left|F^m(\phi_1, \phi_2, \ldots, \phi_n)(t)\right. \\ &\qquad \left. -F(\phi_1, \phi_2, \ldots, \phi_n)(t)\right| e^{-\mu|t|} \\ &\leq 2Ke^{-\mu m} \longrightarrow 0, \quad m \longrightarrow \infty.\end{aligned} \tag{31}$$

By Proposition 2.12 in [22], we have that $F : \Gamma((\underline{\phi}_1, \underline{\phi}_2, \ldots, \underline{\phi}_n), (\overline{\phi}_1, \overline{\phi}_2, \ldots, \overline{\phi}_n)) \to \Gamma((\underline{\phi}_1, \underline{\phi}_2, \ldots, \underline{\phi}_n), (\overline{\phi}_1, \overline{\phi}_2, \ldots, \overline{\phi}_n))$ is compact. This completes the proof. □

3. Main Results

Theorem 6. *Assume that (H_1), (H_2), and (PQM) hold. Moreover, suppose that there is a pair of upper-lower solutions $\overline{\Phi} = (\overline{\phi}_1, \overline{\phi}_2, \ldots, \overline{\phi}_n)$ and $\underline{\Phi} = (\underline{\phi}_1, \underline{\phi}_2, \ldots, \underline{\phi}_n)$ for (2) satisfying (P_1) and (P_2). Then, system (2) has a travelling wave solution.*

Proof. Following Lemmas 2 to 5, we see that all the condition in Schauder's fixed point theorem hold. Then we know that there exists a fixed point $(\phi_1^*, \phi_2^*, \ldots, \phi_n^*)$ of F in $\Gamma((\underline{\phi}_1, \underline{\phi}_2, \ldots, \underline{\phi}_n), (\overline{\phi}_1, \overline{\phi}_2, \ldots, \overline{\phi}_n))$. Now we need to verify the asymptotic boundary conditions. By (P_2) we notice the fact that

$$\begin{aligned}(0, 0, \ldots, 0) &\leq \left(\underline{\phi}_1, \underline{\phi}_2, \ldots, \underline{\phi}_n\right) \leq (\phi_1^*, \phi_2^*, \ldots, \phi_n^*) \\ &\leq \left(\overline{\phi}_1, \overline{\phi}_2, \ldots, \overline{\phi}_n\right) \leq (k_1, k_2, \ldots, k_n);\end{aligned} \tag{32}$$

we get that

$$\begin{aligned}\lim_{t\to-\infty} (\phi_1^*, \phi_2^*, \ldots, \phi_n^*) &= (0, 0, \ldots, 0), \\ \lim_{t\to\infty} (\phi_1^*, \phi_2^*, \ldots, \phi_n^*) &= (k_1, k_2, \ldots, k_n).\end{aligned} \tag{33}$$

Therefore, the fixed point $(\phi_1^*, \phi_2^*, \ldots, \phi_n^*)$ satisfies the asymptotic boundary conditions. This completes the proof. □

4. Conclusion

In this paper, we studied an n-species food chain model with spatial diffusion and time delays. By using Schauder's fixed point theorem and cross-iteration methods, we reduced the existence of the travelling wave solutions to the existence of a pair of upper-lower solutions. Finally, we proved that the system (2) has a travelling wave solution. However, in order to investigate the specific form of the travelling wave solution of (2), we still have a lot of work to do in the future.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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