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Research Article

Homomorphisms between Algebras of Holomorphic Functions

Verónica Dimant,[1] Domingo García,[2] Manuel Maestre,[2] and Pablo Sevilla-Peris[3]

[1] *Departamento de Matemática, Universidad de San Andrés, Vito Dumas 284, Victoria, B1644BID Buenos Aires, Argentina*
[2] *Departamento de Análisis Matemático, Universidad de Valencia, Doctor Moliner 50, Burjasot, 46100 Valencia, Spain*
[3] *Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, 46022 Valencia, Spain*

Correspondence should be addressed to Pablo Sevilla-Peris; psevilla@mat.upv.es

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For two complex Banach spaces X and Y, in this paper, we study the generalized spectrum $\mathscr{M}_b(X,Y)$ of all nonzero algebra homomorphisms from $\mathscr{H}_b(X)$, the algebra of all bounded type entire functions on X, into $\mathscr{H}_b(Y)$. We endow $\mathscr{M}_b(X,Y)$ with a structure of Riemann domain over $\mathscr{L}(X^*,Y^*)$ whenever X is symmetrically regular. The size of the fibers is also studied. Following the philosophy of (Aron et al., 1991), this is a step to study the set $\mathscr{M}_{b,\infty}(X,B_Y)$ of all nonzero algebra homomorphisms from $\mathscr{H}_b(X)$ into $\mathscr{H}_\infty(B_Y)$ of bounded holomorphic functions on the open unit ball of Y and $\mathscr{M}_\infty(B_X,B_Y)$ of all nonzero algebra homomorphisms from $\mathscr{H}_\infty(B_X)$ into $\mathscr{H}_\infty(B_Y)$.

1. Introduction

The study of homomorphisms between topological algebras is one of the basic issues in this theory. Two are the main topological algebras that we come across when we deal with holomorphic functions on infinite dimensional spaces (see Section 2 for precise definitions): $\mathscr{H}_b(X)$, the holomorphic functions of bounded type (which is a Fréchet algebra), and $\mathscr{H}_\infty(B_X)$, the bounded holomorphic functions on the open unit ball (which is a Banach algebra). Here, as a first step in the study of the set of homomorphisms between $\mathscr{H}_\infty(B_X)$ spaces, we mainly focus on algebras of holomorphic functions of bounded type and homomorphisms between them; $L : \mathscr{H}_b(X) \to \mathscr{H}_b(Y)$ (i.e., continuous, linear, and multiplicative mappings). These were already considered in [1]. There the focus was to study the homomorphisms as "individuals," seeking properties of single ones. We have here a different interest: we treat them as a whole, considering the set

$$\begin{aligned}\mathscr{M}_b(X,Y) &= \mathscr{M}(\mathscr{H}_b(X),\mathscr{H}_b(Y)) \\ &= \{\Phi : \mathscr{H}_b(X) \longrightarrow \mathscr{H}_b(Y) \\ &\quad \text{algebra homomorphisms}\} \setminus \{0\}.\end{aligned} \tag{1}$$

We will call this set the *generalized spectrum* or simply the spectrum. Our main aim is to study $\mathscr{M}_b(X,Y)$ and to define on it a topological and a differential structure.

This problem has the same flavor as considering $\mathscr{M}_b(X)$, the spectrum of the algebra $\mathscr{H}_b(X)$ (i.e., the set of nonzero continuous, linear, and multiplicative $\Phi : \mathscr{H}_b(X) \to \mathbb{C}$). This was studied in [2, 3], where a structure of Riemannian manifold over the bidual X^{**} was defined on it (see also [4, Section 3.6] for a very neat and nice presentation and [5–7] for similar results). Our approach is very much indebted to that in [2] and we get up to some point analogous results, defining on $\mathscr{M}_b(X,Y)$ a Riemann structure over $\mathscr{L}(X^*,Y^*)$ (note that $X^{**} = \mathscr{L}(X^*,\mathbb{C})$). We will also be interested in the fibers of elements of $\mathscr{L}(X^*,Y^*)$.

The outline of the paper is the following. In Section 3, for two complex Banach spaces X and Y, we study the generalized spectrum $\mathscr{M}_b(X,Y)$ of all nonzero algebra homomorphisms from $\mathscr{H}_b(X)$ to $\mathscr{H}_b(Y)$. We endow it with a structure of Riemann domain over $\mathscr{L}(X^*,Y^*)$ whenever X is symmetrically regular. In Section 4, we focus on the sets (fibers) of elements in $\mathscr{M}_b(X,Y)$ that are projected on the same element u of $\mathscr{L}(X^*,Y^*)$. The size of these fibers is studied and we prove that they are big by showing that they contain big sets. Following the philosophy of [2], all about $\mathscr{M}_b(X,Y)$ is a step to study in Section 5 the spectrum

$\mathscr{M}_{b,\infty}(X, B_Y)$ of all nonzero algebra homomorphisms from $\mathscr{H}_b(X)$ to $\mathscr{H}_\infty(B_Y)$ of bounded holomorphic functions on the open unit ball of Y. Finally, in Section 6, we deal with the generalized spectrum $\mathscr{M}_\infty(B_X, B_Y)$ of all nonzero algebra homomorphisms from $\mathscr{H}_\infty(B_X)$ to $\mathscr{H}_\infty(B_Y)$.

2. Definitions and Preliminaries

Unless otherwise stated capital letters such as $X, Y, \ldots$ will denote complex Banach spaces. The dual will be denoted by X^* and the open ball of center x_0 and radius $r > 0$ by $B_X(x_0, r)$. If $x_0 = 0$ and $r = 1$ we just write B_X. The space of continuous, linear operators from X to Y will be denoted by $\mathscr{L}(X, Y)$; this is a Banach space with the norm $\|u\| = \sup_{x \in B_X} \|u(x)\|$. The adjoint operator of $u \in \mathscr{L}(X, Y)$ will be denoted by $u^* \in \mathscr{L}(Y^*, X^*)$.

We will denote the canonical inclusion of a space into its bidual by $J_X : X \hookrightarrow X^{**}$.

Given two complex locally convex spaces E and F, a mapping $P : E \to F$ is a *continuous m-homogeneous polynomial* if there is a continuous m-linear mapping $L : E \times \cdots \times E \to F$ such that

$$P(x) = L(x, \ldots, x) \quad \text{for every } x. \tag{2}$$

Throughout the paper every polynomial and multilinear mapping will be assumed to be continuous.

An m-linear mapping L is called *symmetric* if $L(x_{\sigma 1}, \ldots, x_{\sigma m}) = L(x_1, \ldots, x_m)$ for every permutation σ of $\{1, \ldots, m\}$. Each m-homogeneous polynomial has a unique symmetric mapping (which we denote by $\overset{\vee}{P}$) satisfying (2). If $P : X \to Y$ is a continuous m-homogeneous polynomial between Banach spaces, the following expressions define norms for m-linear mappings and for m-homogeneous polynomials, respectively:

$$\begin{gathered} \|L\| = \sup\left\{\left\|L(x_1, \ldots, x_m)\right\| : \left\|x_j\right\| \le 1, j = 1, \ldots, m\right\}, \\ \|P\| = \sup\{\|P(x)\| : \|x\| \le 1\}. \end{gathered} \tag{3}$$

The polarization formula gives [4, Corollary 1.8]

$$\|P\| \le \left\|\overset{\vee}{P}\right\| \le \frac{m^m}{m!}\|P\|. \tag{4}$$

Given an m-linear mapping L, the notation $L(x^{(k)}, y^{(m-k)})$ will mean that x is repeated k times and y is repeated $m-k$ times.

A mapping $f : U \subseteq E \to F$ is *holomorphic* on the open set U if for every x_0 in U, there exist $(P_m f(x_0))_m$, each one of them an m-homogeneous polynomial, such that the series

$$f(x) = \sum_{m=0}^{\infty} P_m f(x_0)(x - x_0) \tag{5}$$

converges uniformly in some neighborhood of x_0 contained in U. This is called the "Taylor series expansion" of f at x_0.

If E and F are Fréchet spaces, then $f : E \to F$ is *Gâteaux holomorphic* if for every $x, y \in E$ the function $\mathbb{C} \to F$, $t \mapsto f(x + ty)$ is holomorphic on some neighborhood of 0. It is known [4, page 152-153] that f is holomorphic if and only if it is Gâteaux holomorphic and continuous.

In the case that $f : X \to Y$, X and Y being complex Banach spaces, f is holomorphic on X if and only if it is Fréchet differentiable on X and in that case $P_1(x) = df(x)$ for every x. Also, by the Cauchy inequalities [4, Proposition 3.2] we have, for every m,

$$\sup_{x \in B_X(0,R)} \left\|P_m(x_0)(x)\right\| \le \sup_{x \in B_X(x_0,R)} \|f(x)\|. \tag{6}$$

A holomorphic function $f : E \to F$ is of *bounded type* if it sends bounded subsets of E to bounded sets of F. We denote by $\mathscr{H}_b(E, F)$ the space of holomorphic functions of bounded type from E to F. If $F = \mathbb{C}$ we simply write $\mathscr{H}_b(E)$. Given $U \subseteq E$ we write

$$\|f\|_U = \sup_{x \in U} |f(x)|. \tag{7}$$

With this notation for X a Banach space $\mathscr{H}_b(X)$ is a Fréchet algebra endowed with the topology τ_b of uniform convergence on the bounded sets, whose seminorms are (for $R > 0$)

$$q_R(f) = \|f\|_{B_X(0,R)} = \sup\{|f(x)| : \|x\| < R\}. \tag{8}$$

Let M be a differential manifold on a complex Banach space X and F a locally convex space. A mapping $f : M \to F$ is said to be holomorphic (of bounded type) if $f \circ \varphi^{-1} : \Omega \to F$ is holomorphic (of bounded type) for every chart (φ, Ω) of M.

Given $x \in X$, we write δ_x for the evaluation mapping at x; that is, $\delta_x(f) = f(x)$ for all holomorphic f.

Let $L : X \times X \to \mathbb{C}$ be a continuous bilinear form. Fix $x \in X$ and for $w \in X^{**}$ let (y_β) be a net in X weak-star convergent to w. Since $L(x, -) \in X^*$, then there exists $\lim_\beta L(x, y_\beta) := \overline{L}(x, w)$. Now, fix $w \in X^{**}$ and for $z \in X^{**}$ let (x_α) be a net in X weak-star convergent to z. Since $\overline{L}(-, w) \in X^*$, then there exists

$$\widetilde{L}(z, w) := \lim_\alpha \overline{L}(x_\alpha, w) = \lim_\alpha \lim_\beta L(x_\alpha, y_\beta). \tag{9}$$

Following this idea Aron and Berner showed in [8] that every function $f \in \mathscr{H}_b(X)$ admits an extension to the bidual, called the *Aron-Berner extension*, $\overline{f} \in \mathscr{H}_b(X^{**})$. By [9, Theorem 3], for every m-homogeneous polynomial P, we have $\|P\| = \|\overline{P}\|$.

A Banach space X is *symmetrically regular* if for all continuous symmetric bilinear form $L : X \times X \to \mathbb{C}$ it follows that

$$\widetilde{L}(z, w) = \lim_\beta \lim_\alpha L(x_\alpha, y_\beta). \tag{10}$$

We refer the reader to [4, 10, 11] for the general background on the theory of holomorphic functions on infinite dimensional spaces.

We are going to work with the set $\mathscr{M}_b(X, Y)$ of nonzero algebra homomorphisms between spaces of holomorphic functions of bounded type defined in (1). Observe that an idempotent element g in $\mathscr{H}_b(Y)$ satisfies that $g(Y)$ is a subset of $\{0, 1\}$ and $g(Y)$ is a connected set. Hence, either $g \equiv 0$ or $g \equiv 1$. So, for any $\Phi \in \mathscr{M}_b(X, Y)$, we should have $\Phi(\mathbf{1}_X) = \mathbf{1}_Y$.

3. The Differential Structure of $\mathscr{M}_b(X,Y)$

Our aim in this section is to endow $\mathscr{M}_b(X,Y)$ with a structure of Riemann domain over $\mathscr{L}(X^*,Y^*)$. On a first step, for each linear operator $u \in \mathscr{L}(X^*,Y^*)$, we define, inspired by [12, Lemma 1],

$$\begin{gathered}\bar{u} : \mathscr{H}_b(X) \longrightarrow \mathscr{H}_b(Y), \\ f \longmapsto \bar{f} \circ u^* \circ J_Y.\end{gathered} \tag{11}$$

It is plain that $\bar{u}$ is an algebra homomorphism and $\bar{u}|_{X^*} = u$. This defines a natural inclusion:

$$\begin{gathered}i : \mathscr{L}(X^*,Y^*) \longrightarrow \mathscr{M}_b(X,Y), \\ u \longmapsto \bar{u}.\end{gathered} \tag{12}$$

On the other hand there is also a projection:

$$\begin{gathered}\pi : \mathscr{M}_b(X,Y) \longrightarrow \mathscr{L}(X^*,Y^*), \\ \Phi \longmapsto [x^* \longmapsto d(\Phi(x^*))(0)].\end{gathered} \tag{13}$$

These clearly satisfy $\pi(i(u)) = u$ for every u.

Given $z \in X^{**}$ we consider the mapping $\tau_z^* : \mathscr{H}_b(X) \to \mathscr{H}_b(X)$ defined by $\tau_z^*(f)(x) = \bar{f}(J_X x + z)$. It is a well-known fact that fixed f in $\mathscr{H}_b(X)$ the mapping $T : X^{**} \to \mathscr{H}_b(X)$ given by $T(z) = \tau_z^*(f)$ is a holomorphic function of bounded type. See [4, Proposition 6.30] or more in general the proof of [3, Theorem 2.2].

With this, for each $\Phi \in \mathscr{M}_b(X,Y)$ and each $u \in \mathscr{L}(X^*,Y^*)$, we can define $\Phi^u \in \mathscr{M}_b(X,Y)$ by

$$\begin{aligned}\Phi^u(f)(y) &= \Phi\left(\tau^*_{u^* J_Y y}(f)\right)(y) \\ &= \Phi\left[x \longmapsto \bar{f}(J_X x + u^* J_Y y)\right](y),\end{aligned} \tag{14}$$

for all $f \in \mathscr{H}_b(X)$ and $y \in Y$. Let us see that Φ^u is well defined. To do that we need to check that the function

$$y \longmapsto \Phi\left(\tau^*_{u^* J_Y y}(f)\right)(y) \tag{15}$$

belongs to $\mathscr{H}_b(Y)$. We see first that it is holomorphic. The following function of two variables

$$\begin{gathered}Y \times Y \longrightarrow \mathbb{C}, \\ (y,z) \longmapsto \Phi\left(\tau^*_{u^* J_Y z}(f)\right)(y),\end{gathered} \tag{16}$$

is holomorphic on each variable. Then Hartogs' theorem gives that it is holomorphic and hence it is so when restricted to the diagonal $\{(y,y) : y \in Y\}$. This gives that (15) is holomorphic.

We see now that $\Phi^u(f)$ is of bounded type. Given $R > 0$ there exists $S = S(R) > 0$ such that

$$\|\Phi(g)\|_{B_Y(0,R)} \le \|g\|_{B_X(0,S)}, \tag{17}$$

for all $g \in \mathscr{H}_b(X)$. Hence

$$\begin{aligned}\sup_{\|y\|<R} &\left|\Phi\left(\tau^*_{u^* J_Y y}(f)\right)(y)\right| \\ &\le \sup_{\|z\|<R} \left\|\Phi\left(\tau^*_{u^* J_Y z}(f)\right)\right\|_{B_Y(0,R)} \\ &\le \sup_{\|z\|<R} \left\|\tau^*_{u^* J_Y z}(f)\right\|_{B_X(0,S)} \\ &\le \|f\|_{B_X(0,S+\|u\|R)} < \infty,\end{aligned} \tag{18}$$

since $\|J_X x + u^* J_Y z\| \le \|x\| + \|u^*\|\|z\| < S + \|u\|R$ for all $x \in B_X(0,S)$ and all $z \in B_Y(0,R)$.

On the other hand it is easy to see that Φ^u is an algebra homomorphism, which finally gives $\Phi^u \in \mathscr{M}_b(X,Y)$.

As a second step we show that

$$\pi(\Phi^u) = \pi(\Phi) + u. \tag{19}$$

Indeed, for each $x^* \in X^*$ and $y \in Y$, we have

$$\begin{aligned}\Phi^u(x^*)(y) &= \Phi\left[x \longmapsto \overline{x^*}(J_X x + u^* J_Y y)\right](y) \\ &= \Phi\left[x \longmapsto x^*(x) + u(x^*)(y)\right](y) \\ &= (\Phi(x^*) + u(x^*))(y).\end{aligned} \tag{20}$$

Thus,

$$\begin{aligned}\pi(\Phi^u)(x^*) &= d(\Phi^u(x^*))(0) \\ &= d(\Phi(x^*) + u(x^*))(0) \\ &= d(\Phi(x^*))(0) + u(x^*) \\ &= (\pi(\Phi) + u)(x^*).\end{aligned} \tag{21}$$

Finally, given $\Phi \in \mathscr{M}_b(X,Y)$ and $\varepsilon > 0$, we consider the set

$$V_{\Phi,\varepsilon} = \{\Phi^u : u \in \mathscr{L}(X^*,Y^*), \|u\| < \varepsilon\}. \tag{22}$$

Now we assume that X is symmetrically regular. We proceed as in the definition of the Riemann domain structure of the spectrum of $\mathscr{H}_b(X)$ and we omit the details (see [4, Section 6.3] for a complete and detailed explanation of the procedure). First of all, if $\Psi \in V_{\Phi,\varepsilon}$, then $\Psi = \Phi^u$, with $\|u\| < \varepsilon$. Hence, for any $v \in \mathscr{L}(X^*,Y^*)$,

$$\begin{aligned}\Psi^v(f)(y) &= (\Phi^u)^v(f)(y) = \Phi^u\left(\tau^*_{v^* J_Y y}(f)\right)(y) \\ &= \Phi\left(\tau^*_{u^* J_Y y} \circ \tau^*_{v^* J_Y y}(f)\right)(y) \\ &= \Phi\left(\tau^*_{(u^*+v^*) J_Y y}(f)\right)(y) = \Phi^{u+v}(f)(y).\end{aligned} \tag{23}$$

Therefore, for $\delta = \varepsilon - \|u\|$, we have $V_{\Psi,\delta} \subset V_{\Phi,\varepsilon}$ and $\{V_{\Phi,\varepsilon}\}_{\varepsilon>0}$ is a neighborhood basis at Φ.

Also, for $\Phi \ne \Psi \in \mathscr{M}_b(X,Y)$, we have that if $\pi(\Phi) = \pi(\Psi)$, then $V_{\Phi,\varepsilon} \cap V_{\Psi,\delta} = \emptyset$, for all $\varepsilon, \delta > 0$ and if $\pi(\Phi) \ne \pi(\Psi)$, then $V_{\Phi,\varepsilon} \cap V_{\Psi,\varepsilon} = \emptyset$ for $\varepsilon = \|\pi(\Phi) - \pi(\Psi)\|/2$. This gives that the topology generated by $\{V_{\Phi,\varepsilon}\}_{\varepsilon>0}$ is Hausdorff.

Let us note that for each Φ in $\mathscr{M}_b(X,Y)$ the subset $V_\Phi = \{\Phi^u : u \in \mathscr{L}(X^*,Y^*)\}$ is the connected component containing Φ.

Summing all this up we have proved the following result.

Proposition 1. *If X is a symmetrically regular Banach space and Y is any Banach space, $(\mathscr{M}_b(X,Y),\pi)$ is a Riemann domain over $\mathscr{L}(X^*,Y^*)$ and each connected component of $(\mathscr{M}_b(X,Y),\pi)$ is homeomorphic to $\mathscr{L}(X^*,Y^*)$.*

Our aim now is to show that each function $f \in \mathscr{H}_b(X)$ can be extended, in some sense, to a function on $\mathscr{M}_b(X,Y)$ of bounded type. We do it with the following sort of Gelfand transform:

$$\begin{gathered}\widehat{f} : \mathscr{M}_b(X,Y) \longrightarrow \mathscr{H}_b(Y),\\ \Phi \longmapsto \Phi(f),\end{gathered} \tag{24}$$

and showing that this, when restricted to each connected component, is holomorphic of bounded type. To do that we need the following lemma.

Lemma 2. *If X and Y are complex Banach spaces and G is an element of $\mathscr{H}_b(X\times Y)$, then the mapping g defined by $g(x)(y) = G(x,y)$ for $(x,y)\in X\times Y$ belongs to $\mathscr{H}_b(X,\mathscr{H}_b(Y))$. Conversely, given g in $\mathscr{H}_b(X,\mathscr{H}_b(Y))$ the mapping $G(x,y) = g(x)(y)$ belongs to $\mathscr{H}_b(X\times Y)$.*

Proof. Let G be in $\mathscr{H}_b(X\times Y)$, and let $\sum_{m=0}^{\infty} P_m f$ be the Taylor series expansion of G at $(0,0)$. We have

$$\begin{aligned}G(x,y) &= \sum_{m=0}^{\infty} P_m f(x,y)\\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}\overset{\vee}{P_m} f\left((x,0)^{(k)},(0,y)^{(m-k)}\right).\end{aligned} \tag{25}$$

Note that if we take $R,S>0$ by the polarization formula (4) and Cauchy inequalities (6), we have

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}\sup_{\|x\|\le R}\sup_{\|y\|\le S}\left\|\overset{\vee}{P_m} f\left((x,0)^{(k)},(0,y)^{(m-k)}\right)\right\|\\ &\quad\le \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}\sup_{\|x\|\le R}\sup_{\|y\|\le S}\frac{m^m}{m!}\|P_m f\|\,\|x\|^k\|y\|^{m-k}\\ &\quad\le \sum_{m=0}^{\infty}(e(R+S))^m\\ &\qquad\times\sup_{\|x\|\le 2e(R+S)}\sup_{\|y\|\le 2e(R+S)}\frac{1}{(2e(R+S))^m}|G(x,y)|\\ &\quad= 2\sup_{\|x\|\le 2e(R+S)}\sup_{\|y\|\le 2e(R+S)}|G(x,y)|<\infty.\end{aligned} \tag{26}$$

By the properties of convergent double series of nonnegative numbers we obtain that, for each fixed k, the series $Q_k(x)(y) := \sum_{m=k}^{\infty}\binom{m}{k}\overset{\vee}{P_m} f((x,0)^{(k)},(0,y)^{(m-k)})$ converges absolutely and uniformly on any product of balls in X and Y with finite radii. Hence $Q_k : X \to \mathscr{H}_b(Y)$ is a continuous k-homogeneous polynomial and actually $g = \sum_{k=0}^{\infty} Q_k$ is an entire function from X to $\mathscr{H}_b(Y)$ that, by above inequalities, is of bounded type.

Conversely, consider g in $\mathscr{H}_b(X,\mathscr{H}_b(Y))$ and define $G : X\times Y \to \mathbb{C}$ by $G(x,y) = g(x)(y)$. By definition, for each $x\in X$, $G(x,-)$ belongs to $\mathscr{H}_b(Y)$. If we fix now $y\in Y$, we have that δ_y is a continuous linear form on $\mathscr{H}_b(Y)$ and $G(x,y) = \delta_y(g(x))$, implying that $G(-,y)$ is the composition of holomorphic mappings. Thus $G(-,y)$ is holomorphic for every $y\in Y$. By Hartogs' theorem, $G\in\mathscr{H}(X\times Y)$. Finally, for fixed $R,S>0$ we have that $g(B_X(0,R))$ is a bounded subset of $\mathscr{H}_b(Y)$. Hence

$$\sup_{\|x\|\le R,\|y\|\le S}|G(x,y)| = \sup_{\|y\|\le S}\sup_{\|x\|\le R}|g(x)(y)|<\infty, \tag{27}$$

and G is bounded on the bounded subsets of $X\times Y$. □

Proposition 3. *Let X be a symmetrically regular Banach space and let Y be any Banach space. Given a function $f\in\mathscr{H}_b(X)$ consider its extension $\widehat{f} : \mathscr{M}_b(X,Y) \to \mathscr{H}_b(Y)$ defined in (24). We have that $\widehat{f}$ is a holomorphic function of bounded type. That is, $\widehat{f}\circ(\pi|_{V_\Phi})^{-1}\in\mathscr{H}_b(\mathscr{L}(X^*,Y^*),\mathscr{H}_b(Y))$ for every Φ.*

Proof. The point is to prove that the function

$$\begin{gathered}\mathscr{L}(X^*,Y^*) \longrightarrow \mathscr{H}_b(Y),\\ u \longmapsto \Phi^u(f)\end{gathered} \tag{28}$$

is holomorphic of bounded type. For that, we introduce an auxiliary mapping $M : \mathscr{L}(X^*,Y^*)\times Y^{**}\times Y \to \mathbb{C}$, defined by

$$M(u,z,y) = \Phi\left[x\longmapsto \overline{f}(x+u^*(z))\right](y). \tag{29}$$

As above, we only need to check that it is separately holomorphic to conclude that M is holomorphic. First we fix $u\in\mathscr{L}(X^*,Y^*)$ and $z\in Y^{**}$ and denote $M_{u,z}(y) := M(u,z,y)$. We have $M_{u,z} = \Phi(\tau^*_{u^*(z)}(f))$ and this belongs to $\mathscr{H}_b(Y)$. Now we fix z,y and take $M_{z,y}(u) := M(u,z,y) = \delta_y(\Phi(\tau^*_{u^*(z)}(f)))$. This mapping is holomorphic (of bounded type) since it is the composition of the linear mapping $\mathscr{L}(X^*,Y^*) \to X^{**}$ defined by $u\mapsto u^*(z)$ with the holomorphic mapping of bounded type:

$$\begin{gathered}X^{**}\longrightarrow\mathbb{C},\\ v\longmapsto(\delta_y\circ\Phi)(\tau^*_v(f)).\end{gathered} \tag{30}$$

Finally we fix u,y and denote $M_{u,y}(z) := M(u,z,y) = \delta_y(\Phi(\tau^*_{u^*(z)}(f)))$. Again, this is the composition of a linear mapping $Y^{**} \to X^{**}$ defined by $z\to u^*(z)$ with the same holomorphic mapping (30). We conclude that M is holomorphic.

Let now $R, S, T > 0$. Given $T > 0$ there exists $U > 0$ such that $\|\Phi(h)\|_{B_Y(0,T)} \leq \|h\|_{B_X(0,U)}$ for every $h \in \mathscr{H}_b(X)$. Hence

$$\begin{aligned}\sup_{\|u\|\leq R, \|z\|\leq S, \|y\|\leq T} & |M(u,z,y)| \\ &= \sup_{\|u\|\leq R} \sup_{\|z\|\leq S} \left\|\Phi\left(\tau^*_{u^*(z)}(f)\right)\right\|_{B_Y(0,T)} \\ &\leq \sup_{\|u\|\leq R} \sup_{\|z\|\leq S} \left\|\tau^*_{u^*(z)}(f)\right\|_{B_X(0,U)} \\ &\leq \|f\|_{B_X(0,U+RS)} < \infty,\end{aligned} \tag{31}$$

and M is of bounded type. Since the mapping $Y \to Y^{**} \times Y$ defined by $y \mapsto (J_Y y, y)$ is obviously holomorphic of bounded type we have that

$$G : \mathscr{L}(X^*, Y^*) \times Y \longrightarrow \mathbb{C} \tag{32}$$

defined by $G(u, y) = M(u, y, y)$ is holomorphic of bounded type. Now a direct application of Lemma 2 gives that the mapping $u \mapsto \Phi^u(f)$ belongs to $\mathscr{H}_b(\mathscr{L}(X^*, Y^*), \mathscr{H}_b(Y))$. □

The above proposition is related to the study of extension of functions of bounded type given in [13].

4. The Size of the Fibers of $\mathscr{M}_b(X,Y)$

We focus now on the sets of elements in $\mathscr{M}_b(X, Y)$ that are projected on the same element u of $\mathscr{L}(X^*, Y^*)$. This is called the "fiber" of u and is defined by

$$\mathscr{F}(u) = \{\Phi \in \mathscr{M}_b(X,Y) : \pi(\Phi) = u\}. \tag{33}$$

Our aim in this section is to find out how big these fibers can be. To begin with, each fixed $u \in \mathscr{L}(X^*, Y^*)$ defines a set

$$A_u = \{g \in \mathscr{H}_b(Y, X^{**}) : dg(0) = u^* \circ J_Y\}. \tag{34}$$

On the other hand every $g \in \mathscr{H}_b(Y, X^{**})$ defines a composition homomorphism $\Phi_g \in \mathscr{M}_b(X, Y)$ given by $\Phi_g(f) = \overline{f} \circ g$. This gives an inclusion

$$\begin{aligned} j : \mathscr{H}_b(Y, X^{**}) &\longrightarrow \mathscr{M}_b(X,Y), \\ g &\longmapsto \Phi_g, \end{aligned} \tag{35}$$

which maps the set A_u into the fiber $\mathscr{F}(u)$. Let us check that j is injective. Given $g_1, g_2 \in \mathscr{H}_b(Y, X^{**}) \setminus \{0\}$, $g_1 \neq g_2$, there exists $y_0 \in Y$ such that $g_1(y_0) \neq g_2(y_0)$ and since X^* separates points of X^{**} we can find $x_0^* \in X^*$ with $g_1(y_0)(x_0^*) \neq g_2(y_0)(x_0^*)$. Thus $\Phi_{g_1}(x_0^*)(y_0) \neq \Phi_{g_2}(x_0^*)(y_0)$ and this gives $\Phi_{g_1} \neq \Phi_{g_2}$.

There is also a projection

$$\begin{aligned} \zeta : \mathscr{M}_b(X,Y) &\longrightarrow \mathscr{H}_b(Y, X^{**}), \\ \Phi &\longmapsto [y \longmapsto (x^* \longmapsto \Phi(x^*)(y))]. \end{aligned} \tag{36}$$

The mapping $\zeta(\Phi)$ belongs to $\mathscr{H}_b(Y, X^{**})$ (and hence ζ is well defined). This follows from the fact that

$$\begin{aligned} Y \times X^* &\longrightarrow \mathbb{C}, \\ (y, x^*) &\longmapsto \Phi(x^*)(y) \end{aligned} \tag{37}$$

is holomorphic. Clearly, $(\zeta \circ j)(g) = g$. Also, note that $\zeta(\Phi)$ determines the values that takes Φ when restricted to X^*. This means that when finite type polynomials are dense in $\mathscr{H}_b(X)$, $\zeta(\Phi)$ determines Φ. So, the only homomorphisms in $\mathscr{M}_b(X, Y)$ are the Φ_g's and we have the following result, which is closely related to [13, Lemmas 4.5 and 4.6] and in [1, Theorem 21].

Proposition 4. *Let X and Y be Banach spaces. If finite type polynomials are dense in $\mathscr{H}_b(X)$ then for each $\Phi \in \mathscr{M}_b(X, Y)$ there exists $g \in \mathscr{H}_b(Y, X^{**})$ such that $\Phi = \Phi_g$. Also, $\mathscr{F}(u) = \{\Phi_g : g \in A_u\}$.*

Let us note that the mapping $j|_{A_u} : A_u \to \mathscr{F}(u)$ is actually injective and, by Proposition 4, if finite type polynomials are dense, surjective. This means that even in the case when finite type polynomials are dense in $\mathscr{H}_b(X)$ (i.e., the space $\mathscr{H}_b(X)$ is rather small), the fibers are thick. Let us see now that if this is not the case (i.e., when there is a polynomial in X that is not weakly continuous on bounded sets), this mapping is no longer surjective and we can find even more homomorphisms inside each fiber.

Proposition 5. *If X is symmetrically regular and there exists a polynomial on X not weakly continuous on bounded sets at a point x_0, then, for each $u \in \mathscr{L}(X^*, Y^*)$, there is $\Phi \in \mathscr{F}(u)$ such that $\Phi \neq \Phi_g$, for all $g \in A_u$.*

Proof. It is enough to prove the result for $F(0)$ because we can change fibers through the mapping

$$\begin{aligned} F(0) &\longrightarrow F(u), \\ \Phi &\longmapsto \Phi^u. \end{aligned} \tag{38}$$

Also, if $\Phi_g \in F(0)$, then $(\Phi_g)^u = \Phi_h$, with $h(y) = g(y) + u^* J_Y y$.

Let P be a polynomial that is not weakly continuous on bounded sets at x_0 and $\{x_\alpha\}$ a bounded net weakly convergent to x_0 such that $|P(x_\alpha) - P(x_0)| > 1$, for all α. For an ultrafilter $\mathscr{U}$ containing the sets $\{\alpha : \alpha \geq \alpha_0\}$, let Φ be given by

$$\Phi(f)(y) = \lim_{\mathscr{U}} f(x_\alpha), \quad \forall f \in \mathscr{H}_b(X), \ y \in Y. \tag{39}$$

Then Φ is a homomorphism in $\mathscr{M}_b(X, Y)$ (actually in $F(0)$), that is, not of composition type. Indeed, since $\Phi(f)$ is a constant function on Y it follows that $d(\Phi(f))(0) = 0$ for every f in $\mathscr{H}_b(X)$ and so $\Phi \in F(0)$. If $\Phi = \Phi_g$ for certain $g \in \mathscr{H}_b(Y, X^{**})$, then we have

$$x^*(x_0) = \lim_{\mathscr{U}} x^*(x_\alpha) = x^*(g(y)), \quad \forall x^* \in X^*, \ y \in Y. \tag{40}$$

This says that $g(y) = J_X x_0$, for all y. Hence,

$$\lim_{\mathscr{U}} P(x_\alpha) = \overline{P}(g(y)) = \overline{P}(J_X x_0) = P(x_0), \tag{41}$$

which is a contradiction. □

Something more can be said when there is a polynomial that is not weakly continuous on bounded sets. In this case, we can insert in each fiber a big set of homomorphisms that are not of composition type. We do it in detail in the fiber of 0.

First, note that if there is a polynomial not weakly continuous on bounded sets, then there is a homogeneous polynomial P which is not weakly continuous on bounded sets at 0. So, the Aron-Berner extension of P is not w^*-continuous at any point $x^{**} \in X^{**}$ [14, Corollary 2, Proposition 3] and [15, Proposition 1] (see also [16, Proof of Proposition 2.6]). Thus, we fix, for each $x^{**} \in X^{**}$ a bounded net $\{x_\alpha^{**}\}$ in X^{**} which w^*-converges to x^{**}, but $|\overline{P}(x_\alpha^{**}) - \overline{P}(x^{**})| > 1$, for all α. For each $x^{**} \in X^{**}$ we fix also an ultrafilter $\mathscr{U}$ containing the sets $\{\alpha : \alpha \geq \alpha_0\}$. Consider the set

$$A = \{g \in \mathscr{H}_b(Y, X^{**}) : g(0) = 0, dg(0) \equiv 0\} \tag{42}$$

and define the following mapping

$$\begin{aligned} A \times X^{**} &\longrightarrow F(0), \\ (g, x^{**}) &\longmapsto \Psi_{g,x^{**}}, \end{aligned} \tag{43}$$

where

$$\Psi_{g,x^{**}}(f)(y) = \lim_{\mathscr{U}} \overline{f}(x_\alpha^{**} + g(y)). \tag{44}$$

This mapping is well defined because $\Psi_{g,x^{**}}(x^*)(y) = \overline{x^*}(x_\alpha^{**} + g(y)) = x^{**}(x^*) + \overline{x^*}(g(y))$; then $\pi(\Psi_{g,x^{**}})(x^*) = d(\overline{x^*} \circ g)(0) = \overline{x^*} \circ dg(0) \equiv 0$. The mapping is also injective. Indeed, if $\Psi_{g,x^{**}} = \Psi_{h,z^{**}}$, then $\Psi_{g,x^{**}}(x^*)(y) = \Psi_{h,z^{**}}(x^*)(y)$, for all $x^* \in X^*$ and $y \in Y$. Then,

$$\begin{aligned} x^{**}(x^*) + \overline{x^*}(g(y)) &= z^{**}(x^*) + \overline{x^*}(h(y)), \\ &\forall x^* \in X^*, \quad y \in Y. \end{aligned} \tag{45}$$

Therefore, $h(y) = g(y) + x^{**} - z^{**}$, for all y and, evaluating at 0, we obtain that $x^{**} = z^{**}$ and, hence, $g = h$.

Note, also, that the homomorphisms $\Psi_{g,x^{**}}$ are not of composition type. Indeed, if $\Psi_{g,x^{**}} = \Phi_h$, for certain $g \in A$, $x^{**} \in X^{**}$, and $h \in \mathscr{H}_b(Y, X^{**})$, then, for all $x^* \in X^*$,

$$x^{**}(x^*) + \overline{x^*} \circ g = \Psi_{g,x^{**}}(x^*) = \Phi_h(x^*) = \overline{x^*} \circ h. \tag{46}$$

If this were the case we would have $h(y) = x^{**} + g(y)$, for all y. Now,

$$\begin{aligned} &\lim_{\mathscr{U}} \overline{P}(x_\alpha^{**} + g(y)) \\ &\quad = \Psi_{g,x^{**}}(P)(y) = \Phi_h(P)(y) = \overline{P}(x^{**} + g(y)). \end{aligned} \tag{47}$$

Evaluating at 0 leads to a contradiction.

Let us finish this analysis by studying Λ: the composition of the inclusion j defined in (35) with the inclusion of $\mathscr{M}_b(X, Y)$ into $\mathscr{L}(\mathscr{H}_b(X), \mathscr{H}_b(Y))$. Then the mapping Λ turns out to be holomorphic (endowing $\mathscr{L}(\mathscr{H}_b(X), \mathscr{H}_b(Y))$ with an appropriate topology). We prepare the proof of this fact with a lemma that is a variant of a classical Dunford result; see also [17, Theorem 3].

Lemma 6. *let $\Psi : E \to X$ be a Gâteaux holomorphic mapping from a complex Fréchet space E to a Banach space X such that $x^* \circ \Psi$ is holomorphic for every $x^* \in A$, where A is a norming subset of the closed unit ball of X^* (i.e., $\|x\| = \sup_{x^* \in A}\{|x^*(x)|\}$ for every $x \in X$). Then Ψ is holomorphic on E.*

Proof. Let K be a finite dimensional compact subset of E. By hypothesis, $\Psi(K)$ is a compact subset of X and hence it is bounded. Thus, there exists $M > 0$ such that

$$\begin{aligned} &\sup\{|x^* \circ \Psi(y)| : x^* \in A, y \in K\} \\ &\quad = \sup\{\|\Psi(y)\| : y \in K\} < M. \end{aligned} \tag{48}$$

Hence, the family of scalar valued holomorphic functions $\{x^* \circ \Psi : x^* \in A\}$ is bounded on the finite dimensional compact subsets of E. But as E is a Fréchet space, then it is also Baire, and by [11] this family is locally bounded: given $y_0 \in E$ there exists an open neighborhood V of y_0 such that

$$\begin{aligned} &\sup\{\|\Psi(y)\| : y \in V\} \\ &\quad = \sup\{|x^* \circ \Psi(y))| : x^* \in A, y \in V\} < \infty. \end{aligned} \tag{49}$$

We have obtained that the Gâteaux holomorphic function Ψ is locally bounded. Then it is holomorphic by [4, Proposition 3.7]. □

The following proposition gives that Λ is holomorphic. The proof may seem at some points similar to that of Proposition 3, but the fact that in the target space we have now a nonmetrizable locally convex topology makes the whole situation much more delicate. We are going to consider now in $\mathscr{L}(\mathscr{H}_b(X), \mathscr{H}_b(Y))$ the topology τ_β defined by the following fundamental system of seminorms:

$$q_{R,\mathscr{B}}(T) = \sup\{|T(f)(y)| : y \in Y, \|y\| \leq R, f \in \mathscr{B}\}, \tag{50}$$

for $T \in \mathscr{L}(\mathscr{H}_b(X), \mathscr{H}_b(Y))$, where $R > 0$ and $\mathscr{B}$ is a bounded subset of $\mathscr{H}_b(X)$.

Proposition 7. *The mapping*

$$\begin{aligned} \Lambda : \mathscr{H}_b(Y, X^{**}) &\longrightarrow \mathscr{L}(\mathscr{H}_b(X), \mathscr{H}_b(Y)), \\ g &\longmapsto \Lambda(g) = \Phi_g \end{aligned} \tag{51}$$

is injective and holomorphic if we consider in $\mathscr{L}(\mathscr{H}_b(X), \mathscr{H}_b(Y))$ the τ_β-topology.

Proof. Clearly Λ is well defined. Our first step is to prove that $\Lambda : \mathscr{H}_b(Y, X^{**}) \to \mathscr{L}_{\tau_\beta}(\mathscr{H}_b(X), \mathscr{H}_b(Y))$ is Gâteaux holomorphic. Let $g_1, g_2 \in \mathscr{H}_b(Y, X^{**})$, $f \in \mathscr{H}_b(X)$, $y \in Y$, and $t \in \mathbb{C}$. Here we have $\Lambda(g_1 + tg_2)(f)(y) = \overline{f}(g_1(y) + tg_2(y))$. If $\sum_{m=0}^{\infty} P_m f$ is the Taylor series expansion of f at 0 on X, then

$$\overline{f}(x^{**}) = \sum_{m=0}^{\infty} \overline{P_m f}(x^{**}), \tag{52}$$

for every $x^{**} \in X^{**}$ and the convergence is absolute and uniform on the bounded subsets of X^{**} [8, 9]. Thus

$$\begin{aligned}
&\Lambda(g_1 + tg_2)(f)(y) \\
&\quad = \sum_{m=0}^{\infty} \overline{P_m f}(g_1(y) + tg_2(y)) \\
&\quad = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \overset{\vee}{\overline{P_m f}} \left(g_1(y)^{(m-k)}, (tg_2(y))^{(k)}\right) \\
&\quad = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} t^k \overset{\vee}{\overline{P_m f}} \left(g_1(y)^{(m-k)}, g_2(y)^{(k)}\right) \\
&\quad = \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \binom{m}{k} \overset{\vee}{\overline{P_m f}} \left(g_1(y)^{(m-k)}, g_2(y)^{(k)}\right) t^k,
\end{aligned} \tag{53}$$

where our last step is simply formal. We are going to concentrate our effort now to show that this formal last equality holds in our setting. If we denote by $\overline{B_Y(0, R)}$ the closure of $B_Y(0, R)$, then $g_j(\overline{B_Y(0, R)})$ is a bounded subset of X^{**} for $j = 1, 2$. Thus there exists $M > 0$ such that $\|g_j(y)\| \le M$, for every y in $\overline{B_Y(0, R)}$ and $j = 1, 2$. We fix $s_0 > 1$ and we take $S > 0$ such that $(1 + s_0)eM < S$. We have

$$\begin{aligned}
&\sum_{m=0}^{\infty} \sum_{k=0}^{m} e^m \binom{m}{k} \left(\frac{M}{S}\right)^{m-k} \left(\frac{s_0 M}{S}\right)^k \\
&\quad = \sum_{m=0}^{\infty} \left(\frac{e(1 + s_0) M}{S}\right)^m \\
&\quad = \frac{S}{S - e(1 + s_0) M}.
\end{aligned} \tag{54}$$

Hence, by the properties of summability of double series of nonnegative numbers, the double series below is convergent in $\mathbb{R}$:

$$\sum_{k=0}^{\infty} \sum_{m=k}^{\infty} e^m \binom{m}{k} \left(\frac{M}{S}\right)^{m-k} \left(\frac{s_0 M}{S}\right)^k < \infty, \tag{55}$$

and its sum is again $S/(S - e(1 + s_0)M)$. On the other hand, by using the polarization formula (4), [9, Theorem 3], and Cauchy's inequalities (6) we get

$$\begin{aligned}
&\sup_{\|y\| \le R} \left| \overset{\vee}{\overline{P_m f}} \left(g_1(y)^{(m-k)}, g_2(y)^{(k)}\right) \right| \\
&\quad \le \sup_{\|y\| \le R} e^m \|P_m f\| \|g_1(y)\|^{m-k} \|g_2(y)\|^k \\
&\quad \le \left(\frac{eM}{S}\right)^m \|f\|_{B_X(0,S)}.
\end{aligned} \tag{56}$$

By applying now (55) we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \binom{m}{k} \sup_{\|y\| \le R} \left| \overset{\vee}{\overline{P_m f}} \left(g_1(y)^{(m-k)}, g_2(y)^{(k)}\right) \right| s_0^k \\
&\quad \le \frac{S}{S - e(1 + s_0) M} \|f\|_{B_X(0,S)}.
\end{aligned} \tag{57}$$

Since the function $y \in Y \to \overset{\vee}{\overline{P_m f}} (g_1(y)^{(m-k)}, g_2(y)^{(k)})$ is the composition of a continuous multilinear mapping and two holomorphic mappings of bounded type, it is holomorphic of bounded type on Y. By using (57), we obtain that the series

$$T_k(f) := \sum_{m=k}^{\infty} \overset{\vee}{\overline{P_m f}} \left(g_1(\cdot)^{(m-k)}, g_2(\cdot)^{(k)}\right) \tag{58}$$

τ_b-converges in $\mathscr{H}_b(Y)$; hence $T_k(f)$ belongs to $\mathscr{H}_b(Y)$ for every f in $\mathscr{H}_b(X)$. Actually, if we consider $T_k : \mathscr{H}_b(X) \to \mathscr{H}_b(Y)$, this is a linear operator. By (57), it is also continuous, since given $R > 0$ there exists $S > 0$ such that

$$\sup_{\|y\| \le R} |T_k(f)(y)| \le \frac{S}{S - e(1 + s_0) M} \|f\|_{B_X(0,S)}, \tag{59}$$

for every $f \in \mathscr{H}_b(X)$. Then

$$\begin{aligned}
q(T_k) &= \sup_{\|y\| \le R} \sup_{f \in \mathscr{B}} |T_k(f)(y)| \\
&\le \sum_{m=k}^{\infty} \binom{m}{k} \left(\frac{e(1 + M)}{S}\right)^m \sup_{f \in \mathscr{B}} \|f\|_{B_X(0,S)}.
\end{aligned} \tag{60}$$

We have, for $t \in \mathbb{C}$ with $|t| \le s_0$, again by (57),

$$\begin{aligned}
&\sup_{|t| \le s_0} \sum_{k=0}^{\infty} q(T_k) |t|^k \\
&\quad = \sum_{k=0}^{\infty} q(T_k) s_0^k \le \frac{S}{S - e(1 + s_0) M} \\
&\quad \times \sup_{f \in \mathscr{B}} \|f\|_{B_X(0,S)}.
\end{aligned} \tag{61}$$

As a consequence the series $\sum_{k=0}^{\infty} T_k t^k$ defined on $\mathbb{C}$ with values in $\mathscr{L}(\mathscr{H}_b(X), \mathscr{H}_b(Y))$, τ_β-converges uniformly on the compacts of $\mathbb{C}$. Hence it is an entire function. Since we have proved that all series involved converge absolutely, we can apply the reordering of absolutely convergent double series to conclude that the last formal equality of (53) actually holds and then

$$\Lambda(g_1 + tg_2)(f)(y) = \sum_{k=0}^{\infty} T_k t^k \tag{62}$$

is an entire function on $\mathbb{C}$ for every g_1, g_2. This gives that Λ is Gâteaux holomorphic.

We fix now $q = q_{R,\mathscr{B}}$, a continuous seminorm of the fundamental system defined in (50) and denote by Z_q the completion of the normed space $(\mathscr{L}(\mathscr{H}_b(X), \mathscr{H}_b(Y))/\operatorname{Ker} q, \widehat{q})$. Given $y \in Y$ and $f \in \mathscr{H}_b(X)$ we define the continuous linear functional

$$\delta_{f,y} : \mathscr{L}(\mathscr{H}_b(X), \mathscr{H}_b(Y)) \longrightarrow \mathbb{C} \tag{63}$$

by $\delta_{f,y}(T) = T(f)(y)$. Clearly the quotient mapping $\widehat{\delta}_{f,y} : Z_q \to \mathbb{C}$, defined by $\widehat{\delta}_{f,y}(\widehat{T}) = T(f)(y)$, belongs to Z_q^*. On the other hand the set $\{\widehat{\delta}_{f,y} : \|y\| \leq R, f \in \mathscr{B}\}$ is a norming subset of Z_q since

$$q(T) = \sup\left\{\left|\delta_{f,y}(T)\right| : y \in Y, \|y\| \leq R, f \in \mathscr{B}\right\}. \tag{64}$$

We can consider $\widehat{\Lambda} : \mathscr{H}_b(Y, X^{**}) \to Z_q$ that remains Gâteaux holomorphic. Thus $\delta_{f,y} \circ \Lambda = \widehat{\delta}_{f,y} \circ \widehat{\Lambda} : \mathscr{H}_b(Y, X^{**}) \to \mathbb{C}$ is Gâteaux holomorphic for every $y \in Y$ and every $f \in \mathscr{H}_b(X)$. If we show that it is continuous, then it will be holomorphic and, by Lemma 6, we will get that $\widehat{\Lambda} : \mathscr{H}_b(Y, X^{**}) \to Z_q$ is holomorphic for every seminorm q. Since $\mathscr{L}_{\tau_\beta}(\mathscr{H}_b(X), \mathscr{H}_b(Y))$ is a complete space, we can conclude that $\Lambda : \mathscr{H}_b(Y, X^{**}) \to \mathscr{L}_{\tau_\beta}(\mathscr{H}_b(X), \mathscr{H}_b(Y))$ is holomorphic.

Let $g \in \mathscr{H}_b(Y, X^{**})$. Now for fixed $f \in \mathscr{H}_b(X)$ and $y_0 \in Y$, we consider $R > \|y_0\|$ and we choose $S > 0$ such that

$$g(B_Y(0, R)) \subset B_{X^{**}}(0, S). \tag{65}$$

As $\overline{f}$ is uniformly continuous on bounded subsets of X^{**}, for a given $\varepsilon > 0$, there exists $0 < \delta < 1$ such that if $z_1, z_2 \in B_{X^{**}}(0, S+1)$ with $\|z_1 - z_2\| < \delta$, then

$$\left|\overline{f}(z_1) - \overline{f}(z_2)\right| < \varepsilon. \tag{66}$$

Let $h \in \mathscr{H}_b(Y, X^{**})$ with $\sup_{\|y\|<R}\|g(y) - h(y)\| < \delta$. Clearly $h(y_0) \in B_{X^{**}}(0, S+1)$. Thus

$$\left|\delta_{f,y_0} \circ \Lambda(g) - \delta_{f,y_0} \circ \Lambda(h)\right| = \left|\overline{f}(g(y_0)) - \overline{f}(h(y_0))\right| < \varepsilon. \tag{67}$$

This gives that $\delta_{f,y_0} \circ \Lambda$ is continuous for every y_0 and f and completes the proof. □

We recall that the τ_p topology on $\mathscr{L}(\mathscr{H}_b(X), \mathscr{H}_b(Y))$ is the topology of the pointwise convergence on the points of $\mathscr{H}_b(X)$.

Corollary 8. *The mapping*

$$\begin{aligned} j : \mathscr{H}_b(Y, X^{**}) &\longrightarrow \mathscr{M}_b(X, Y), \\ g &\longmapsto j(g) = \Phi_g \end{aligned} \tag{68}$$

is injective and holomorphic when $\mathscr{M}_b(X, Y)$ is endowed with the topology induced by τ_p.

5. Some Properties of $\mathscr{M}_{b,\infty}(X, B_Y)$

When Aron et al. undertook in [2] the study of $\mathscr{M}_b(X)$, the spectra of the Fréchet algebra of holomorphic functions of bounded type they explicitly stated that it was a step to study the spectra of the Banach algebra $\mathscr{H}_\infty(B_X)$ of bounded holomorphic functions on the open unit ball of X (endowed with the supremum norm). Following this philosophy we now study the spectrum consisting of nonzero continuous homomorphisms $\Phi : \mathscr{H}_b(X) \to \mathscr{H}_\infty(B_Y)$ that we will denote by $\mathscr{M}_{b,\infty}(X, B_Y)$.

As above we can define

$$\begin{aligned} \pi : \mathscr{M}_{b,\infty}(X, B_Y) &\longrightarrow \mathscr{L}(X^*, Y^*), \\ \Phi &\longmapsto [x^* \longmapsto d(\Phi(x^*))(0)]. \end{aligned} \tag{69}$$

Proposition 9. *If X is a symmetrically regular Banach space and Y is any Banach space, then $(\mathscr{M}_{b,\infty}(X, B_Y), \pi)$ is a Riemann domain over $\mathscr{L}(X^*, Y^*)$ and each connected component of $(\mathscr{M}_{b,\infty}(X, B_Y), \pi)$ is homeomorphic to $\mathscr{L}(X^*, Y^*)$.*

Proof. The proof follows almost word by word that of Proposition 1. The basis of neighborhoods of a point Φ in $\mathscr{M}_{b,\infty}(X, B_Y)$ is given by $V_{\Phi,\varepsilon} = \{\Phi^u : u \in \mathscr{L}(X^*, Y^*), \|u\| < \varepsilon\}$, for $\varepsilon > 0$, where

$$\begin{aligned} \Phi^u(f)(y) &= \Phi\left(\tau^*_{u^* J_Y y}(f)\right)(y) \\ &= \Phi\left[x \longmapsto \overline{f}(J_X x + u^* J_Y y)\right](y), \end{aligned} \tag{70}$$

for all $f \in \mathscr{H}_b(X)$ and $y \in B_Y$. The fact that $\Phi^u(f)$ is in $\mathscr{H}_\infty(B_Y)$ follows from a similar argument to that in (18) taking $R = 1$. □

Now, as in (24), we can define a Gelfand transform of $f \in \mathscr{H}_b(X)$ by

$$\begin{aligned} \widehat{f} : \mathscr{M}_{b,\infty}(X, B_Y) &\longrightarrow \mathscr{H}_\infty(B_Y), \\ \Phi &\longmapsto \widehat{f}(\Phi) = \Phi(f), \end{aligned} \tag{71}$$

and we can see that this is a holomorphic extension of f to $\mathscr{M}_{b,\infty}(X, B_Y)$.

Proposition 10. *Let X be a symmetrically regular Banach space and let Y be any Banach space. Given a function $f \in \mathscr{H}_b(X)$ consider its extension $\widehat{f}$ defined in (71). Then the restriction of $\widehat{f}$ to each connected component of $\mathscr{M}_{b,\infty}(X, B_Y)$ is a holomorphic function of bounded type.*

Proof. Like in Proposition 3 it is enough to prove that for each fixed Φ in $\mathscr{M}_{b,\infty}(X, B_Y)$, the mapping $T : \mathscr{L}(X^*, Y^*) \to \mathscr{H}_\infty(B_Y)$ defined as $T(u) = \Phi^u(f)$ for u in $\mathscr{L}(X^*, Y^*)$ is holomorphic of bounded type.

First we check that it is uniformly continuous on any bounded subset of $\mathscr{L}(X^*, Y^*)$. By definition, there exists $R > 0$ such that

$$\|\Phi(h)\| = \sup_{\|z\|<1} |\Phi(h)(z)| \leq \sup_{\|x\|<R} |h(x)|, \quad (72)$$

for every h in $\mathscr{H}_b(X)$. Let $M > 0$. Since $\overline{f}$ is uniformly continuous on $B_{X^{**}}(0, R+M)$, given $\varepsilon > 0$ there exists $\delta > 0$ such that if z_1, z_2 are in $B_{X^{**}}(0, R+M)$ with $\|z_1 - z_2\| < \delta$, then $|\overline{f}(z_1) - \overline{f}(z_2)| < \varepsilon$. Consider now u_1, u_2 in $\mathscr{L}(X^*, Y^*)$ with $\|u_j\| < M$ for $j = 1, 2$ and $\|u_1 - u_2\| < \delta$. We have

$$\begin{aligned}
&\|T(u_1) - T(u_2)\| \\
&\quad = \sup_{\|y\|<1} \left|\Phi^{u_1}(f)(y) - \Phi^{u_2}(f)(y)\right| \\
&\quad = \sup_{\|y\|<1} \left|\Phi\left(\tau^*_{u_1^* J_Y y}(f) - \tau^*_{u_2^* J_Y y}(f)\right)(y)\right| \\
&\quad \leq \sup_{\|y\|<1} \sup_{\|z\|<1} \left|\Phi\left(\tau^*_{u_1^* J_Y y}(f) - \tau^*_{u_2^* J_Y y}(f)\right)(z)\right| \\
&\quad \leq \sup_{\|y\|<1} \sup_{\|x\|<R} \left|\left(\tau^*_{u_1^* J_Y y}(f) - \tau^*_{u_2^* J_Y y}(f)\right)(x)\right| \\
&\quad = \sup_{\|y\|<1} \sup_{\|x\|<R} \left|\overline{f}(J_X x + u_1^* J_Y y) - \overline{f}(J_X x + u_2^* J_Y y)\right| < \varepsilon.
\end{aligned} \quad (73)$$

The last inequality holds because $J_X x + u_j^* J_Y y \in B_{X^{**}}(0, R + M)$ for $j = 1, 2$ and $\|J_X x + u_1^* J_Y y - (J_X x + u_2^* J_Y y)\| \leq \|u_1 - u_2\| < \delta$.

Now we prove that T is Gâteaux holomorphic. Let us denote by $\sum_{m=0}^{\infty} P_m f(x)$ the Taylor series expansion of f at a point x in X. Consider u_1, u_2 in $\mathscr{L}(X^*, Y^*)$. Since X is symmetrically regular we have that

$$\begin{aligned}
&\overline{f}(J_X x + u_1^*(z) + \lambda u_2^*(z)) \\
&\quad = \sum_{m=0}^{\infty} \overline{P_m f(x)}(u_1^*(z) + \lambda u_2^*(z)) \\
&\quad = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \overline{\overset{\vee}{P_m f(x)}}\left((u_1^*(z))^{(m-k)}, (u_2^*(z))^{(k)}\right) \lambda^k,
\end{aligned} \quad (74)$$

for every z in Y^{**} and λ in $\mathbb{C}$. Now we fix z and take $R, M > 1$ with $\|x\| \leq R$ and $|\lambda| \leq M$. We have, by using the polarization constants and Cauchy inequalities, that

$$\begin{aligned}
&\sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \left|\overline{\overset{\vee}{P_m f(x)}}\left((u_1^*(z))^{(m-k)}, (u_2^*(z))^{(k)}\right)\right| |\lambda|^k \\
&\quad \leq \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} e^m \|P_m f(x)\|_{B_X} \times \|u_1^*\|^{m-k} \|u_2^*\|^k \|v\|^m M^k \\
&\quad = \left(eM\|v\|\left(\|u_1^*\| + \|u_2^*\|\right)\right)^m \times \|P_m f(x)\|_{B_X} = \frac{1}{2^m} \|P_m f(x)\|_{B_X(0,2K)} \\
&\quad \leq \frac{1}{2^m} \|f(x)\|_{B_X(0,R+2K)},
\end{aligned} \quad (75)$$

where $K = eM\|v\|(\|u_1^*\| + \|u_2^*\|)$. As a consequence

$$\begin{aligned}
&\sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \sup_{\|x\|\leq R} \left|\overline{\overset{\vee}{P_m f(x)}}\left((u_1^*(z))^{(m-k)}, (u_2^*(z))^{(k)}\right)\lambda^k\right| \\
&\quad \leq \sum_{m=0}^{\infty} \frac{1}{2^m} \|f(x)\|_{B_X(0,R+2K)} \\
&\quad = 2\|f(x)\|_{B_X(0,R+2K)}.
\end{aligned} \quad (76)$$

Hence we have that the function $g_{z,k}(x) = \sum_{k=m}^{\infty} \binom{m}{k} \overline{\overset{\vee}{P_m f(z)}}((u_1^*(z))^{(m-k)}, (u_2^*(z))^{(k)})$ is well defined and actually belongs to $\mathscr{H}_b(X)$ for every nonnegative integer k and every z in Y^{**}. Moreover, the equality

$$\overline{f}(J_X x + u_1^*(z) + \lambda u_2^*(z)) = \sum_{k=0}^{\infty} g_{z,k}(x) \lambda^k \quad (77)$$

holds for every x in X, z in Y^{**} and λ in $\mathbb{C}$. Actually, (76) implies also that the series $\sum_{k=0}^{\infty} g_{z,k}(\cdot)\lambda^k$ converges in $\mathscr{H}_b(X)$ for each z in Y^{**}. Hence,

$$\Phi\left[x \longmapsto \overline{f}(J_X x + u_1^*(z) + \lambda u_2^*(z))\right] = \sum_{k=0}^{\infty} \Phi(g_{z,k}) \lambda^k, \quad (78)$$

with $\Phi(g_{z,k})$ in $\mathscr{H}_\infty(B_Y)$ for every z and k and the series converges in $\mathscr{H}_\infty(B_Y)$ for each λ in $\mathbb{C}$. Now, if we take $z = y \in B_Y$, we have

$$T(u_1 + \lambda u_2)(y) = \sum_{k=0}^{\infty} \Phi(g_{y,k})(y) \lambda^k. \quad (79)$$

To finish we just have to observe that the function $y \mapsto \Phi(g_{y,k})(y)$ belongs to $\mathscr{H}_\infty(B_Y)$ for every k. This is an immediate consequence of the fact that $\Phi(g_{y,k})(y) = G(y, y)$, where G is the function

$$\begin{aligned}
G : Y^{**} \times B_Y &\longrightarrow \mathbb{C}, \\
(z, y) &\longmapsto G(z, y) = \Phi(g_{z,k})(y),
\end{aligned} \quad (80)$$

which clearly is separately holomorphic. Thus, by Hartogs' theorem, G is holomorphic. □

Proceeding now as in Section 4 we are going to see how we can insert big sets into the fibers of $\mathcal{M}_{b,\infty}(X, B_Y)$. Indeed, given $g \in \mathcal{H}_\infty(B_Y, X^{**})$ we consider $\Phi_g \in \mathcal{M}_{b,\infty}(X, B_Y)$ given by $\Phi_g(f) = \overline{f} \circ g$. This is well defined, since $g(B_Y)$ is bounded in X^{**} and $\overline{f} \in \mathcal{H}_b(X^{**})$. This again gives an inclusion

$$j : \mathcal{H}_\infty(B_Y, X^{**}) \longrightarrow \mathcal{M}_{b,\infty}(X, B_Y), \quad g \longmapsto \Phi_g \tag{81}$$

that for each $u \in \mathcal{L}(X^*, Y^*)$ maps the set

$$\{g \in \mathcal{H}_\infty(B_Y, X^{**}) : dg(0) = u^* \circ J_Y\} \tag{82}$$

into the fiber of u. Also in this case we have a projection

$$\zeta : \mathcal{M}_{b,\infty}(X, B_Y) \longrightarrow \mathcal{H}_\infty(B_Y, X^{**}), \quad \Phi \longmapsto [y \longmapsto (x^* \longmapsto \Phi(x^*)(y))]. \tag{83}$$

Again like in Section 4 the map $[y \in B_Y \mapsto (x^* \mapsto \Phi(x^*)(y))]$ is holomorphic. The fact that it is also bounded follows immediately from the fact that B_{X^*} is a bounded subset in $\mathcal{H}_b(X)$ and then $\{\Phi(x^*) : x^* \in B_{X^*}\}$ is bounded in $\mathcal{H}_\infty(B_Y)$.

Like in the case of $\mathcal{M}_b(X, Y)$ we have that if finite type polynomials are dense in $\mathcal{H}_b(X)$ then for each $\Phi \in \mathcal{M}_{b,\infty}(X, B_Y)$ there exists $g \in \mathcal{H}_\infty(B_Y, X^{**})$ such that $\Phi = \Phi_g$.

On the other hand for every nontrivial Banach space Y, there exists an element in $\mathcal{M}_{b,\infty}(\ell_2, B_Y)$ that does not belong to $\{\Phi_g : g \in \mathcal{H}_\infty(B_Y, \ell_2)\}$. Indeed, in [18] it is shown that there exists a continuous homomorphism $\varphi : \mathcal{H}_b(\ell_2) \to \mathbb{C}$ such that $\varphi(Q) = 1$, where $Q = \sum_{n=1}^{\infty} x_n^2$, and vanishes on all continuous homogeneous polynomial on ℓ_2 of odd degree. We define $\varphi \otimes \mathbf{1}_Y$ as $\varphi \otimes \mathbf{1}_Y(f) = \phi(f)\mathbf{1}_Y$ for $f \in \mathcal{H}_b(\ell_2)$. Clearly $\varphi \otimes \mathbf{1}_Y$ is in $\mathcal{M}_{b,\infty}(\ell_2, B_Y)$. Observe that if $\varphi \otimes \mathbf{1}_Y = \Phi_g$ for a certain $g \in \mathcal{H}_\infty(B_Y, \ell_2)$, then

$$0 = \varphi(x^*) = (\varphi \otimes \mathbf{1}_Y)(y)(x^*) = g(y)(x^*), \tag{84}$$

for every $y \in B_Y$ and every $x^* \in (\ell_2)^* = \ell_2$, and we have obtained that $g = 0$. Hence $\Phi_g(f)(y) = \overline{f} \circ g(y) = \overline{f} \circ 0(y) = f(0)$ for every $y \in B_Y$ and every $f \in \mathcal{H}_b(\ell_2)$. In particular, $1 = \varphi(Q) = Q(0) = 0$, a contradiction.

6. Homomorphisms between Algebras of Bounded Functions

In this last section we introduce, for two given X and Y complex Banach spaces,

$$\begin{aligned}\mathcal{M}_\infty(B_X, B_Y) &= \mathcal{M}(\mathcal{H}_\infty(B_X), \mathcal{H}_\infty(B_Y)) \\ &= \{\Phi : \mathcal{H}_\infty(B_X) \longrightarrow \mathcal{H}_\infty(B_Y) \\ &\quad \text{algebra homomorphisms}\} \setminus \{0\}.\end{aligned} \tag{85}$$

In this case, no Riemann manifold structure is known, but on the other hand we are going to show that some results very close to the Gelfand transform of the elements of a uniform algebra can be obtained.

The space $\mathcal{H}_\infty(B_Y)$ is a dual space. That was proved by Mujica in [19], where he found a topological predual $\mathcal{G}_\infty(B_Y)$ that is a subspace of $\mathcal{H}_\infty(B_Y)^*$ such that $\varphi \in \mathcal{G}_\infty(B_Y)$ if and only if the restriction of φ to B_Y is continuous when endowed with the compact open topology τ_0. Let us observe that $\{\delta_y : y \in B_Y\}$ is a subset of $\mathcal{G}_\infty(B_Y)$. We denote by w^* the weak-star topology $w(\mathcal{H}_\infty(B_Y), \mathcal{G}_\infty(B_Y))$.

Theorem 11. *$\mathcal{M}_\infty(B_X, B_Y)$ is a τ_p-compact subset of $\mathcal{L}(\mathcal{H}_\infty(B_X), (\mathcal{H}_\infty(B_Y), w^*))$.*

Proof. An application of the Banach Steinhaus theorem yields that $\mathcal{L}(\mathcal{H}_\infty(B_X), \mathcal{H}_\infty(B_Y))$ and $\mathcal{L}(\mathcal{H}_\infty(B_X), (\mathcal{H}_\infty(B_Y), w^*))$ coincide as sets.

By Alaoglu-Bourbaki theorem any bounded subset of $(\mathcal{H}_\infty(B_Y), w^*)$ is weak-star relatively compact. Thus, by [20, § 39.4(5)] the space $\mathcal{L}(\mathcal{H}_\infty(B_X), (\mathcal{H}_\infty(B_Y), w^*))$ has the property that every equicontinuous subset of it is relatively τ_p-compact. As the spectrum $\mathcal{M}_\infty(B_X, B_Y)$ is equicontinuous, we obtain that it is relatively τ_p-compact. Now we check that the spectrum is τ_p-closed. Take T in the τ_p closure of $\mathcal{M}_\infty(B_X, B_Y)$ and let (Φ_α) be a net in the spectrum τ_p convergent to T. Then, given f, g in $\mathcal{H}_\infty(B_X)$ we have that $\Phi_\alpha(f)$, $\Phi_\alpha(g)$, and $\Phi_\alpha(fg) = \Phi_\alpha(f)\Phi_\alpha(g)$ w^*-converge to $T(f)$, $T(g)$, and $T(fg)$, respectively. But the w^* topology in $\mathcal{H}_\infty(B_Y)$ coincides with the topology of uniform convergence on the compact subsets of $\mathcal{G}_\infty(B_Y)$. On the other hand in [19, 2.1 Theorem] it is proved that the mapping

$$g_{B_Y} : B_Y \longrightarrow \mathcal{G}_\infty(B_Y) \tag{86}$$

defined by $g_{B_Y}(y) = \delta_y$ is holomorphic. Hence if K is a compact subset of B_Y, then $\{\delta_y : y \in K\}$ is a compact subset of $\mathcal{G}_\infty(B_Y)$ and we obtain that $\Phi_\alpha(f)$ converges uniformly to $T(f)$ on K. Also $\Phi_\alpha(g)$ converges uniformly to $T(g)$ on K. As a consequence the net $\Phi_\alpha(f)\Phi_\alpha(g)$ converges to $T(f)T(g)$ on the compact open topology of B_Y. This implies, by the definition of $\mathcal{G}_\infty(B_Y)$, that $\varphi(\Phi_\alpha(f)\Phi_\alpha(g))$ converges to $\varphi(T(f)T(g))$ for every φ in $\mathcal{G}_\infty(B_Y)$. In other words the net $\Phi_\alpha(f)\Phi_\alpha(g)$ w^*-converges to $T(f)T(g)$ and we have

$$T(fg) = T(f)T(g), \tag{87}$$

for every f, g in $\mathcal{H}_\infty(B_X)$. Thus T belongs to $\mathcal{M}_\infty(B_X, B_Y)$ and the conclusion follows. □

In our setting, taking $\mathcal{M}_\infty(B_X, B_Y)$ as a τ_p-compact set, we are going to extend the concept of Gelfand transform of an element of a Banach algebra in the following way. Given $f \in \mathcal{H}_\infty(B_X)$, define

$$\widehat{f} : \mathcal{M}_\infty(B_X, B_Y) \longrightarrow \mathcal{H}_\infty(B_Y) \tag{88}$$

by $\widehat{f}(\Phi) = \Phi(f)$.

Proposition 12. *The mapping*

$$\widehat{\ } : \mathcal{H}_\infty(B_X) \longrightarrow C(\mathcal{M}_\infty(B_X, B_Y), \mathcal{H}_\infty(B_Y)), \quad f \longmapsto \widehat{f} \tag{89}$$

is an isometry of algebras.

Proof. Clearly it is an algebra homomorphism and

$$\left\|\widehat{f}\right\| = \sup_{\Phi\in\mathscr{M}_\infty(B_X,B_Y)} \left\|\Phi\left(f\right)\right\| \leq \left\|f\right\|, \tag{90}$$

for every f in $\mathscr{H}_\infty(B_X)$. For a fixed f, given $\varepsilon > 0$ there exists $x_0 \in B_X$ such that $|f(x_0)| > \|f\| - \varepsilon$. Let $1 < R < 1/\|x_0\|$ and consider any $y_0^* \in X^*$ with $\|y_0^*\| = 1$ that attains its norm at a certain $y_0 \in Y$ with $\|y_0\| = 1$. Let $g : B_Y \to B_X$ be defined by $g(y) = y_0^*(y)Rx_0$. We have that Φ_g belongs to $\mathscr{M}_\infty(B_X, B_Y)$ and

$$\begin{aligned}\left\|\widehat{f}\right\| &\geq \left\|\Phi_g\left(f\right)\right\| \geq \left|f\circ g\left(\frac{y_0}{R}\right)\right| \\ &= \left|f\left(\frac{y_0^*\left(y_0\right)}{R}Rx_0\right)\right| = \left|f\left(x_0\right)\right| > \left\|f\right\| - \varepsilon.\end{aligned} \tag{91}$$

Hence the inequality $\|\widehat{f}\| \geq \|f\|$ also holds. □

The scalar spectrum of $\mathscr{H}_\infty(B_X)$, denoted by $\mathscr{M}_\infty(B_X)$, can be considered as a subset of $\mathscr{M}_\infty(B_X, B_Y)$ by associating to each φ in $\mathscr{M}_\infty(B_X)$ the homomorphism $\varphi\otimes\mathbf{1}_Y$ that belongs to $\mathscr{M}_\infty(B_X, B_Y)$. In that way the new Gelfand transform $\widehat{f}$ of each f in $\mathscr{H}_\infty(B_X)$ can be considered and extension of the classical $\widehat{f}$ by the equality

$$\widehat{f}\left(\varphi\right)\mathbf{1}_Y = \widehat{f}\left(\varphi\otimes\mathbf{1}_Y\right). \tag{92}$$

It is natural to consider again $\mathscr{M}_b(X, Y)$ and $\mathscr{M}_{b,\infty}(X, B_Y)$ under the light of Proposition 12. We cannot give to any of these two spectra a topology that makes them compact sets. But in the case of $\mathscr{M}_{b,\infty}(X, B_Y)$ it can be endowed with a topology in such a way that this set is a countable union of compact sets. Indeed, as above, an application of the Banach Steinhaus theorem, now for Fréchet spaces, yields that $\mathscr{L}(\mathscr{H}_b(X), \mathscr{H}_\infty(B_Y))$ and $\mathscr{L}(\mathscr{H}_b(X), (\mathscr{H}_\infty(B_Y), w^*))$ coincide as sets and, also as above, any equicontinuous subset of $\mathscr{L}(\mathscr{H}_b(X), (\mathscr{H}_\infty(B_Y), w^*))$ is relatively τ_p-compact. Given $R > 0$, we denote

$$\begin{aligned}\mathscr{M}_{b,\infty}(X, B_Y)_R = \{&\Phi \in \mathscr{M}_{b,\infty}\left(X, B_Y\right) : \left\|\Phi\left(f\right)\right\| \\ &\leq \left\|f\right\|_{B_X(0,R)}\ \forall f \in \mathscr{H}_b\left(X\right)\}.\end{aligned} \tag{93}$$

Then, $\mathscr{M}_{b,\infty}(X, B_Y)_R$ is an equicontinuous set, and a similar argument of one given in the proof of Proposition 12 implies that $\mathscr{M}_{b,\infty}(X, B_Y)_R$ is a τ_p compact set. Finally $\mathscr{M}_{b,\infty}(X, B_Y) = \bigcup_{n=1}^{\infty} \mathscr{M}_{b,\infty}(X, B_Y)_n$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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