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Research Article

Rough Operations and Uncertainty Measures on MV-Algebras

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We define a lower approximate operation and an upper approximate operation based on a partition on MV-algebras and discuss their properties. We then introduce a belief measure and a plausibility measure on MV-algebras and investigate the relationship between rough operations and uncertainty measures.

1. Introduction

The rough set theory, introduced by Pawlak, has been conceived as a tool to conceptualize, organize, and analyze various types of data, in particular, to deal with inexact, uncertain, or vague knowledge in applications related to artificial intelligence. Since then, the subject has been investigated in many studies and various rough set models have been used in machine learning, knowledge discovery, decision support systems, and pattern recognition. In Pawlak's rough set model, a key concept is an equivalence relation, and given an equivalence relation on a universe, we can define a pair of rough approximations which provide a lower bound and an upper bound for each subset of the universe. Rough approximations can also be defined equivalently by a partition of the universe which is corresponding to the equivalence relation [1–6].

Dempster-Shafer theory of evidence is a method developed to model and manipulate uncertain, imprecise, incomplete, and even vague information. It was originated by Dempster's concept of lower and upper probabilities and extended by Shafer as a theory. The basic representational structure in this theory is a belief structure, which consists of a family of subsets, called focal elements, with associated individual positive weights summing to 1. The fundamental numeric measures derived from the belief structure are a dual pair of belief and plausibility functions [7]. Combining the Dempster-Shafer theory and fuzzy set theory has been suggested to be a way to deal with different kinds of uncertain information in intelligent systems in a number of studies [8–10]. In [11–14], by introducing a pair of dual rough operations on Boolean algebras and using them to interpret some uncertainty measures on Boolean algebras, Bayesian theory and Dempster-Shafer theory are extended to be constructed on Boolean algebras. This provides a more general framework to deal with uncertainty reasoning and a better understanding of both rough operations and uncertainty measures on Boolean algebras.

The present paper extends the rough operations and Dempster-Shafer theory with respect to Boolean algebras to MV-algebra. The paper is arranged as follows. In Section 2, some results of MV-algebra which will be used in this paper are recollected. In Section 3, we introduce a pair of approximate operations based on a partition on MV-algebra, which is the generation of rough operations on Boolean algebras, and discuss their properties. In Section 4, we define a belief measure and a plausibility measure on MV-algebras and investigate the relationship between rough operations and uncertainty measures. In Section 5 concludes.

2. MV-Algebra and Its Partitions

In this section, we recall firstly the basic notions on MV-algebras. See [15–18] for further results on MV-algebras. An MV-algebra $M = (M; 0, \neg, \oplus)$ is an algebra where $\oplus$ is an associative and commutative binary operation on M having 0 as the neutral element, a unary operation $\neg$ is involutive with $a \oplus \neg 0 = \neg 0$ for all $a \in M$, and moreover the identity $a \oplus \neg(a \oplus \neg b) = b \oplus \neg(b \oplus \neg a)$ is satisfied for all $a, b \in M$.

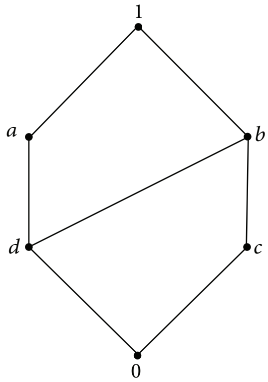

Figure 1

A partial order is defined on M by $a \leq b$ if and only if $\neg a \oplus b = 1$. An additional constant 1 and two binary operations $\otimes$, $\rightarrow$, $\vee$ and $\wedge$ are defined as follows:

$$1 = \neg 0, \qquad a \otimes b = \neg(\neg a \oplus \neg b), \qquad a \longrightarrow b = \neg a \oplus b,$$
$$a \vee b = (a \otimes \neg b) \oplus b, \qquad a \wedge b = (a \oplus \neg b) \otimes b. \tag{1}$$

If M is also totally ordered then M is called a totally ordered MV-algebra. An MV-algebra M is called σ-complete if every nonempty countable subset of M has a supremum in M.

An MV-subalgebra of M is a subset M_1 of M containing the neutral element 0 of M, closed under the operations of M and endowed with the restriction of these operations to M_1.

The (finite) Cartesian product $M_1 \times M_2 \times \cdots \times M_k$ of MV-algebras M_i, endowed with the partial order and the MV-algebra operations defined pointwise, really is also an MV-algebra and will be called product, for short.

An element $a \in M$ satisfying $a \oplus a = a$ is called an idempotent element (Boolean element); the set $B(M) = \{a \in M \mid a \oplus a = a\}$ of all idempotent elements of M endowed with the natural restriction of operations inherited from M is a Boolean algebra.

Example 1. The real unit interval $[0, 1]$ with Łukasiewicz operation $a \rightarrow b = R_L(a, b) = (1 - a + b) \wedge 1$, and $a \otimes b = (a + b - 1) \vee 0$, where $a, b \in [0, 1]$, is a σ-complete MV-algebra called the MV-unit interval.

Example 2. Let $X \neq \emptyset$, $M = [0, 1]^X$, where $[0, 1]$ is the MV-unit interval. The order and the operation $\oplus, \neg$ on M are defined in pointwise: for $A, B \in M$

$$A \leq B \quad \text{iff } x \in X,\ A(x) \leq B(x),$$
$$(A \oplus B)(x) = A(x) \otimes B(x), \qquad (\neg A)(x) = 1 - A(x),$$
$$x \in X. \tag{2}$$

Then M is an MV-algebra called the MV-cube.

Table 1

	0	a	b	c	d	1
$\neg$	1	c	d	a	b	0

Table 2

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	1	1	a	1
b	b	1	1	b	1	1
c	c	1	b	c	b	1
d	d	a	1	b	a	1
1	1	1	1	1	1	1

Example 3. Let $M = \{0, a, b, c, d, 1\}$. The order and the operations $\vee, \wedge$ on M are defined as Figure 1, and the operations $\otimes$, $\rightarrow$ on M are defined as Tables 1 and 2, respectively. Then M is an MV-algebra.

Proposition 4 (see [15–18]). *Let M be an MV-algebra. For any $x, y, z \in M$,*

(P1) $x \leq y$ *if and only if* $x \rightarrow y = 1$;

(P2) $1 \rightarrow x = x$, $1 \otimes x = x$;

(P3) $x \otimes y \rightarrow z = x \rightarrow (y \rightarrow z)$;

(P4) $y \rightarrow y \otimes x = x \vee \neg y$;

(P5) $x \wedge y \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$;

(P6) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x = x \vee y$;

(P7) $(M, \oplus, 0)$ *is a commutative semigroup with unit element 0;*

(P8) $(\bigvee_{i \in N} x_i) \otimes a = \bigvee_{i \in N}(x_i \otimes a)$, *if the supremum exists in equality;*

(P9) $a \oplus (\bigwedge_{i \in N} y_i) = \bigwedge_{i \in N}(a \oplus y_i)$, *if the infimum exists in equality.*

Two elements a and b in an MV-algebra M are called orthogonal (denoted as $a \perp b$) if $a \leq \neg b$. Obviously, $a \perp b$ if and only if $a \otimes b = 0$. A finite subset $\xi = \{a_1, a_2, \ldots, a_n\}$ of elements of an MV-algebra M is said to be $\oplus$-orthogonal if $\forall I \subset \{1, 2, \ldots, n\}$, $\bigoplus_{i \in I} a_i \perp a_j$, $j \in \{1, 2, \ldots, n\} - I$.

Definition 5. A finite collection $\xi = \{a_1, a_2, \ldots, a_n\}$ of nonneutral elements of an MV-algebra M is said to be a partition of M if and only if

(i) ξ is $\oplus$-orthogonal;

(ii) $\bigvee_{i=1}^{n} a_i = 1$.

Lemma 6. *Every element in partition $\xi = \{a_1, a_2, \ldots, a_n\}$ is idempotent element; that is, $\forall a_i \in \xi$, $a_i \otimes a_i = a_i$, $a_i \oplus a_i = a_i$.*

Proof. $\forall a_i \in \xi$, we have $a_i = a_i \otimes 1 = a_i \otimes \bigvee_{j=1}^{n} a_j = \bigvee_{j=1}^{n}(a_i \otimes a_j) = a_i \otimes a_i$, and it also means that $a_i \rightarrow \neg a_i = \neg a_i$. It follows that $a_i \wedge \neg a_i = a_i \otimes (a_i \rightarrow \neg a_i) = a_i \otimes \neg a_i = 0$. By

Table 3

x	$(1,1)$	$(1,0)$	$(a,1)$	$(a,0)$	$(b,1)$	$(b,0)$	$(c,1)$	$(c,0)$	$(d,1)$	$(d,0)$	$(0,1)$	$(0,0)$
$\mathrm{Pl}(x)$	1	1/2	3/4	1/4	1	1/2	3/4	1/4	3/4	1/4	1/2	0
$\mathrm{Bel}(x)$	1	1/2	3/4	1/4	3/4	1/4	3/4	1/4	1/2	0	1/2	0
$m^*(x)$	1	1/2	3/4	1/4	1	1/2	3/4	1/4	3/4	1/4	1/2	0
$m_*(x)$	1	1/2	3/4	1/4	3/4	1/4	3/4	1/4	1/2	0	1/2	0

$(a_i \oplus a_i) \otimes \neg a_i = \neg a_i \otimes (\neg a_i \rightarrow a_i) = \neg a_i \wedge a_i = 0$ we know $a_i \oplus a_i \leq \neg a_i \rightarrow 0 = a_i$. Obviously, $a_i \leq a_i \oplus a_i$. Hence, $a_i \oplus a_i = a_i$. □

Theorem 7. *Let $\xi = \{a_1, a_2, \ldots, a_n\}$ be a partition of M. Suppose that*

$$M_0 = \left\{ x \mid \text{there exist finite elements } \{a_{i_1}, \ldots, a_{i_l}\} \subseteq \xi \text{ such that } x = \bigoplus_{k \in I_x} a_k \right\} \cup \{0\}, \tag{3}$$

where $I_x = \{i_1, \ldots, i_l\} \subseteq \{1, 2, \ldots, n\}$. Then M_0 is a Boolean algebra.

Proof. At first, we prove that $\neg x = \bigoplus_{k \in \{1,2,\ldots,n\} - I_x} a_k$ for $x = \bigoplus_{k \in I_x} a_k \in M_0$, where $I_x = \{i_1, \ldots, i_l\} \subseteq \{1, 2, \ldots, n\}$. From $x \oplus \bigoplus_{k \in \{1,2,\ldots,n\} - I_x} a_k = \bigoplus_{k \in \{1,2,\ldots,n\}} a_k = 1$ we know $\neg x \leq \bigoplus_{k \in \{1,2,\ldots,n\} - I_x} a_k$. On other hand, from $a_i \otimes \bigoplus_{k \in \{1,2,\ldots,n\} - I_x} a_k = 0$ we know $\bigoplus_{k \in \{1,2,\ldots,n\} - I_x} a_k \leq \neg a_i$ for any $i \in I_x$. Hence, $\bigoplus_{k \in \{1,2,\ldots,n\} - I_x} a_k \leq \bigwedge_{i \in I_x} \neg a_i = \neg(\bigvee_{i \in I_x} a_i) \leq \neg(\bigoplus_{i \in I_x} a_i) = \neg x$. This proves that $\neg x = \bigoplus_{k \in \{1,2,\ldots,n\} - I_x} a_k$. Hence, $\neg x \in M_0$ for every $x \in M_0$.

Secondly, we prove that M_0 is closed under $\wedge$ and $\vee$. Let $x = \bigoplus_{k \in I_x} a_k$, $y = \bigoplus_{i \in I_y} a_i \in M_0$. Note that $a_i \wedge a_j = (a_i \oplus \neg a_j) \otimes a_j = (a_i \oplus \bigoplus_{k \in \{1,2,\ldots,n\} - \{j\}} a_k) \otimes a_j = (\bigoplus_{k \in \{1,2,\ldots,n\} - \{j\}} a_k) \otimes a_j = \neg a_j \otimes a_j = 0$ for $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$. If $I_x \cap I_y \neq \emptyset$ then $x \wedge y = (\bigoplus_{k \in I_x} a_k) \wedge (\bigoplus_{i \in I_y} a_i) = \bigoplus_{k \in I_x, i \in I_y} (a_k \wedge a_i) = \bigoplus_{k \in I_x \cap I_y} a_k$ or else $x \wedge y = 0$. Hence, $x \wedge y \in M_0$ for all $x, y \in M_0$. By the duality of $\wedge$ and $\vee$ we know $x \vee y \in M_0$ for all $x, y \in M_0$.

Finally, from the above proof we easily obtain that $x \wedge \neg x = 0$ and $x \vee \neg x = 1$. Therefore, M_0 is a Boolean algebra. □

Example 8. Let $M = [0,1]^X$ be the MV-cube in Example 2 and $\{A_1, A_2, \ldots, A_n\}$ be a partition of X in the usual sense. Denote

$$\chi_{A_i}(x) = \begin{cases} 1, & x \in A_i, \\ 0, & x \notin A_i. \end{cases} \tag{4}$$

Then $\xi = \{\chi_{A_1}, \chi_{A_2}, \ldots, \chi_{A_n}\}$ is a partition of M.

Example 9. Let M be the MV-algebra in Example 3. Then $\{a, c\}$ is a partition of M.

Example 10. Let M be the MV-algebra in Example 3 and $B = \{0, 1\}$ the classic Boolean algebra. Then $M \times B$ is an MV-algebra, and $\{(a, 0), (c, 0), (0, 1)\}$ is a partition of $M \times B$.

3. Approximate Operations on MV-Algebras

In Pawlak's rough set theory, a rough set is induced by a partition of the universe. In this section, we will extend Pawlak's rough set theory by defining a pair of approximate operations induced by a partition of the unity of an MV-algebra.

Definition 11. Let M be an MV-algebra and $\xi = \{a_1, a_2, \ldots, a_n\}$ a partition of M. Then a pair of operations $H_\xi : M \rightarrow M$ and $L_\xi : M \rightarrow M$, such that

$$\begin{aligned} H_\xi(x) &= \vee \{a_i \mid a_i \in \xi, a_i \otimes x \neq 0\}, \\ L_\xi(x) &= \vee \{a_i \mid a_i \in \xi, a_i \longrightarrow x \neq 1\} \longrightarrow 0, \end{aligned} \tag{5}$$

are called a lower approximation and an upper approximation based on partition ξ, respectively. If $H_\xi(x) = L_\xi(x)$ then x is called a definable element with respect to (M, ξ) else x is called a rough element. If no confusion arisen then the operations H_ξ and L_ξ can be abbreviated as H and L, respectively.

Remark 12. Let M be a Boolean algebra of some subsets of a nonempty set U and let ξ be a partition of U in the usual sense. In this case, the conditions $a_i \otimes x \neq 0$ and $a_i \rightarrow x \neq 1$ in Definition 11 identity with that $a_i \cap x \neq \emptyset$ and $a_i \subseteq x$ do not hold in the usual set meaning, respectively. Hence,

$$\begin{aligned} H(x) &= \vee \{a_i \mid a_i \in \xi, a_i \otimes x \neq 0\} \\ &= \cup \{a_i \mid a_i \in \xi, a_i \cap x \neq \emptyset\}, \\ L(x) &= \vee \{a_i \mid a_i \in \xi, a_i \longrightarrow x \neq 1\} \longrightarrow 0 \\ &= M - \cup \{a_i \mid a_i \in \xi, a_i \subseteq x \text{ does not hold}\} \\ &= \cup \{a_i \mid a_i \in \xi, a_i \subseteq x\}. \end{aligned} \tag{6}$$

This means that the operations H and L introduced for MV-algebras in Definition 11 are extensions of the operations H and L in the typical set notations, respectively.

Theorem 13. *If we denote $\neg\xi = \{\neg a_1, \neg a_2, \ldots, \neg a_n\}$, then*

$$L(x) = \wedge \{a_i \mid a_i \in \neg\xi, x \oplus a_i \neq 1\}. \tag{7}$$

Proof. Consider the following:

$$\begin{aligned} L(x) &= \vee\{a_i \mid a_i \in \xi, a_i \longrightarrow x \neq 1\} \longrightarrow 0 \\ &= \wedge\{a_i \longrightarrow 0 \mid a_i \in \xi, a_i \longrightarrow x \neq 1\} \\ &= \wedge\{\neg a_i \mid a_i \in \xi, x \oplus \neg a_i \neq 1\} \\ &= \wedge\{\neg a_i \mid \neg a_i \in \neg\xi, x \oplus \neg a_i \neq 1\} \\ &= \wedge\{a_i \mid a_i \in \neg\xi, x \oplus a_i \neq 1\}. \end{aligned} \tag{8}$$

□

Theorem 14. *Let* $\xi = \{a_1, a_2, \ldots, a_n\}$ *be a partition of MV-algebra* M *and* L *and* H *the lower and upper approximations induced by* ξ*, respectively. Then*

(1) $L(1) = 1$, $H(0) = 0$;

(2) $\forall a_i \in \xi$, $L(a_i) = a_i$, $H(a_i) = a_i$;

(3) *if* $x \leq y$ *then* $L(x) \leq L(y)$, $H(x) \leq H(y)$;

(4) $L(x) = \neg(H(\neg x))$;

(5) $L(x) \leq x \leq H(x)$;

(6) $L(x \wedge y) = L(x) \wedge L(y)$, $H(x) \vee H(y) = H(x \vee y)$;

(7) $H(H(x)) = H(x)$, $L(L(x)) = L(x)$;

(8) $H(L(x)) = L(x)$, $L(H(x)) = H(x)$.

Proof. The proof of (1), (2), and (3) is obvious as follows.

(4) $\neg(H(\neg x)) = \vee\{a_i \mid a_i \in \xi, a_i \otimes \neg x \neq 0\} \rightarrow 0 = \vee\{a_i \mid a_i \in \xi, a_i \rightarrow x \neq 1\} \rightarrow 0 = L(x)$.

(5) $x = x \otimes \vee\{a_i \mid a_i \in \xi\} = \vee\{a_i \otimes x \mid a_i \in \xi\} = \vee\{a_i \otimes x \mid a_i \in \xi, a_i \otimes x \neq 0\} \leq \vee\{a_i \mid a_i \in \xi, a_i \otimes x \neq 0\} = H(x)$.

By the duality of L and H, we have $L(x) \leq x \leq H(x)$.

(6) By (3) we have $H(x) \vee H(y) \leq H(x \vee y)$. For any $a_i \in \xi$, if $a_i \otimes (x \vee y) \neq 0$ then it follows from $a_i \otimes (x \vee y) = (a_i \otimes x) \vee (a_i \otimes y)$ that $a_i \otimes x \neq 0$ or $a_i \otimes y \neq 0$. Hence,

$$\begin{aligned} \{a_i \mid a_i \in \xi, a_i \otimes (x \vee y) \neq 0\} &\subseteq \{a_i \mid a_i \in \xi, a_i \otimes x \neq 0\} \\ &\quad \cup \{a_i \mid a_i \in \xi, a_i \otimes y \neq 0\}. \end{aligned} \tag{9}$$

Thus,

$$\begin{aligned} \vee\{a_i \mid a_i \in \xi, a_i \otimes (x \vee y) \neq 0\} &\leq (\vee\{a_i \mid a_i \in \xi, a_i \otimes x \neq 0\}) \\ &\quad \vee (\vee\{a_i \mid a_i \in \xi, a_i \otimes y \neq 0\}). \end{aligned} \tag{10}$$

This shows that $H(x \vee y) \leq H(x) \vee H(y)$. Therefore, $H(x) \vee H(y) = H(x \vee y)$.

By the duality of L and H we know that another equation holds.

(7) By (3) and (5) we have $H(x) \leq H(H(x))$. For any $a_j \in \xi$,

$$\begin{aligned} a_j \otimes H(x) &= a_j \otimes \vee\{a_i \mid a_i \in \xi, a_i \otimes x \neq 0\} \\ &= \vee\{a_j \otimes a_i \mid a_i \in \xi, a_i \otimes x \neq 0\} \neq 0, \end{aligned} \tag{11}$$

if and only if there exists $a_{i_0} \in \xi$ and $a_{i_0} \otimes x \neq 0$ such that $a_j \otimes a_{i_0} \neq 0$. Hence, $\{a_j \in \xi, a_j \otimes H(x) \neq 0\} \subseteq \{a_i \in \xi, a_i \otimes x \neq 0\}$. It follows that

$$\begin{aligned} H(H(x)) &= \vee\{a_j \mid a_j \in \xi, a_j \otimes H(x) \neq 0\} \\ &\leq \vee\{a_i \mid a_i \in \xi, a_i \otimes x \neq 0\} = H(x). \end{aligned} \tag{12}$$

This shows that $H(H(x)) = H(x)$.

(8) By (3) we have $L(x) \leq H(L(x))$. If $a_{j_0} \in \{a_j \mid a_j \in \xi, a_j \otimes L(x) \neq 0\}$, then $a_{j_0} \otimes L(x) = a_{j_0} \otimes \wedge\{\neg a_i \mid a_i \in \xi, a_i \rightarrow x \neq 1\} \neq 0$. It follows from $a_{j_0} \otimes \neg a_{j_0} = 0$ that $\neg a_{j_0} \notin \{\neg a_i \mid a_i \in \xi, a_i \rightarrow x \neq 1\}$. Since ξ is a partition we have $a_{j_0} \leq \neg a_i$ $(i \neq j_0)$. Hence, $a_{j_0} \leq \wedge\{\neg a_i \mid a_i \in \xi, a_i \rightarrow x \neq 1\}$. This means that $\vee\{a_j \mid a_j \in \xi, a_j \otimes L(x) \neq 0\} \leq \wedge\{\neg a_i \mid a_i \in \xi, a_i \rightarrow x \neq 1\}$. This shows that $H(L(x)) \leq L(x)$. Therefore, $H(L(x)) = L(x)$. □

Example 15. Let the MV-algebra M and the partition of M be as defined in Example 9. Then $H(b) = 1$, $H(d) = a$; $L(b) = c$, $L(d) = 0$.

Example 16. The MV-algebra $M \times B$ and the partition $\xi = \{(a, 0), (c, 0), (0, 1)\}$ of $M \times B$ as defined in Example 10. Then $H((1, 0)) = (1, 0)$, $H((a, 1)) = (a, 1)$, $H((b, 1)) = (1, 1)$, $H((b, 0)) = (1, 0)$, $H((c, 1)) = (c, 1)$, $H((d, 1)) = (a, 1)$, $H((d, 0)) = (a, 0)$; $L((1, 0)) = (1, 0)$, $L((a, 1)) = (a, 1)$, $L((b, 1)) = (c, 1)$, $L((b, 0)) = (c, 0)$, $L((c, 1)) = (c, 1)$, $L((d, 1)) = (0, 1)$, $L((d, 0)) = (0, 0)$.

By Definition 11 we know that every element in ξ is definable and the definable elements of M can also be obtained by the following theorem.

Theorem 17. *Let* M *be an MV-algebra and* ξ *a partition of* M*. The definable elements of* M *with respect to* (M, ξ) *are*

$$\begin{aligned} L^* &= \{x \in M \mid \forall a_i \in \xi, a_i \otimes x \neq 0 \text{ implies } a_i \leq x\} \\ &= \{x \in M \mid \forall a_i \in \xi, a_i \otimes x = 0 \text{ or } a_i \leq x\}, \end{aligned} \tag{13}$$

and L^* *forms a Boolean algebra.*

Proof. At first, we prove that the definable elements set with respect to (M, ξ) are L^*. Suppose that x is a definable element. Then $x = H(x) = \vee\{a_i \mid a_i \in \xi, a_i \otimes x \neq 0\}$. $\forall a_i \in \xi$, if $a_i \otimes x \neq 0$ then $a_i \leq H(x) = x$. This means that $x \in L^*$. Conversely, let $x \in L^*$. Then $a_i \otimes x \neq 0$ implies $a_i \leq x$. It follows that $H(x) = \vee\{a_i \mid a_i \in \xi, a_i \otimes x \neq 0\} \leq x$. By Theorem 14(4) we have $H(x) = x$.

Since x is a definable element we also know $x = L(x) = \wedge\{a_i \mid a_i \in \neg\xi, x \oplus a_i \neq 1\}$. Hence, for any $a_i \in \neg\xi$, if $x \oplus a_i \neq 1$ then $x \leq a_i$. Let $a_i \in \xi$ and $a_i \otimes x \neq 0$. Then $\neg a_i \oplus \neg x \neq 1$. Hence, $\neg x \leq \neg a_i$; that is $a_i \leq x$. This means that $x \in L^*$. Conversely, let $x \in L^*$. Then $a_i \otimes x \neq 0$ implies $a_i \leq x$. If $a_i \in \neg\xi$ and $x \oplus a_i \neq 1$, then $\neg a_i \otimes \neg x \neq 0$. Hence, $\neg a_i \leq x$ does not hold. By $\neg a_i \in \xi$ and the definition of L^* we know $\neg a_i \otimes x = 0$; hence, $x \leq a_i$. This shows that $x \leq \wedge\{a_i \mid a_i \in \neg\xi, x \oplus a_i \neq 1\} = L(x)$. By Theorem 14(4) we have $L(x) = x$.

This shows that the definable elements set with respect to (M, ξ) are L^*.

Next we prove that L^* forms a Boolean algebra. Let $x_1, x_2 \in L^*$. For any $a_i \in \xi$, if $a_i \otimes (x_1 \wedge x_2) = (a_i \otimes x_1) \wedge$

$(a_i \otimes x_2) \neq 0$, then $a_i \otimes x_1 \neq 0$ and $a_i \otimes x_2 \neq 0$. By the definition of L^* we know $a_i \leq x_1$ and $a_i \leq x_2$; that is $a_i \leq x_1 \wedge x_2$. Hence, $x_1 \wedge x_2 \in L^*$. This shows that L^* is closed under operation $\wedge$. If $a_i \otimes (x_1 \vee x_2) = (a_i \otimes x_1) \vee (a_i \otimes x_2) \neq 0$, then $a_i \otimes x_1 \neq 0$ or $a_i \otimes x_2 \neq 0$. By the definition of L^* we know $a_i \leq x_1$ or $a_i \leq x_2$; that is $a_i \leq x_1 \vee x_2$. Hence, $x_1 \vee x_2 \in L^*$. This shows that L^* is closed under operation $\vee$. This shows that L^* forms a sublattice of M.

By Theorem 14 we know $0, 1 \in L^*$. Let $x \in L^*$. Then $\forall a_i \in \xi$, $a_i \otimes x = 0$ or $a_i \leq x$. This means that $\forall a_i \in \xi$, $a_i \otimes x = 0$ or $a_i \otimes \neg x = 0$. It follows from $\forall a_i \in \xi, a_i \otimes \neg x = 0$ or $a_i \otimes \neg\neg x = a_i \otimes x = 0$ that $\neg x \in L^*$. Since $\forall a_i \in \xi, a_i \otimes x = 0$ or $a_i \otimes \neg x = 0$, we have $\forall a_i \in \xi, (a_i \otimes x) \wedge (a_i \otimes \neg x) = a_i \otimes (x \wedge \neg x) = 0$. Hence, $0 = \bigvee_{i=1}^{n} (a_i \otimes (x \wedge \neg x)) = (\bigvee_{i=1}^{n} a_i) \otimes (x \wedge \neg x) = 1 \otimes (x \wedge \neg x) = x \wedge \neg x$. Analogously, $1 = x \vee \neg x$.

This shows that L^* is a Boolean algebra. □

4. Approximate Operations and Uncertainty Measures

In this section, we will discuss the relationship between rough operations and uncertainty measures. Given an MV-algebra, we may only know the measures of some elements when information is absent in an MV-algebra. For those elements that we do not know the measure, what we can do is to define belief measure and plausibility measure on them.

Definition 18. The function $m : M \to [0, 1]$ such that

(1) $m(0) = 0$,

(2) $m(1) = 1$,

(3) m is finitely additive, that is, if $\xi = \{a_1, a_2, \ldots, a_n\}$ is $\oplus$-orthogonal then $m(\bigoplus_{k=1}^{n} a_k) = \sum_{k=1}^{n} m(a_k)$

is called a finitely additive measure on MV-algebra M.

For finitely additive measure, we have the following conclusion.

Theorem 19 (see [19, 20]). *If m is a finitely additive measure on the MV-algebra M then*

(1) *if $a \leq b$ then $m(b \ominus a) = m(b) - m(a)$, where $b \ominus a = b \otimes \neg a$;*

(2) *if $a \leq b$ then $m(a) \leq m(b)$;*

(3) *$m(a) + m(b) = m(a \oplus b) + m(a \otimes b)$;*

(4) *$m(a) + m(b) = m(a \vee b) + m(a \wedge b)$.*

Example 20. Let M be an MV-algebra and $[0, 1]$ the MV-unit interval. If a mapping $\nu : M \to [0, 1]$ is a homomorphism of type $(\neg, \oplus)$, that is, $\nu(\neg a) = \neg \nu(a)$, $\nu(a \oplus b) = \nu(a) \oplus \nu(b)$, for any $a, b \in M$, then ν is called a Łukasiewicz-valuation (see [21]). Then it is easy to prove that the Łukasiewicz-valuation ν is a finitely additive measure.

Example 21. Let M be an MV-algebra, let $\xi = \{a_1, a_2, \ldots, a_n\}$ be a partition of M, let M_0 be the smallest subalgebra of M containing ξ, and let Ω_0 be the set of all homomorphisms of type $(\oplus, \neg)$ from M_0 to the MV-unit interval $[0, 1]$. An element x in M_0 can be viewed as a function from Ω_0 to $[0, 1]$; that is, for any $x \in M_0$, a function $x : \Omega_0 \to [0, 1]$, $x(\nu) = \nu(x)$ can be defined. Suppose that $(\Omega_0, \mathscr{A}, \mu)$ is a probability measure space, which satisfies the fact that x is a $\mathscr{A}$-measurable function on Ω_0; that is, $\forall \alpha \in [0, 1], \{\nu \in \Omega_0 \mid x(\nu) \geq \alpha\} \in \mathscr{A}$ for every $x \in M_0$. In this case, the element x can also be viewed as a random variable from Ω_0 to $[0, 1]$ (see [21]). Hence, we can define $m(x) = \int_{\Omega_0} x d\mu$ for every $x \in M_0$. It is easy to prove that $m : M_0 \to [0, 1]$ is a finitely additive measure.

Example 22. Let M be an MV-algebra. A state m on a σ-complete MV-algebra M is a mapping $m : M \to [0, 1]$ such that for all $a, b, a_n \in M$:

(1) $m(1) = 1$,

(2) if $a \otimes b = 0$ then $m(a \oplus b) = m(a) + m(b)$,

(3) if $a_n \nearrow a$ then $m(a_n) \nearrow m(a)$,

where $a_n \nearrow a$ stands for a_n is a nondecreasing sequence and $a = \bigvee_{n \in N} a_n$ (see [20, 21]). Then a state m is a finitely additive measure by the following Lemma 23.

Lemma 23. *For any $\oplus$-orthogonal subset $\xi = \{a_1, a_2, \ldots, a_n\}$ and any state m of M, it holds that*

$$m\left(\bigoplus_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} m(a_i). \quad (14)$$

Proof. Since $\xi = \{a_1, a_2, \ldots, a_n\}$ is $\oplus$-orthogonal we know $(\bigoplus_{i \in I} a_i) \otimes a_j = 0$. From the definition of a state, we have inductively

$$m\left(\bigoplus_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} m(a_i). \quad (15)$$

□

Proposition 24. *Let $\xi = \{a_1, a_2, \ldots, a_n\}$ be a partition of MV-algebra M and let M_0 be the Boolean algebra defined in Theorem 7. If function $m : \xi \to [0, 1]$ satisfies $\sum_{i=1}^{n} m(a_i) = 1$ then we can extend m to M_0 by defining $m(0) = 0$, $m(x) = m(\bigoplus_{k \in I_x} a_k) = \sum_{k \in I_x} m(a_k)$, for every $x = \bigoplus_{k \in I_x} a_k \in M_0$, and m is a finitely additive measure on M_0.*

Proof. Obviously $m(0) = 0$ and $m(1) = 1$. For $x, y \in M_0$ then there are $I_x, I_y \subseteq \{1, 2, \ldots, n\}$ such that $x = \bigoplus_{k \in I_x} a_k$ and $y = \bigoplus_{k \in I_y} a_k$. If $x \otimes y = 0$ then we assert $I_x \cap I_y = \emptyset$. In fact, if $I_x \cap I_y \neq \emptyset$ then there is $i_0 \in I_x \cap I_y$ such that $x \otimes y \geq a_{i_0} \otimes a_{i_0} \neq 0$. Hence, $m(x \oplus y) = m(\bigoplus_{k \in I_x} a_k \oplus \bigoplus_{k \in I_y} a_k) = m(\bigoplus_{k \in I_x \cup I_y} a_k) = \sum_{k \in I_x \cup I_y} m(a_k)$. This shows that m is a finitely additive measure on M_0. □

Definition 25. (1) A function Bel : $M \to [0,1]$ is called a belief measure if Bel(0) = 0, Bel(1) = 1, and

$$\mathrm{Bel}\left(x_1 \vee x_2 \vee \cdots \vee x_l\right) \geq \sum\left\{(-1)^{|J|+1}\mathrm{Bel}\left(\bigwedge_{j\in J} x_j\right) \mid \emptyset \neq J \subseteq \{1,2,\ldots,l\}\right\}, \tag{16}$$

for every positive integer l and for every l-tuple $x_1, x_2, \ldots, x_l$, of subsets of M.

(2) A function Pl : $M \to [0,1]$ is called a plausibility measure if Pl(0) = 0, Pl(1) = 1, and

$$\mathrm{Pl}\left(x_1 \wedge x_2 \wedge \cdots \wedge x_l\right) \leq \sum\left\{(-1)^{|J|+1}\mathrm{Pl}\left(\bigvee_{j\in J} x_j\right) \mid \emptyset \neq J \subseteq \{1,2,\ldots,l\}\right\}, \tag{17}$$

for every positive integer l and for every l-tuple $x_1, x_2, \ldots, x_l$, of subsets of M.

Theorem 26. *Suppose that $\xi = \{a_1, a_2, \ldots, a_n\}$ is a partition of MV-algebra M and function $m : \xi \to [0,1]$ satisfies $\sum_{i=1}^{n} m(a_i) = 1$. By using function m we define a function $Pl : M \to [0,1]$ on M as follows:*

$$Pl(x) = \sum\left\{m(a_i) \mid a_i \in \xi, a_i \otimes x \neq 0\right\}, \quad x \in M. \tag{18}$$

Then Pl is a plausibility measure on M.

Proof. It is easy to check Pl(0) = 0, Pl(1) = 1, and $0 \leq \mathrm{Pl}(x) \leq 1$. For any $x_1, x_2, \ldots, x_l \in M$, it is easy to check

$$\begin{aligned}\left\{a_i \mid a_i \otimes \left(\bigwedge_{j=1}^{l} x_j\right) \neq 0\right\} &= \left\{a_i \mid \bigwedge_{j=1}^{l}\left(a_i \otimes x_j\right) \neq 0\right\} \\ &\subseteq \bigcap_{j=1}^{l}\left\{a_i \mid a_i \otimes x_j \neq 0\right\}.\end{aligned} \tag{19}$$

By the well-known inclusion-exclusion formulas in probability theory we have

$$\begin{aligned}&\mathrm{Pl}\left(\bigwedge_{j=1}^{l} x_j\right) \\ &= \sum\left\{m(a_i) \mid a_i \otimes \left(\bigwedge_{j=1}^{l} x_j\right) \neq 0\right\} \\ &\leq \sum\left\{m(a_i) \mid a_i \in \bigcap_{j=1}^{l}\left\{a_i \mid a_i \otimes x_j \neq 0\right\}\right\} \\ &= \sum\Big\{(-1)^{|J|+1} \\ &\qquad \times \left(\sum\left\{m(a_i) \mid \exists j \in J, a_i \otimes x_j \neq 0\right\}\right) \\ &\qquad \mid \emptyset \neq J \subseteq \{1,2,\ldots,l\}\Big\} \\ &= \sum\Big\{(-1)^{|J|+1} \\ &\qquad \times \left(\sum\left\{m(a_i) \mid \bigvee_{j\in J}\left(a_i \otimes x_j\right) \neq 0\right\}\right) \\ &\qquad \mid \emptyset \neq J \subseteq \{1,2,\ldots,l\}\Big\} \\ &= \sum\left\{(-1)^{|J|+1}\mathrm{Pl}\left(\bigvee_{j\in J} x_j\right) \mid \emptyset \neq J \subseteq \{1,2,\ldots,l\}\right\}.\end{aligned} \tag{20}$$

This shows that Pl is a plausibility measure on M. □

Theorem 27. *Suppose that $\xi = \{a_1, a_2, \ldots, a_n\}$ is a partition of MV-algebra M and function $m : \xi \to [0,1]$ satisfies $\sum_{i=1}^{n} m(a_i) = 1$. By using function m we define a function $Pl : M \to [0,1]$ on M as follows:*

$$Bel(x) = 1 - \sum\left\{m(a_i) \mid a_i \in \xi, a_i \longrightarrow x \neq 1\right\}. \tag{21}$$

Then Bel is a belief measure on M.

Proof. It is easy to check Bel(0) = 0, Bel(1) = 1, and $0 \leq \mathrm{Bel}(1) \leq 1$. For any $x_1, x_2, \ldots, x_l \in M$, it is easy to check

$$\begin{aligned}\left\{a_i \mid a_i \longrightarrow \bigvee_{j=1}^{l} x_j \neq 1\right\} &= \left\{a_i \mid \bigvee_{j=1}^{l}\left(a_i \longrightarrow x_j\right) \neq 1\right\} \\ &\subseteq \bigcap_{j=1}^{l}\left\{a_i \mid a_i \longrightarrow x_j \neq 1\right\}.\end{aligned} \tag{22}$$

By the well-known inclusion-exclusion formulas in probability theory we have

$$\begin{aligned}&\mathrm{Bel}\left(\bigvee_{j=1}^{l} x_i\right) \\ &= 1 - \sum\left\{m(a_i) \mid a_i \longrightarrow \bigvee_{j=1}^{l} x_i \neq 1\right\} \\ &\geq 1 - \sum\left\{m(a_i) \mid a_i \in \bigcap_{j=1}^{l}\left\{a_i \mid a_i \longrightarrow x_j \neq 1\right\}\right\} \\ &= 1 - \sum\Big\{(-1)^{|J|+1} \\ &\qquad \times \left(\sum\left\{m(a_i) \mid \exists j \in J, a_i \longrightarrow x_j \neq 1\right\}\right) \\ &\qquad \mid \emptyset \neq J \subseteq \{1,2,\ldots,l\}\Big\}\end{aligned}$$

$$= 1 - \sum \Big\{ (-1)^{|J|+1} \times \Big(\sum \Big\{ m(a_i) \mid a_i \longrightarrow \bigwedge_{j\in J} x_j \neq 1 \Big\} \Big) \mid \emptyset \neq J \subseteq \{1,2,\ldots,l\} \Big\}$$
$$= 1 - \sum \Big\{ (-1)^{|J|+1} \times \Big(1 - \mathrm{Bel}\Big(\bigwedge_{j\in J} x_j \Big) \Big) \mid \emptyset \neq J \subseteq \{1,2,\ldots,l\} \Big\}$$
$$= 1 - \sum \{ (-1)^{|J|+1} \mid \emptyset \neq J \subseteq \{1,\ldots,l\} \} + \sum \Big\{ (-1)^{|J|+1} \mathrm{Bel}\Big(\bigwedge_{j\in J} x_j \Big) \mid \emptyset \neq J \subseteq \{1,\ldots,l\} \Big\}$$
$$= \sum \Big\{ (-1)^{|J|+1} \mathrm{Bel}\Big(\bigwedge_{j\in J} x_j \Big) \mid \emptyset \neq J \subseteq \{1,2\ \ldots,l\} \Big\}. \tag{23}$$

This shows that Bel is a belief measure on M. □

Let M be an MV-algebra and $\xi = \{a_1, a_2, \ldots, a_n\}$ a partition of M. Suppose that function $m : \xi \to [0,1]$ satisfies $\sum_{i=1}^{n} m(a_i) = 1$. Then we can extend m to M by defining

$$m_*(x) = \mathrm{Bel}(L(x)), \qquad m^*(x) = \mathrm{Pl}(H(x)), \tag{24}$$

where $x \in M$, $L(x)$, and $H(x)$ are lower and upper approximations of x, respectively. The values $m_*(x)$ and $m^*(x)$ of an element x of M can be viewed as our best estimate of the *true* measures of x, given our lack of knowledge. Moreover, we can get the following theorem.

Theorem 28. *m_* and m^* defined above are belief measure and plausibility measure on M, respectively.*

Proof. (1) We prove $\mathrm{Pl}(H(x)) = \mathrm{Pl}(x)$. Note that $\mathrm{Pl}(H(x)) = \sum\{m(a_i) \mid a_i \in \xi, a_i \otimes H(x) \neq 0\}$. In the following we prove that $a_i \otimes H(x) \neq 0$ if and only if $a_i \otimes x \neq 0$. In fact, $a_i \otimes H(x) = a_i \otimes \vee\{a_j \mid a_j \in \xi, a_j \otimes x \neq 0\} = \vee\{a_i \otimes a_j \mid a_j \in \xi, a_j \otimes x \neq 0\} = \{a_i \otimes a_i \mid a_i \otimes x \neq 0\}$. This shows that $a_i \otimes H(x) \neq 0$ if and only if $a_i \otimes x \neq 0$. Hence, $m^*(x) = \mathrm{Pl}(H(x)) = \sum\{m(a_i) \mid a_i \in \xi, a_i \otimes H(x) \neq 0\} = \sum\{m(a_i) \mid a_i \in \xi, a_i \otimes x \neq 0\} = \mathrm{Pl}(x)$. It follows from Theorem 26 that m^* is a plausibility measure on M.

(2) We prove $\mathrm{Bel}(L(x)) = \mathrm{Bel}(x)$. Note that $\mathrm{Bel}(L(x)) = 1 - \sum\{m(a_i) \mid a_i \in \xi, a_i \to L(x) \neq 1\}$. In the following we prove that $a_i \to L(x) \neq 1$ if and only if $a_i \to x \neq 1$. In fact, $a_i \to L(x) = a_i \to (\vee\{a_j \mid a_j \in \xi, a_j \to x \neq 1\} \to 0) = a_i \otimes \vee\{a_j \mid a_j \in \xi, a_j \to x \neq 1\} \to 0 = \vee\{a_i \otimes a_j \mid a_j \in \xi, a_j \to x \neq 1\} \to 0 = \{a_i \otimes a_i \mid a_i \to x \neq 1\} \to 0$. This shows that $a_i \to L(x) \neq 1$ if and only if $a_i \to x \neq 1$. Hence, $m_*(x) = \mathrm{Bel}(H(x)) = 1 - \sum\{m(a_i) \mid a_i \in \xi, a_i \to L(x) \neq 1\} = 1 - \sum\{m(a_i) \mid a_i \in \xi, a_i \to x \neq 1\} = \mathrm{Bel}(x)$. It follows from Theorem 27 that m_* is a belief measure on M. □

Example 29. The MV-algebra $M \times B$ and the partition $\xi = \{(a,0), (c,0), (0,1)\}$ of $M \times B$ as defined in Example 10. Suppose that function $m : \xi \to [0,1]$, $m((a,0)) = 1/4$, $m((c,0)) = 1/4$, $m((0,1)) = 1/2$. The uncertainty measures Pl, Bel, m^*, m_* on M with respect to ξ are shown in Table 3.

5. Conclusion

In this paper, a pair of dual rough operations on MV-algebras is introduced. The properties of rough operations and the relationship between rough operations and uncertainty measures are discussed. If information is absent in an MV-algebra and we may only know the measures of some elements, then what we can do is to define belief measure and plausibility measure on MV-algebra, which is used to interpret some uncertainty measures on MV-algebras.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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