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Research Article

Exact Finite Difference Scheme and Nonstandard Finite Difference Scheme for Burgers and Burgers-Fisher Equations

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We present finite difference schemes for Burgers equation and Burgers-Fisher equation. A new version of exact finite difference scheme for Burgers equation and Burgers-Fisher equation is proposed using the solitary wave solution. Then nonstandard finite difference schemes are constructed to solve two equations. Numerical experiments are presented to verify the accuracy and efficiency of such NSFD schemes.

1. Introduction

During the last few decades, nonlinear diffusion equation (1)

$$u_t + \alpha u u_x - u_{xx} = f(u, x, t) \tag{1}$$

has played an important role in nonlinear physics. Recently, it also began to become important in various other fields of science, for example, biology, chemistry, and economics [1–3].

When $f(u, x, t) = 0$, (1) is reduced to the famous Burgers equation (2)

$$u_t = u_{xx} - \alpha u u_x. \tag{2}$$

This equation is the simplest equation combining both nonlinear propagation effects and diffusive effects [3]. It has been used in many fields especially for describing wave processes in acoustics and hydrodynamics [2]. Researchers have devoted a lot of efforts to studying the solutions of this equation [1–6]. A. van Niekerk and F. D. van Niekerk [4] applied Galerkin methods to the nonlinear Burgers equation and obtained implicit and explicit algorithms using different higher order rational basis functions. Hon and Mao [5] applied the multiquadric as a spatial approximation scheme for solving the nonlinear Burgers equation. Biazar and Aminikhah [6] considered the variational iteration method to solve nonlinear Burgers equation.

If we take $f(u, x, t) = u(1 - u)$, (1) becomes the Burgers-Fisher equation (3)

$$u_t + \alpha u u_x - u_{xx} = u(1 - u). \tag{3}$$

Burgers-Fisher equation is very important in fluid dynamic model. There have been extensive studies and applications of this model. A nonstandard finite difference scheme for the Burgers-Fisher equation was given by Mickens and Gumel [7]. In [8], Kaya and El-Sayed constructed a numerical simulation and explicit solutions of the generalized Burgers-Fisher equation. Ismail et al. [9] obtained the approximate solutions for the Burgers-Huxley and Burgers-Fisher equations by using the Adomian decomposition method. Wazwaz [10] presented the tanh method for generalized forms of nonlinear heat conduction and Burgers-Fisher equations. Javidi and Golbabai [11, 12] studied spectral collocation method and spectral domain decomposition method for the solution of the generalized Burgers-Fisher equation. Numerical solution of Burgers-Fisher equation is presented based on the cubic B-spline quasi-interpolation by Zhu and Kang [13]. Kocacoban et al. [14] solved Burgers-Fisher equation by using a different numerical approach that shows rather rapid convergence than other methods.

Among various techniques for solving partial differential equations, the nonstandard finite difference (NSFD) schemes have been proved to be one of the most efficient approaches in recent years [15, 16]. Exact finite difference scheme [17–22] is a special NSFD method. The exact discretization method was first discussed by Potts [23] in 1982. Potts considered the question that whether a linear ordinary difference equation that has the same general solution with the given linear ordinary differential equation (ODE) can be determined. A detailed description of subsequent developments can be found in Agarwal's book [24]. In this book, Agarwal said that any ODE has the exact discretization if its solution exists. More importantly, studies have shown that this statement is also true for partial differential equations [20].

The exact discretization is very important for the construction of new numerical algorithms. Mickens et al. [17] considered a second-order, linear equation $(d^2x/dt^2) + a(t)(dx/dt) + b(t)x = f(t)$ with constant coefficients and gave an exact finite difference scheme of the equation. Rucker [18] constructed an exact finite difference for a nonlinear PDE having linear advection and an odd-cubic reaction term $u_t + au_x = \lambda_1 u - \lambda_2 u^3$. Roeger and Mickens [19] gave NSFD schemes that provide exact numerical methods for a first-order differential equation having three distinct fixed points. And they also constructed a nonexact NSFD scheme preserving the critical properties of the original differential equation. Then Roeger [20] studied a two-dimensional linear system with constant coefficients and constructed exact finite-difference scheme for the system. Roeger [21] raised an exact nonstandard finite-difference methods for a linear system with a certain coefficient matrix. Cieśliński [22] discussed the exact finite difference scheme of classical harmonic oscillator equation and its various extensions cases.

The objective of this paper is twofold. The first objective is to consider the Burgers and Burgers-Fisher equations

$$u_t + uu_x - u_{xx} = 0, \tag{4}$$

$$u_t + uu_x - u_{xx} = u(1-u), \tag{5}$$

with the finite difference schemes. We obtain the exact finite difference schemes based on the solitary wave solutions of two equations. The other objective is to construct new NSFD schemes for solving Burgers equation (4) and Burgers-Fisher equation (5). The NSFD method of Burgers equation (4) and Burgers-Fisher equations (5) is constructed using a method generated by the work of Mickens et al. [17, 19, 25–29] and Roeger and Mickens [19–21]. In numerical simulation, we compare our scheme with Adomian decomposition method (ADM) [9, 30]. It is shown that ADM will have to consume more computations for derivative and integral when aiming to achieve the same accuracy with our method. And we also compare the numerical solution with the exact solitary wave solution. The numerical solutions meet the properties that the "physically" relevant solutions have.

The present paper is built up as follows. In the next section, we begin with proposing the exact difference scheme for the Burgers equation (4) and Burgers-Fisher equation (5). Then we give nonstandard finite difference schemes for two equations in Section 3. Numerical experiments are then presented in the final section, showing that our proposed approach is efficient and accurate.

2. Exact Finite Difference Scheme

In this section, we illustrate the exact finite difference schemes for Burgers equation (4) and Burgers-Fisher equation (5).

2.1. Exact Finite Difference Scheme for Burgers Equation. The exact solitary wave solution to (4) is given by [1]

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left[-\frac{1}{4}\left(x-\frac{t}{2}\right)\right] = \frac{1}{1+e^{(1/2)(x-(t/2))}}. \tag{6}$$

Pay attention to the solitary wave solution, $0 \le u(x,t) \le 1$. If we choose $\Delta t = 2h$, then it can easily obtain $u(x+h,t) = u(x,t-\Delta t)$ and the following equations:

$$\begin{aligned} \frac{1}{u(x,t)} &= 1 + e^{(1/2)(x-(1/2)t)}, \\ \frac{1}{u(x+h,t)} &= 1 + e^{(1/2)(x+h-(1/2)t)}, \\ \frac{1}{u(x-h,t)} &= 1 + e^{(1/2)(x-h-(1/2)t)}. \end{aligned} \tag{7}$$

According (7), we can write

$$\begin{aligned} \frac{1}{u(x,t)} - \frac{1}{u(x+h,t)} &= e^{(1/2)(x-(1/2)t)}\left(1-e^{(1/2)h}\right) \\ &= \left(1-\frac{1}{u(x,t)}\right)\left(e^{(1/2)h}-1\right), \\ \frac{1}{u(x,t)} - \frac{1}{u(x-h,t)} &= e^{(1/2)(x-(1/2)t)}\left(1-e^{-(1/2)h}\right) \\ &= \left(\frac{1}{u(x,t)}-1\right)\left(1-e^{-(1/2)h}\right). \end{aligned} \tag{8}$$

Let the step functions are $\psi_1 = (1-e^{-(1/2)h})/(1/2)$, $\psi_2 = (e^{(1/2)h}-1)/(1/2)$, $\phi_1 = (1-e^{-(1/4)\Delta t})/(1/4)$ and $\phi_2 = (e^{(1/4)\Delta t}-1)/(1/4)$, so $\phi_1 = 2\psi_1$, and $\phi_2 = 2\psi_2$. Thus, we can have the forward and backward difference quotients with the special stepsize functions:

$$\begin{aligned} \partial u &= \frac{u(x+h,t)-u(x,t)}{\psi_2} = \frac{1}{2}u(x+h,t)(u(x,t)-1), \\ \bar{\partial} u &= \frac{u(x,t)-u(x-h,t)}{\psi_1} = \frac{1}{2}u(x-h,t)(u(x,t)-1). \end{aligned} \tag{9}$$

If we select $u_{xx} = \partial\bar{\partial}u$, then using the first equation of (9) we can get

$$\begin{aligned}
\partial\bar{\partial}u &= \frac{((u(x+h,t)-u(x,t))/\psi_2)-((u(x,t)-u(x-h,t))/\psi_2)}{\psi_1} \\
&= \frac{u(x+h,t)(u(x,t)-1)-u(x,t)(u(x-h,t)-1)}{2\psi_1} \\
&= \frac{u(x,t)(u(x+h,t)-u(x-h,t))+u(x,t)-u(x+h,t)}{2\psi_1} \\
&= \frac{u(x,t)(u(x+h,t)-u(x-h,t))}{2\psi_1}+\frac{u(x,t)-u(x+h,t)}{2\psi_1} \\
&= u(x,t)\frac{u(x+h,t)-u(x-h,t)}{2\psi_1}+\frac{u(x,t)-u(x,t-\Delta t)}{\phi_1}.
\end{aligned} \tag{10}$$

When we choose $u_{xx} = \bar{\partial}\partial u$, using the second equation of (9), we can receive

$$\begin{aligned}
\bar{\partial}\partial u &= \frac{((u(x+h,t)-u(x,t))/\psi_1)-((u(x,t)-u(x-h,t))/\psi_1)}{\psi_2} \\
&= \frac{u(x,t)(u(x+h,t)-1)-u(x-h,t)(u(x,t)-1)}{2\psi_2} \\
&= \frac{u(x,t)(u(x+h,t)-u(x-h,t))+u(x-h,t)-u(x,t)}{2\psi_2} \\
&= \frac{u(x,t)(u(x+h,t)-u(x-h,t))}{2\psi_2}+\frac{u(x-h,t)-u(x,t)}{2\psi_2} \\
&= u(x,t)\frac{u(x+h,t)-u(x-h,t)}{2\psi_2}+\frac{u(x,t+\Delta t)-u(x,t)}{\phi_2}.
\end{aligned} \tag{11}$$

Based upon the solitary wave solution (6), we write U_j^n as

$$U_j^n = u(x_j, t_n) = \frac{1}{1+e^{(1/2)(x_j-(t_n/2))}}. \tag{12}$$

Then we can write an implicit exact finite difference scheme according to (10) as

$$\frac{U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1}}{\psi_2\psi_1} = U_j^{n+1}\frac{U_{j+1}^{n+1}-U_{j-1}^{n+1}}{2\psi_1}+\frac{U_j^{n+1}-U_j^n}{\phi_1}. \tag{13}$$

And we can also obtain an explicit exact finite difference scheme based on (11) as

$$\frac{U_{j+1}^{n}-2U_j^{n}+U_{j-1}^{n}}{\psi_2\psi_1} = U_j^{n}\frac{U_{j+1}^{n}-U_{j-1}^{n}}{2\psi_2}+\frac{U_j^{n+1}-U_j^n}{\phi_2}. \tag{14}$$

Thus the stepsize functions depend on h and Δt. Then we can obtain the following theorem.

Theorem 1. *An implicit exact finite difference scheme and an explicit exact finite difference scheme for Burgers equation* (4) *are given by* (13) *and* (14)*, respectively. The stepsize satisfies $2h = \Delta t$, and the stepsize functions satisfy*

$$\begin{aligned}
\psi_1 &= \frac{\left(1-e^{-(1/2)h}\right)}{(1/2)}, & \psi_2 &= \frac{\left(e^{(1/2)h}-1\right)}{(1/2)}, \\
\phi_1 &= \frac{\left(1-e^{-(1/4)\Delta t}\right)}{(1/4)}, & \phi_2 &= \frac{\left(e^{(1/4)\Delta t}-1\right)}{(1/4)}.
\end{aligned} \tag{15}$$

2.2. Exact Finite Difference Scheme for Burgers-Fisher Equation. In this section, we will obtain the exact finite difference scheme for Burgers-Fisher equation (5). For Burgers-Fisher equation (5), the exact solitary wave solution is

$$u(x,t) = \frac{1}{1+e^{(1/2)(x-(5t/2))}}. \tag{16}$$

The exact solution (16) to (5) satisfies $0 \le u(x,0) \le 1$.

On the basis of the solitary wave solution (16), set $\Delta t = (2/5)h$, so $u(x+h,t) = u(x,t-\Delta t)$ holds. Thus we can have

$$\begin{aligned}
\frac{1}{u(x,t)} &= 1+e^{(1/2)(x-(5/2)t)}, \\
\frac{1}{u(x+h,t)} &= 1+e^{(1/2)(x+h-(5/2)t)}, \\
\frac{1}{u(x-h,t)} &= 1+e^{(1/2)(x-h-(5/2)t)}.
\end{aligned} \tag{17}$$

According to (17), we can write

$$\begin{aligned}
\frac{1}{u(x,t)}-\frac{1}{u(x+h,t)} &= e^{(1/2)(x-(5/2)t)}\left(1-e^{(1/2)h}\right) \\
&= \left(1-\frac{1}{u(x,t)}\right)\left(e^{(1/2)h}-1\right), \\
\frac{1}{u(x,t)}-\frac{1}{u(x-h,t)} &= e^{(1/2)(x-(5/2)t)}\left(1-e^{-(1/2)h}\right) \\
&= \left(\frac{1}{u(x,t)}-1\right)\left(1-e^{-(1/2)h}\right).
\end{aligned} \tag{18}$$

Let the step functions are $\psi_1 = (1-e^{-(1/2)h})/(1/2)$, $\psi_2 = (e^{(1/2)h}-1)/(1/2)$, $\phi_1 = (1-e^{-(5/4)\Delta t})/(5/4)$, and $\phi_2 = (e^{(5/4)\Delta t}-1)/(5/4)$. Thus, we can have the forward and backward difference quotients with the special stepsize functions:

$$\begin{aligned}
\partial u &= \frac{u(x+h,t)-u(x,t)}{\psi_2} = \frac{1}{2}u(x+h,t)(u(x,t)-1), \\
\bar{\partial}u &= \frac{u(x,t)-u(x-h,t)}{\psi_1} = \frac{1}{2}u(x-h,t)(u(x,t)-1).
\end{aligned} \tag{19}$$

By the same way in Section 2.1, if we choose $u_{xx} = \partial\bar{\partial}u$, then using the first equation of (19) we can get

$$\begin{aligned}
&\partial\bar{\partial}u \\
&= \frac{((u(x+h,t)-u(x,t))/\psi_2)-((u(x,t)-u(x-h,t))/\psi_2)}{\psi_1} \\
&= \frac{u(x,t)(u(x+h,t)-u(x-h,t))}{2\psi_1} + \frac{u(x,t)-u(x+h,t)}{2\psi_1} \\
&= u(x,t)\frac{u(x+h,t)-u(x-h,t)}{2\psi_1} + \frac{u(x,t)-u(x+h,t)}{2\psi_1}.
\end{aligned} \tag{20}$$

We can notice that $1/2\varphi_1 = 1/5\phi_1 = (1/\phi_1) - (4/5\phi_1) = (1/\phi_1) - (2/\varphi_1)$. So we can have

$$\begin{aligned}
&\frac{u(x,t)-u(x+h,t)}{2\psi_1} \\
&\quad= \frac{u(x,t)-u(x+h,t)}{\phi_1} + 2\frac{u(x+h,t)-u(x,t)}{\varphi_1} \\
&\quad= \frac{u(x,t)-u(x,t-\Delta t)}{\phi_1} + u(x+h,t)(u(x,t)-1).
\end{aligned} \tag{21}$$

When we choose $u_{xx} = \bar{\partial}\partial u$, using the second equation of (19), we can receive

$$\begin{aligned}
&\bar{\partial}\partial u \\
&= \frac{((u(x+h,t)-u(x,t))/\psi_1)-((u(x,t)-u(x-h,t))/\psi_1)}{\psi_2} \\
&= \frac{u(x,t)(u(x+h,t)-u(x-h,t))}{2\psi_2} + \frac{u(x-h,t)-u(x,t)}{2\psi_2} \\
&= u(x,t)\frac{u(x+h,t)-u(x-h,t)}{2\psi_2} + \frac{u(x-h,t)-u(x,t)}{2\psi_2}.
\end{aligned} \tag{22}$$

And we can also have $1/2\varphi_2 = 1/5\phi_2 = (1/\phi_2) - (4/5\phi_2) = (1/\phi_2) - (2/\varphi_2)$, so

$$\begin{aligned}
&\frac{u(x-h,t)-u(x,t)}{2\psi_2} \\
&\quad= \frac{u(x-h,t)-u(x,t)}{\phi_2} + 2\frac{u(x,t)-u(x-h,t)}{\varphi_2} \\
&\quad= \frac{u(x,t+\Delta t)-u(x,t)}{\phi_2} + u(x-h,t)(u(x,t)-1).
\end{aligned} \tag{23}$$

Using the notation in Section 2.1, we can obtain an exact finite difference scheme according to (20) and (21):

$$\begin{aligned}
\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\psi_2\psi_1} &= U_j^{n+1}\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\psi_1} + \frac{U_j^{n+1} - U_j^n}{\phi_1} \\
&\quad + U_{j+1}^{n+1}\left(U_j^{n+1} - 1\right).
\end{aligned} \tag{24}$$

And we can also obtain an explicit exact finite difference scheme based on (22) and (23) as

$$\begin{aligned}
\frac{U_{j+1}^{n} - 2U_j^{n} + U_{j-1}^{n}}{\psi_2\psi_1} &= U_j^{n}\frac{U_{j+1}^{n} - U_{j-1}^{n}}{2\psi_2} + \frac{U_j^{n+1} - U_j^n}{\phi_2} \\
&\quad + U_{j-1}^{n}\left(U_j^{n} - 1\right).
\end{aligned} \tag{25}$$

Then we can obtain the following theorem.

Theorem 2. *An implicit exact finite difference scheme and an explicit exact finite difference scheme for Burgers-Fisher equation* (5) *are given by* (24) *and* (25), *respectively. The stepsize satisfies* $(2/5)h = \Delta t$, *and the stepsize functions satisfy*

$$\begin{aligned}
\psi_1 &= \frac{\left(1 - e^{-(1/2)h}\right)}{(1/2)}, \qquad \psi_2 = \frac{\left(e^{(1/2)h} - 1\right)}{(1/2)}, \\
\phi_1 &= \frac{\left(1 - e^{-(5/4)\Delta t}\right)}{(5/4)}, \qquad \phi_2 = \frac{\left(e^{(5/4)\Delta t} - 1\right)}{(5/4)}.
\end{aligned} \tag{26}$$

Remark 3. From Theorems 1 and 2, we can see that the values of step functions ψ_1, ψ_2, ϕ_1, and ϕ_2 depend on the values of h and Δt. And the stepsize must satisfy $2h = \Delta t$ and $(2/5)h = \Delta t$, respectively.

3. Nonstandard Finite Difference Scheme

The exact numerical schemes of Burgers equation and Burgers-Fisher equation are obtained in Section 2. Notice that the stepsize for exact schemes in Section 2 must satisfy some fixed conditions. In order to release the conditions for stepsize, we would like to use a general way studying form [17, 19–21, 25–29] to construct nonstandard finite difference schemes for two equations.

3.1. Nonstandard Finite Difference Scheme for Burgers Equation. In the classical sense, the first derivative approximation can be represented as $u_t \to (u_{n+1} - u_n)/\Delta t$, $u_x \to (u_{j+1} - u_j)/h$. In our sense, the discrete derivative is generalized as [28]

$$u_t \longrightarrow \frac{u_{n+1} - u_n}{\phi(\Delta t, \lambda)}, \qquad \phi(\Delta t, \lambda) = \Delta t + O\left(\Delta t^2\right); \tag{27}$$

$$\begin{aligned}
u_x &\longrightarrow \frac{u_{j+1} - u_j}{\psi(h, \chi)}, \qquad u_x \longrightarrow \frac{u_{j+1} - u_{j-1}}{2\psi(h, \chi)}, \\
&\psi(h, \chi) = h + O\left(h^2\right),
\end{aligned} \tag{28}$$

where λ, χ is various parameters appearing in the differential equation. $t_n = n\Delta t$, $x_j = jh$, u_n, u_j is an approximation to $u(t_n), u(x_j)$, respectively. This way also can be extended to construct second discrete partial derivatives.

In the classical sense, a special difference scheme of the Burgers equation can be written as

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} - u_j^{n+1}\frac{u_j^n - u_{j-1}^n}{h}, \tag{29}$$

where h and Δt are the stepsizes.

Similar to the classical difference scheme (29), we set the exact difference scheme as

$$\frac{U_j^{n+1} - U_j^n}{\Phi} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Psi} - U_j^{n+1}\frac{U_j^n - U_{j-1}^n}{\Gamma}, \tag{30}$$

where Φ, Γ, and $\Psi = \Gamma^2$ are the step functions.

According to (29) and (30), we can get

$$\Phi = \frac{\left(U_j^{n+1} - U_j^n\right)\Psi\Gamma}{\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)\Gamma - U_j^{n+1}\left(U_j^n - U_{j-1}^n\right)\Psi}. \tag{31}$$

Define $s_j^n = e^{(1/2)(x_j-(t_n/2))}$. We use s to replace s_j^n in our calculation process for simplicity. Using (29) and (31) we can obtain a more detailed format as follows:

$$\begin{aligned}
\Phi &= \left(\frac{1}{1+e^{-\Delta t/4}s} - \frac{1}{1+s}\right)\Psi\Gamma \\
&\quad\times\left(\left(\frac{1}{1+e^{h/2}s} - \frac{2}{1+s} + \frac{1}{1+e^{-h/2}s}\right)\Gamma \right.\\
&\qquad\left. - \frac{1}{1+e^{-\Delta t/4}s}\left(\frac{1}{1+s} - \frac{1}{1+e^{-h/2}s}\right)\Psi\right)^{-1} \\
&= \left(s - e^{-\Delta t/4}s\right)\left(1+e^{h/2}s\right)\left(1+e^{-h/2}s\right)\Psi\Gamma \\
&\quad\times\left(\left(\left(1+e^{-h/2}s\right)\left(s-e^{h/2}s\right) + \left(1+e^{h/2}s\right)\left(s-e^{-h/2}s\right)\right)\right. \\
&\qquad\left.\times\left(1+e^{-\Delta t/4}s\right)\Gamma + \left(s-e^{-h/2}s\right)\left(1+e^{h/2}s\right)\Psi\right)^{-1} \\
&= \frac{\left(1-e^{-\Delta t/4}\right)\left(1+e^{h/2}s\right)\left(1+e^{-h/2}s\right)\Psi}{e^{h/2}\left(1-e^{-h/2}\right)^2(s-1)\left(1+e^{-\Delta t/4}s\right) + \left(1-e^{-h/2}\right)\left(1+e^{h/2}s\right)\Gamma} \\
&= \frac{\left(1-e^{-\Delta t/4}\right)\left(1+e^{h/2}s\right)\left(1+e^{-h/2}s\right)\Gamma^2}{e^{h/2}\left(1-e^{-h/2}\right)^2(s-1)\left(1+e^{-\Delta t/4}s\right) + \left(1-e^{-h/2}\right)\left(1+e^{h/2}s\right)\Gamma}.
\end{aligned} \tag{32}$$

We select $\Gamma = 2(e^{h/2} - 1) > 0$, and so $\Psi = \Gamma^2 = 4(e^{h/2} - 1)^2 > 0$. Substituting Γ and Ψ into (32), we can get

$$\begin{aligned}
\Phi &= \left(1-e^{-\Delta t/4}\right)\left(1+e^{h/2}s\right)\left(1+e^{-h/2}s\right)4\left(e^{h/2}-1\right)^2 \\
&\quad\times\left(e^{h/2}\left(1-e^{-h/2}\right)^2(s-1)\left(1+e^{-\Delta t/4}s\right)\right. \\
&\qquad\left.+\left(1-e^{-h/2}\right)\left(1+e^{h/2}s\right)2\left(e^{h/2}-1\right)\right)^{-1} \\
&= \left(1-e^{-\Delta t/4}\right)\left(1+e^{h/2}s\right)\left(1+e^{-h/2}s\right)4e^{h}\left(1-e^{-h/2}\right)^2 \\
&\quad\times\left(e^{h/2}\left(1-e^{-h/2}\right)^2(s-1)\left(1+e^{-\Delta t/4}s\right)\right. \\
&\qquad\left.+\left(1-e^{-h/2}\right)\left(1+e^{h/2}s\right)2\left(e^{h/2}-1\right)\right)^{-1} \\
&= \frac{4\left(1-e^{-\Delta t/4}\right)\left(1+e^{h/2}s\right)\left(e^{h/2}+s\right)}{\left(1+e^{-\Delta t/4}s\right)(s-1) + 2\left(1+e^{h/2}s\right)}.
\end{aligned} \tag{33}$$

If $\Gamma = h + O(h^2)$, $h \to 0$, $\Delta t \to 0$, we can easily receive $\Phi \to 4(1 - e^{-\Delta t/4})$, so $\Phi = \Delta t + O(\Delta t^2)$. So when h and Δt approach zero, we can obtain a nonstandard finite difference scheme for Burgers-equation as follows:

$$\begin{gathered}
\frac{U_j^{n+1} - U_j^n}{\Phi} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Psi} - U_j^{n+1}\frac{U_j^n - U_{j-1}^n}{\Gamma}, \\
\Phi = 4\left(1-e^{-\Delta t/4}\right), \qquad \Psi = 4\left(e^{h/2}-1\right)^2, \\
\Gamma = 2\left(e^{h/2}-1\right).
\end{gathered} \tag{34}$$

It can be easily noticed that the scheme is explicit. Solving for U_j^{n+1} and with appropriate $R = \Phi/\Psi$ and $r = \Phi/\Gamma$ gives

$$U_j^{n+1} = \frac{R\left(U_{j+1}^n + U_{j-1}^n\right) + (1-2R)U_j^n}{1 + r\left(U_j^n - U_{j-1}^n\right)}. \tag{35}$$

We can write the following Theorem to ensure the nonnegativity and boundedness.

Theorem 4. *If $1 - 2R - r \geq 0$, the numerical solution U_j^n (35) satisfies*

$$0 \leq U_j^n \leq 1 \Longrightarrow 0 \leq U_j^{n+1} \leq 1, \tag{36}$$

for all relevant values of n and j.

Proof. $1-2R-r \geq 0$ implies that $1-2R \geq r > 0, r < 1$. Using the upside of (35) minus downside, we receive

$$\begin{aligned} &R\left(U_{j+1}^{n}+U_{j-1}^{n}\right)+(1-2R)U_j^n-rU_j^n+rU_{j-1}^n \\ &\quad=R\left(U_{j+1}^{n}+U_{j-1}^{n}\right)+(1-2R-r)U_j^n+rU_{j-1}^n \\ &\quad\leq R(1+1)+(1-2R-r)\cdot 1+r\cdot 1=1, \\ &R\left(U_{j+1}^{n}+U_{j-1}^{n}\right)+(1-2R)U_j^n \geq 0, \\ &1+r\left(U_j^n-U_{j-1}^n\right)+\Phi U_j^n \geq 1-r+rU_j^n \geq 0. \end{aligned} \tag{37}$$

Equation (37) implies that

$$0 \leq U_j^{n+1}=\frac{R\left(U_{j+1}^{n}+U_{j-1}^{n}\right)+(1-2R)U_j^n+\Phi U_j^n}{1+r\left(U_j^n-U_{j-1}^n\right)+\Phi U_j^n} \leq 1. \tag{38}$$

In a word, if the initial data is nonnegative and bounded by one, then the discrete-time solution (35) has this behavior for all subsequent times. This completes the proof. □

3.2. Nonstandard Finite Difference Scheme for Burgers-Fisher Equation. In this section, we will show a nonstandard finite difference scheme for Burgers-Fisher equation. Using the result of Section 3.1, a discrete scheme for the left side of (5) can be constructed by the following form:

$$\frac{U_j^{n+1}-U_j^n}{\Phi}=\frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{\Psi}-U_j^{n+1}\frac{U_j^n-U_{j-1}^n}{\Gamma}, \tag{39}$$

where the forms of Φ, Ψ, and Γ are same as those parameters in (34), and U_j^n is an approximation to $u(x_j,t_n)$. If we ignore the status items, Burgers-Fisher equation is reduced to the logistic growth equation. Referring to the exact scheme of logistic growth equation [29], we can replace the right side of (5) by the "nonlocal" form:

$$u(1-u)=u-u^2 \longrightarrow U_j^n-U_j^{n+1}U_j^n. \tag{40}$$

Based upon (39) and (40), a nonstandard finite difference scheme for (5) is given:

$$\begin{aligned} \frac{U_j^{n+1}-U_j^n}{\Phi}=&\frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{\Psi}-U_j^{n+1}\frac{U_j^n-U_{j-1}^n}{\Gamma} \\ &+U_j^n-U_j^{n+1}U_j^n. \end{aligned} \tag{41}$$

Similar to the result in Section 3.1, the stepsize function for Burgers-Fisher equation (5) could be written as

$$\begin{aligned} \Phi=4\left(1-e^{-\Delta t/4}\right), \qquad \Psi=4\left(e^{h/2}-1\right)^2, \\ \Gamma=2\left(e^{h/2}-1\right). \end{aligned} \tag{42}$$

We can find that $\Phi \to \Delta t$, $\Psi \to h^2$ and $\Gamma \to h$ as h and Δt approach zero.

It can be seen that the scheme is explicit. Solving for U_j^{n+1} and with appropriate $R=\Phi/\Psi$ and $r=\Phi/\Gamma$ gives

$$U_j^{n+1}=\frac{R\left(U_{j+1}^{n}+U_{j-1}^{n}\right)+(1-2R+\Phi)U_j^n}{1+r\left(U_j^n-U_{j-1}^n\right)+\Phi U_j^n}. \tag{43}$$

Similar to Theorem 4, we find at once the following result.

Theorem 5. *If $1-2R-r \geq 0$, the numerical solution (43) satisfies*

$$0 \leq U_j^n \leq 1 \Longrightarrow 0 \leq U_j^{n+1} \leq 1, \tag{44}$$

for all relevant values of n and j.

Proof. As in Theorem 4, $1-2R-r \geq 0$ implies that $1-2R \geq r > 0$, $r < 1$. Using the upside of (43) minus downside, we receive

$$\begin{aligned} &R\left(U_{j+1}^{n}+U_{j-1}^{n}\right)+(1-2R+\Phi)U_j^n-rU_j^n+rU_{j-1}^n-\Phi U_j^n \\ &\quad=R\left(U_{j+1}^{n}+U_{j-1}^{n}\right)+(1-2R-r)U_j^n+rU_{j-1}^n \\ &\quad\leq R(1+1)+(1-2R-r)\cdot 1+r\cdot 1=1, \\ &R\left(U_{j+1}^{n}+U_{j-1}^{n}\right)+(1-2R)U_j^n+\Phi U_j^n \geq 0, \\ &1+r\left(U_j^n-U_{j-1}^n\right)+\Phi U_j^n \geq 1-r+rU_j^n+\Phi U_j^n \geq 0. \end{aligned} \tag{45}$$

So the inequalities (45) imply that

$$0 \leq U_j^{n+1}=\frac{R\left(U_{j+1}^{n}+U_{j-1}^{n}\right)+(1-2R)U_j^n+\Phi U_j^n}{1+r\left(U_j^n-U_{j-1}^n\right)+\Phi U_j^n} \leq 1. \tag{46}$$

So the initial data is nonnegative and bounded by one; then the discrete-time solution (43) has this behavior for all subsequent times. This can ensure that the positivity and boundedness conditions hold. This completes the proof. □

For appropriate R and r, setting $u_j^n=u(x_j,t_n)$ precisely, we have Taylor's formula for the solution of equation (5), with appropriate $\overline{x}_j \in (x_j,x_{j+1})$, $\overline{t}_n \in (t_n,t_{n+1})$. For functions defined on the grid, we introduce these difference quotients:

$$\begin{aligned} \partial_t u_j^n &= \frac{U_j^{n+1}-U_j^n}{\Phi}, \\ \partial_x u_j^n &= \frac{U_{j+1}^n-U_j^n}{\Gamma}, \\ \partial_x\overline{\partial}_x u_j^n &= \frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{\Psi}. \end{aligned} \tag{47}$$

Using the method in [31], the local truncation error (or local discretization error) τ_j^n is shown as follows

$$\begin{aligned}
\tau_j^n &= \partial_t u_j^n + u_j^{n+1}\partial_x u_j^n - \partial_x\bar{\partial}_x u_j^n - u_j^n\left(1-u_j^{n+1}\right) \\
&= \left(\partial_t u_j^n - u_t\left(x_j,t_n\right)\right)+\left(u_j^{n+1}\partial_x u_j^n - u\left(x_j,t_n\right)u_x\left(x_j,t_n\right)\right) \\
&\quad - \left(\partial_x\bar{\partial}_x u_j^n - u_{xx}\left(x_j,t_n\right)\right) \\
&\quad - \left(u_j^n\left(1-u_j^{n+1}\right) - u\left(x_j,t_n\right)\left(1-u\left(x_j,t_n\right)\right)\right) \\
&= u_t\left(x_j,t_n\right)\left(\frac{\Delta t}{\Phi}-1\right)+\frac{\Delta t^2}{2\Phi}u_{tt}\left(x_j,t_n\right)+\frac{\Delta t^3}{6\Phi}u_{ttt}\left(x_j,\bar{t}_n\right) \\
&\quad + u\left(x_j,t_n\right)u_x\left(x_j,t_n\right)\left(\frac{h}{\Gamma}-1\right) \\
&\quad + \frac{h^2}{2\Gamma}u\left(x_j,t_n\right)u_{xx}\left(x_j,t_n\right) \\
&\quad + \frac{h^3}{6\Gamma}u\left(x_j,t_n\right)u_{xxx}\left(\bar{x}_j,t_n\right)+\frac{h\Delta t}{\Gamma}u_t\left(x_j,t_n\right)u_x\left(x_j,t_n\right) \\
&\quad + \frac{h^2\Delta t}{2\Gamma}u_t\left(x_j,t_n\right)u_{xx}\left(x_j,t_n\right) \\
&\quad + \frac{h^3\Delta t}{6\Gamma}u_t\left(x_j,t_n\right)u_{xxx}\left(\bar{x}_j,t_n\right) \\
&\quad + \frac{h\Delta t^2}{2\Gamma}u_{tt}\left(x_j,\bar{t}t_n\right)u_x\left(x_j,t_n\right) \\
&\quad + \frac{h^2\Delta t^2}{4\Gamma}u_{tt}\left(x_j,\bar{t}_n\right)u_{xx}\left(x_j,t_n\right) \\
&\quad + \frac{h^3\Delta t^2}{12\Gamma}u_{tt}\left(x_j,\bar{t}_n\right)u_{xxx}\left(\bar{x},t_n\right)-\left(\frac{h^2}{\Psi}-1\right)u_{xx}\left(x_j,t_n\right) \\
&\quad - \frac{h^4}{12\Psi}u_{xxxx}\left(\bar{x}_j,t_n\right)+u\left(x_j,t_n\right)\Delta t u_t\left(x_j,t_n\right) \\
&\quad + \frac{\Delta t^2}{2}u\left(x_j,t_n\right)u_{tt}\left(x_j,t_n\right)+\frac{\Delta t^3}{6}u\left(x_j,t_n\right)u_{ttt}\left(x_j,\bar{t}_n\right).
\end{aligned} \tag{48}$$

When $h \to 0$ and $\Delta t \to 0$, we have $\Phi \approx \Delta t$, $\Gamma \approx h$ and $\Psi \approx h^2$. Therefore, $\tau_j^n = O(\Delta t + h)$ if $h \to 0$ and $\Delta t \to 0$. We also can say that the exact solution satisfies the difference equation except for a small error.

Remark 6. From (34) and (42), we can see that the value of Φ depends on the value of h and Δt, which implies that R and r also depend on the value of h and Δt. And appropriate R and r that satisfy $1 - 2R - r \geq 0$ (Theorems 4 and 5) can ensure that the positivity and boundedness conditions hold.

4. Numerical Experiments

To verify the effectivity of the NSFD scheme in Section 3, we simulate the initial-boundary value problems:

$$\begin{aligned}
u_t + uu_x - u_{xx} &= 0, \quad 0 \leq x \leq 1,\ t \geq 0, \\
u(x,0) &= \frac{1}{1+e^{x/2}}, \quad 0 \leq x \leq 1, \\
u(0,t) &= \frac{1}{1+e^{-t/4}}, \quad t \geq 0, \\
u(1,t) &= \frac{1}{1+e^{(1/2)-(t/4)}}, \quad t \geq 0.
\end{aligned} \tag{49}$$

We use scheme (34) and give the initial condition as follows

$$\begin{aligned}
U_j^0 &= \frac{1}{1+e^{x_j/2}}, \quad j = 0,1,\ldots,J, \\
U_0^n &= \frac{1}{1+e^{-t_n/4}}, \quad n = 0,1,\ldots,N, \\
U_J^n &= \frac{1}{1+e^{(x_J/2)-(t_n/4)}}, \quad n = 0,1,\ldots,N.
\end{aligned} \tag{50}$$

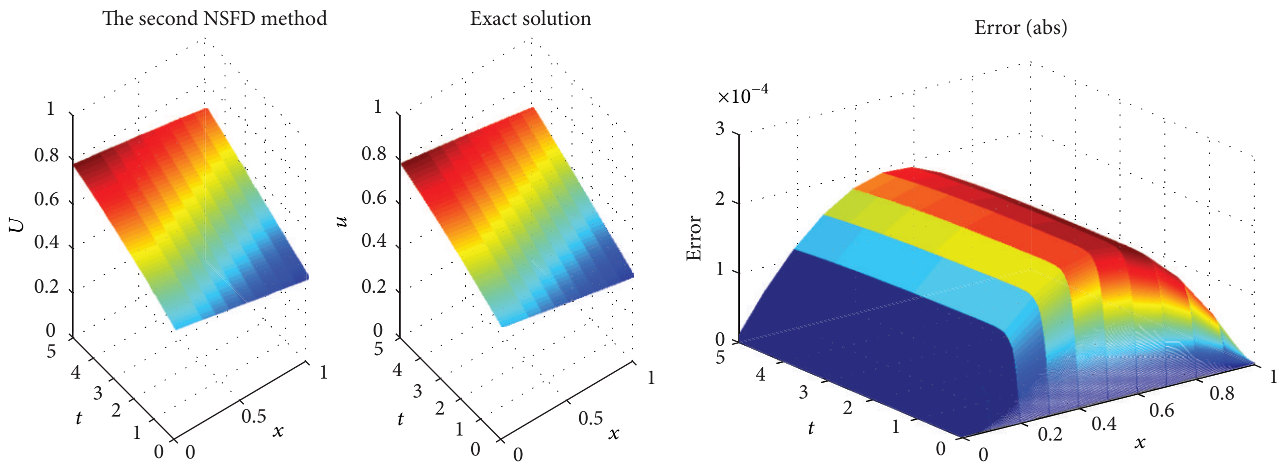
For (49), in order to compare the numerical solution and the solitary wave solution (27), we plot the values of these two solutions in Figure 1(a), in which we set the space step h as 0.1 with the number of space steps as 10, time step Δt as 0.001, and the number of time steps as 5000, respectively. We can see that the values of $2R$ and r ensure that $2R + r < 1$. It ensures the positivity and boundedness of our method. The error of the method is presented in Figure 2(b). For a given fixed value of $x = \bar{x}$, Figure 2(a) shows the values of numerical solution and solitary wave solution and Figure 2(b) shows the error between two solutions of different formats. It also can be found that in Figure 2(a) U is increased from 0 to 1 as the analytical solution at the given fixed value of $x = \bar{x}$. It means that at a fixed $x = \bar{x} > 0$,

$$\lim_{t\to\infty} U(\bar{x},t) = 1. \tag{51}$$

We can see that the result of the calculation is consistent with diffusion phenomena from the physical point of view. Figures 1(a) and 2(a) also show that the positivity and the boundedness hold.

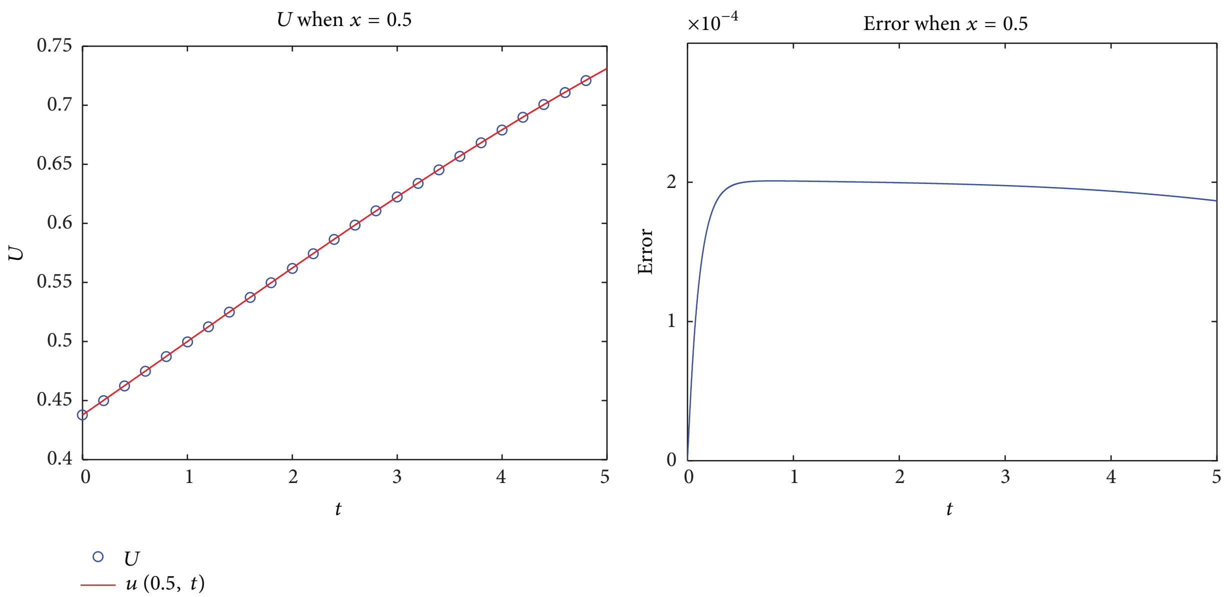
Consider the following problem:

$$\begin{aligned}
u_t + uu_x - u_{xx} &= u(1-u), \quad 0 \leq x \leq 1,\ t \geq 0, \\
u(x,0) &= \frac{1}{1+e^{x/2}}, \quad 0 \leq x \leq 1, \\
u(0,t) &= \frac{1}{1+e^{-5t/4}}, \quad t \geq 0, \\
u(1,t) &= \frac{1}{1+e^{(1/2)-(5t/4)}}, \quad t \geq 0.
\end{aligned} \tag{52}$$



(a) U_j^n and $u(x,t)$ (b) Error between U_j^n and $u(x,t)$

FIGURE 1: Simulations of NSFD scheme (34) for (4) with stepsize $\Delta t = 0.001$ and $h = 0.1$.



(a) U_j^n and $u(x,t)$ (b) Error between U_j^n and $u(x,t)$

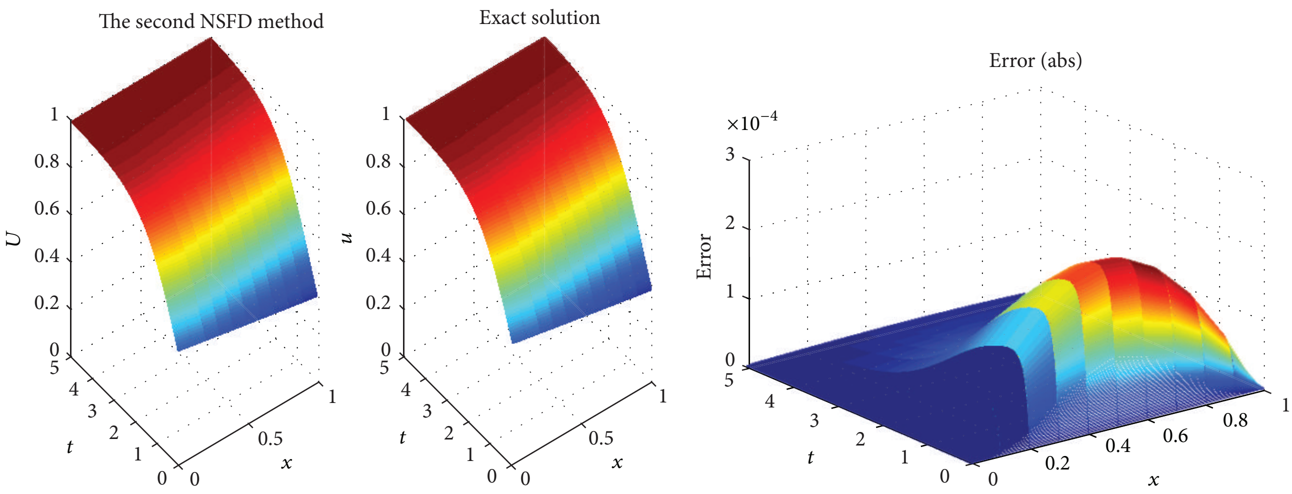
FIGURE 2: U and $u(x,t)$ at a fixed value $x = \overline{x} = 0.5$ for NSFD scheme (34).

We use the two schemes (41), (42). Then give the initial condition as following:

$$U_j^0 = \frac{1}{1+e^{x_j/2}}, \quad j = 0, 1, \ldots, J,$$
$$U_0^n = \frac{1}{1+e^{-5t_n/4}}, \quad n = 0, 1, \ldots, N, \tag{53}$$
$$U_J^n = \frac{1}{1+e^{(x_J/2)-(5t_n/4)}}, \quad n = 0, 1, \ldots, N.$$

For the problem (52), we also use the space step h as 0.1 with the number of space steps as 10, time step Δt as 0.001, and the number of time steps as 5000, respectively. In the simulation, $R = 0.0951$ and $r = 0.0098$, so $2R + r < 1$. It ensures the positivity and boundedness of our method. In the simulation Figure 3(a) indicates the numerical solution and the solitary wave solution. The error of the method is presented in Figure 3(b). For the given fixed value of $x = \overline{x}$, Figure 4(a) also can show that at a fixed $x = \overline{x} > 0$, U is increased from 0 to 1. It just likes a diffusion process expected. The two simulations show that our NSFD schemes are efficient and accurate.

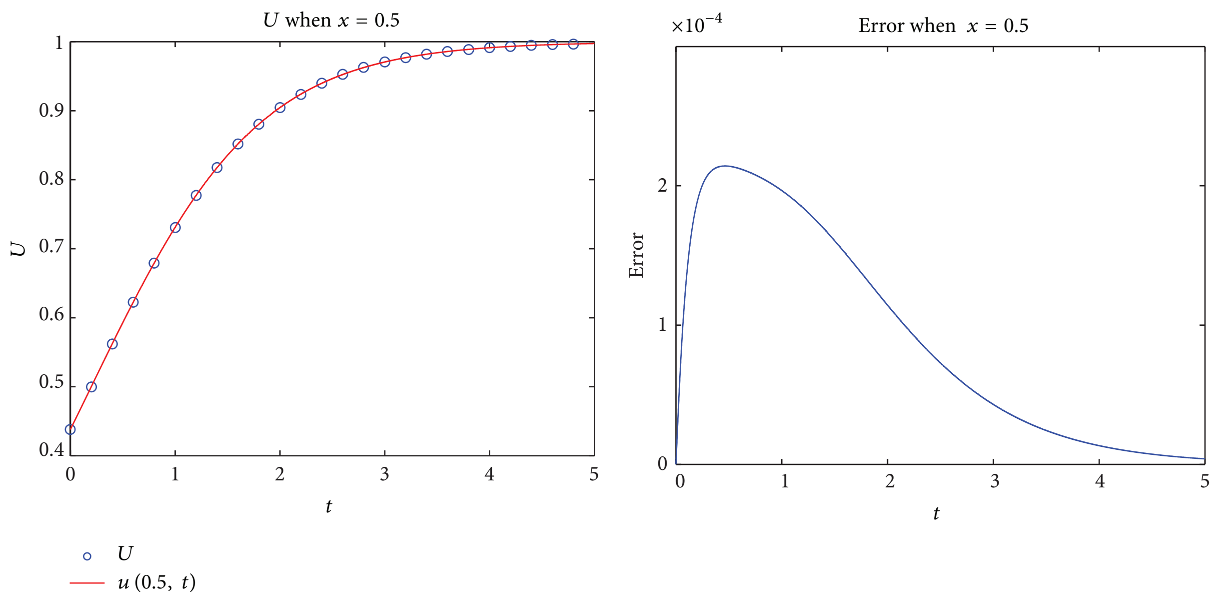
For the exact schemes in Section 2, if we select the stepsize as $h = 0.1$ and $\Delta t = 0.001$, the exact schemes are reduced to NSFD scheme. In Figure 5, we contrast this NSFD (13) with the NSFD scheme in Section 3 for Burgers equation. It shows that this NSFD scheme is also efficient and accurate.



(a) U_j^n and $u(x,t)$

(b) Error between U_j^n and $u(x,t)$

FIGURE 3: Simulations of NSFD scheme (41) for (5) with stepsize $\Delta t = 0.001$ and $h = 0.1$.



(a) U_j^n and $u(x,t)$

(b) Error between U_j^n and $u(x,t)$

FIGURE 4: U and $u(x,t)$ at a fixed value $x = \overline{x} = 0.5$ for NSFD scheme (41).

We compare our methods (41) with Adomain decomposition method [9] for Burgers-Fisher equation, which is shown as follows:

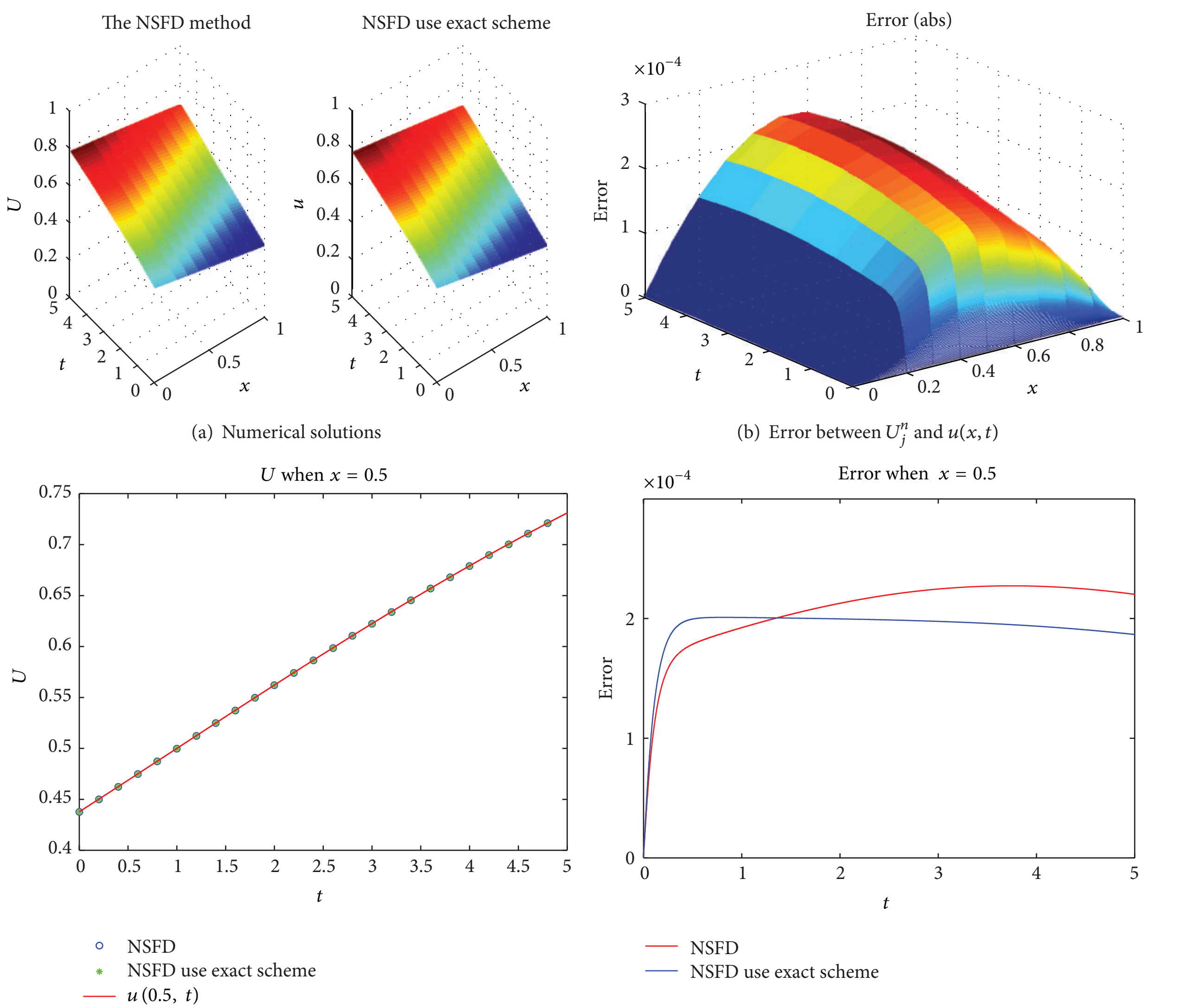
$$\begin{gathered} u_0(x,t) = u(x,0) = f(x), \\ u_{n+1}(x,t) = f(x) + L^{-1}\left(R(u_n) - A_n\right), \\ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \end{gathered} \tag{54}$$

As in paper [9], we use five u_n. By applying the ADM method to the problem (49), we get

$$u_0 = u_0(x,t) = u(x,0) = \frac{1}{1+e^{x/2}},$$

$$u_1 = u_1(x,t) = -\int_0^t \left(A_0 - u_{0xx}\right),$$

$$u_2 = u_2(x,t) = -\int_0^t \left(A_1 - u_{1xx}\right),$$



(a) Numerical solutions

(b) Error between U_j^n and $u(x,t)$

(c) Values at fixed $x = 0.5$

(d) Errors at fixed $x = 0.5$

FIGURE 5: Comparison of NSFD in Section 3 and exact scheme (13) in Section 2 with other stepsizes $h = 0.1$ and Δt for Burgers equation.

TABLE 1: The absolute errors of NSFD method and ADM ($n = 4$) for (52) at $x = 0.1$.

$x = 0.1$	$t = 0.005$	$t = 0.01$	$t = 0.1$	$t = 0.5$
NSFD	7.1788×10^{-6}	1.2226×10^{-5}	5.0003×10^{-5}	7.0794×10^{-5}
ADM	7.3×10^{-3}	1.47×10^{-2}	1.531×10^{-1}	8.911×10^{-1}

$$u_3 = u_3(x,t) = -\int_0^t (A_2 - u_{2xx}),$$
$$u_4 = u_4(x,t) = -\int_0^t (A_3 - u_{3xx}). \tag{55}$$

And Adomain polynomials are given by

$$A_0 = u_0 u_{0x} + u_0 (1 - u_0),$$
$$A_1 = (u_1 u_{0x} + u_0 u_{1x}) - [u_0 (1 - u_1) + u_1 (1 - u_0)],$$
$$A_2 = (u_0 u_{2x} + u_1 u_{1x} + u_{0x} u_2) - [u_0 (1 - u_2) + u_1 (1 - u_1) + u_2 (1 - u_0)],$$
$$A_3 = (u_0 u_{3x} + u_{1x} u_2 + u_{0x} u_3 + u_1 u_{2x}) - [u_0(1 - u_3) + u_1(1 - u_2) + u_2(1 - u_1) + u_3(1 - u_0)]. \tag{56}$$

For each $x = 0.1$, 0.5 and 0.9, NSFD methods and ADM method are applied at different times: $t = 0.005, 0.01, 0.1$, and 0.5 with stepsize $h = 0.1$, $\Delta t = 0.001$. From Tables 1, 2, and 3, we can see that our method is more accurate than ADM

Table 2: The absolute errors of NSFD method and ADM ($n = 4$) for (52) at $x = 0.5$.

$x = 0.5$	$t = 0.005$	$t = 0.01$	$t = 0.1$	$t = 0.5$
NSFD	8.6033×10^{-6}	1.7213×10^{-5}	1.3419×10^{-4}	2.1407×10^{-4}
ADM	6.6×10^{-3}	1.32×10^{-2}	1.382×10^{-1}	8.220×10^{-1}

Table 3: The absolute errors of NSFD method and ADM ($n = 4$) for (52) at $x = 0.9$.

$x = 0.9$	$t = 0.005$	$t = 0.01$	$t = 0.1$	$t = 0.5$
NSFD	7.2255×10^{-6}	1.2430×10^{-5}	5.5237×10^{-5}	8.3619×10^{-5}
ADM	4.7×10^{-3}	1.17×10^{-2}	1.236×10^{-1}	7.502×10^{-1}

($n = 4$) which uses finite $u_n(x,t)$. To achieve better accuracy, ADM will require n to be big enough. In other words, ADM will have to consume more computations for derivative and integral. Hence, our method is superior to ADM in terms of computations when aiming to achieve the same accuracy.

5. Conclusions

In this paper, we present an exact finite difference scheme for a particular Burgers and Burgers-Fisher equation based on the solitary wave solutions. The proposed step function depends on h, Δt. And nonstandard finite difference schemes for Burgers and Burgers-Fisher equations can be constructed using the method in Mickens and Roeger's papers. Numerical experiments for a particular example are given. The results show that the numerical solutions of our methods meet the properties that the "physically" relevant solutions should have. By comparison, our methods are also found to be accurate and effective.

Conflict of Interests

The authors declare that there is no conflict of interests.

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