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Research Article

Positive Solutions for Systems of Nonlinear Higher Order Differential Equations with Integral Boundary Conditions

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By constructing some general type conditions and using fixed point theorem of cone, this paper investigates the existence of at least one and at least two positive solutions for systems of nonlinear higher order differential equations with integral boundary conditions. As application, some examples are given.

1. Introduction

In this paper, we consider the following systems of nonlinear mixed higher order differential equations with integral boundary conditions:

$$\begin{gathered}
u^{(n_1)}(t) + a_1(t) f_1(t, u(t), v(t)) = 0, \quad t \in (0,1), \\
v^{(n_2)}(t) + a_2(t) f_2(t, u(t)) = 0, \quad t \in (0,1), \\
u(0) = u'(0) = \cdots = u^{(n_1-2)}(0) = 0, \\
u(1) = \int_0^1 n_1(t) u(t)\, dt, \\
v(0) = v'(0) = \cdots = v^{(n_2-2)}(0) = 0, \\
v(1) = \int_0^1 n_2(t) v(t)\, dt,
\end{gathered} \tag{1}$$

where $f_1 \in C([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty))$, $f_2 \in C([0,1] \times [0,+\infty), [0,+\infty))$, $a_i \in C([0,1],[0,+\infty))$, $n_i \geq 3$, and $n_i(t) \in L^1[0,1]$ is nonnegative, $i = 1,2$; $f_1(t,0,0) \equiv f_2(t,0) \equiv 0$.

Boundary value problems with integral boundary conditions arise naturally in thermal conduction problems [1], semiconductor problems [2], and hydrodynamic problems [3]. Such problems include two-, three-, and multipoint boundary value problems as special cases and attracted much attention (see [4–12] and the references therein). In particular, we would like to mention the result of Pang et al. [9]. In [9], by applying fixed point index theory, Pang et al. study the expression and properties of Green's function and obtained the existence of positive solutions for nth-order m-point boundary value problems:

$$\begin{gathered}
u^{(n)}(t) + a(t) f(u(t)) = 0, \quad t \in (0,1), \\
u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\
u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i).
\end{gathered} \tag{2}$$

Yang and Wei [10], Feng and Ge [11], and Li and Wei [12] improved and generalized the results of [9] by using different methods.

On the other hand, much effort has been devoted to the study of the existence of positive solutions for systems of nonlinear differential equations (see [13–16] and the references therein). In [13], by applying Krasnoselskii fixed point theorem in a cone, Hu and Wang obtained multiple positive solutions of boundary value problems for systems of nonlinear second-order differential equations. In [14], Henderson and Ntouyas extended the results of [13] to systems of nonlinear nth-order three-point boundary value problems:

$$u^{(n)}(t)+\lambda a(t) f(t, v(t))=0, \quad t \in(0,1),$$
$$v^{(n)}(t)+\lambda b(t) h(t, u(t))=0, \quad t \in(0,1),$$
$$u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \quad u(1)=\alpha u(\eta),$$
$$v(0)=v'(0)=\cdots=v^{(n-2)}(0)=0, \quad v(1)=\alpha v(\eta). \tag{3}$$

In [15], by using fixed point index theory, Xie and Zhu improved the results of [14]. At the same time, boundary value problems with integral boundary conditions have received attention [16, 17].

Motivated by the work of the abovementioned papers, our aim in this paper is to study the existence of positive solutions associated with systems (1) by applying fixed point theorem in cone. Further, we present some general type conditions (H_4)–(H_7) instead of the sublinear or superlinear conditions which are used in [4, 5, 8, 10, 12–14]. Our conditions are applicable for more general functions.

2. Several Lemmas

For convenience, we make the following notations. Let

$$\beta_i=\int_0^1 n_i(t) t^{n_i-1} dt,$$
$$\mu_i=\int_0^1 K_i(s) a_i(s) ds, \tag{4}$$
$$\delta_i=\int_a^b K_i(s) a_i(s) ds,$$
$$i=1,2,$$

where $K_i(s)$ is defined by Lemma 6 and $[a,b]$ is some subset of $(0,1)$.

List the following assumptions:

(H_1) $a_i \in C([0,1],[0,+\infty))$, $a_i(t)$ do not vanish identically for $t \in [a,b]$, $i=1,2$;

(H_2) $f_1 \in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$, $f_2 \in C([0,1]\times[0,+\infty),[0,+\infty))$;

(H_3) $\beta_1, \beta_2 \in [0,1)$;

(H_4) there exist $\alpha \in (0,1]$, $\lambda_1>0$ and a sufficiently large $M_1>1$ such that

(1) $f_1(t,u,v) \geq \lambda_1 v^{\alpha}$, for all $(t,u,v) \in [0,1]\times[0,+\infty)\times[M_1,+\infty)$,

(2) $f_2(t,u) \geq C_1 u^{1/\alpha}$, for all $(t,u) \in [0,1]\times[M_1,+\infty)$,

where $C_1=\max\{(\gamma\delta_2)^{-1},(\gamma\delta_2)^{-1}(\gamma^2\lambda_1\delta_1)^{-1/\alpha}\}$; γ is defined by (21).

(H_5) There exist $\beta \in (0,+\infty)$, $\lambda_2>0$ and a sufficiently small $\rho_2 \in (0,1)$ such that

(1) $f_1(t,u,v) \leq \lambda_2 v^{\beta}$, for all $(t,u,v) \in [0,1]\times[0,+\infty)\times[0,\rho_2]$,

(2) $f_2(t,u) \leq C_2 u^{1/\beta}$, for all $(t,u) \in [0,1]\times[0,\rho_2]$,

where $C_2=\min\{\rho_2\mu_2^{-1},\mu_2^{-1/\beta}(\mu_1\lambda_2)^{-1}\}$.

(H_6) There exist $p \in (0,+\infty)$, $\lambda_3>0$, and $M_2>0$ such that

(1) $f_1(t,u,v) \leq \lambda_3 v^{p}+M_2$, for all $(t,u,v) \in [0,1]\times[0,+\infty)\times[0,+\infty)$,

(2) $f_2(t,u) \leq C_3 u^{1/p}+M_2$, for all $(t,u) \in [0,1]\times[0,+\infty)$,

where $C_3=(2\mu_1\lambda_3)^{-1/p}\mu_2^{-1}$.

(H_7) There exist $q \in (0,1]$, $\lambda_4>0$ and a sufficiently small $\varepsilon>0$ such that

(1) $f_1(t,u,v) \geq \lambda_4 v^{q}$, for all $(t,u,v) \in [0,1]\times[0,+\infty)\times[0,\varepsilon]$,

(2) $f_2(t,u) \geq C_4 u^{1/q}$, for all $(t,u) \in [0,1]\times[0,\varepsilon]$,

where $C_4=\gamma^{-(1/q)(2+q)}(\lambda_4\delta_1)^{-1/q}\delta_2^{-1}$.

(H_8) $f_1(t,u,v)$ and $f_2(t,u)$ are increasing on u, v and there exists $R>0$ such that

$f_1(s,R,\int_0^1 K_2(r)a_2(r)f_2(r,R)dr) < \mu_1^{-1}R$, for all $s,r \in [0,1]$.

Lemma 1. *If $\beta_i \in [0,1)$, for any $y(t) \in C[0,1]$, higher order differential equations*

$$w^{(n_i)}(t)+y(t)=0, \quad t \in(0,1),$$
$$w(0)=w'(0)=\cdots=w^{(n_i-2)}(0)=0, \tag{5}$$
$$w(1)=\int_0^1 n_i(t) w(t) dt$$

have a unique solution

$$w(t)=\int_0^1 K_i(t,s) y(s) ds, \tag{6}$$

where

$$K_i(t,s)=K_{i1}(t,s)+K_{i2}(t,s), \tag{7}$$

$$K_{i1}(t,s)=\frac{1}{(n_i-1)!}\begin{cases} t^{n_i-1}(1-s)^{n_i-1}-(t-s)^{n_i-1}, & 0 \leq s \leq t \leq 1, \\ t^{n_i-1}(1-s)^{n_i-1}, & 0 \leq t \leq s \leq 1,\end{cases} \tag{8}$$

$$K_{i2}(t,s)=\frac{t^{n_i-1}}{1-\beta_i}\int_0^1 n_i(t) K_{i1}(t,s) dt. \tag{9}$$

Proof. By Taylor's formula, we get

$$w(t) = -\frac{1}{(n_i-1)!}\int_0^t (t-s)^{n_i-1} y(s)\,ds + \frac{A}{(n_i-1)!} t^{n_i-1}. \tag{10}$$

Letting $t = 1$ in (10), we have

$$A = (n_i-1)!\,w(1) + \int_0^1 (1-s)^{n_i-1} y(s)\,ds. \tag{11}$$

Substituting $w(1) = \int_0^1 n_i(t)w(t)dt$ and (11) into (10), we obtain

$$\begin{aligned} w(t) &= -\frac{1}{(n_i-1)!}\int_0^t (t-s)^{n_i-1} y(s)\,ds \\ &\quad + \frac{1}{(n_i-1)!}\int_0^1 t^{n_i-1}(1-s)^{n_i-1} y(s)\,ds \\ &\quad + t^{n_i-1}\int_0^1 n_i(s)\,w(s)\,ds \\ &= \frac{1}{(n_i-1)!}\int_0^t \left[t^{n_i-1}(1-s)^{n_i-1} - (t-s)^{n_i-1}\right] y(s)\,ds \\ &\quad + t^{n_i-1}\int_0^1 n_i(s)\,w(s)\,ds \\ &\quad + \frac{1}{(n_i-1)!}\int_t^1 t^{n_i-1}(1-s)^{n_i-1} y(s)\,ds \\ &= \int_0^1 K_{i1}(t,s)\,y(s)\,ds + t^{n_i-1}\int_0^1 n_i(s)\,w(s)\,ds. \end{aligned} \tag{12}$$

Multiplying (12) with $n_i(t)$ and integrating it, we have

$$\begin{aligned} \int_0^1 n_i(t)\,w(t)\,dt &= \int_0^1 n_i(t)\int_0^1 K_{i1}(t,s)\,y(s)\,ds\,dt \\ &\quad + \int_0^1 n_i(t)\,t^{n_i-1}dt \int_0^1 n_i(s)\,w(s)\,ds, \end{aligned} \tag{13}$$

so

$$\begin{aligned} &\int_0^1 n_i(t)\,w(t)\,dt \\ &\quad = \frac{1}{1-\beta_i}\int_0^1 n_i(t)\int_0^1 K_{i1}(t,s)\,y(s)\,ds\,dt. \end{aligned} \tag{14}$$

Substituting (14) into (12), we have

$$\begin{aligned} w(t) &= \int_0^1 K_{i1}(t,s)\,y(s)\,ds \\ &\quad + \frac{t^{n_i-1}}{1-\beta_i}\int_0^1 n_i(t)\int_0^1 K_{i1}(t,s)\,y(s)\,ds\,dt \\ &= \int_0^1 K_{i1}(t,s)\,y(s)\,ds \\ &\quad + \int_0^1 \left(\frac{t^{n_i-1}}{1-\beta_i}\int_0^1 n_i(t)\,K_{i1}(t,s)\,dt\right) y(s)\,ds \\ &= \int_0^1 \left[K_{i1}(t,s) + K_{i2}(t,s)\right] y(s)\,ds \\ &= \int_0^1 K_i(t,s)\,y(s)\,ds, \end{aligned} \tag{15}$$

where $K_i(t,s)$ is defined by (7). □

Definition 2. $(u,v) \in C^{n_1}(0,1)\cap C[0,1]\times C^{n_2}(0,1)\cap C[0,1]$ is said to be a positive solution of systems (1) if and only if (u,v) satisfies systems (1) and $u(t) > 0$, $v(t) > 0$, for any $t \in [0,1]$.

Lemma 3 (see [6]). *If $\beta_i \in [0,1)$, the continuous function $K_{i1}(t,s), i = 1,2$, has the following properties:*

(i) $0 \le K_{i1}(t,s) \le K_{i1}(s)$, *for all* $t,s \in [0,1]$, *where* $K_{i1}(s) := s(1-s)^{n_i-1}/(n_i-2)!$;

(ii) $K_{i1}(t,s) \ge \gamma_i(t)K_{i1}(s)$, *for all* $t,s \in [0,1]$, *where* $\gamma_i(t) := (1/(n_i-1))\min\{t^{n_i-1},(1-t)t^{n_i-2}\}$.

Remark 4. Combining (i) and (ii), we can easily see

$$\min_{t\in[a,b]} K_{i1}(t,s) \ge \gamma_i K_{i1}(s) \ge \gamma_i K_{i1}(t,s), \quad \forall t,s \in [0,1], \tag{16}$$

where $\gamma_i = \min\{\gamma_i(t) : t \in [a,b]\}$.

Lemma 5. *If $\beta_i \in [0,1)$, the continuous function $K_{i2}(t,s)$ has the following property:*

$$0 \le K_{i2}(t,s) \le K_{i2}(1,s) := \frac{1}{1-\beta_i}\int_0^1 n_i(t)\,K_{i1}(t,s)\,dt, \quad \forall t,s \in [0,1]. \tag{17}$$

Proof. From the properties of $K_{i1}(t,s)$ and the definition of $K_{i2}(t,s)$, we can prove easily the results of Lemma 5. □

Lemma 6. *If $\beta_i \in [0,1)$, the continuous function $K_i(t,s)$ defined by (7) satisfies*

(i) $K_i(t,s) \ge 0$, *for all* $t,s \in [0,1]$,

(ii) $K_i(t,s) \le K_i(s)$ *for each* $t,s \in [0,1]$, *and* $\min_{t\in[a,b]}K_i(t,s) \ge \gamma_i^* K_i(s)$, *for all* $s \in [0,1]$,

where $\gamma_i^* = \min\{\gamma_i, a^{n_i-1}\}$, γ_i *is defined in Remark 4 and* $K_i(s) = K_{i1}(s) + K_{i2}(1,s)$.

Proof. (1) From Lemma 5 and (i) of Lemma 3, we get the proof of (i) immediately.

(2) From Lemma 5 and (i) of Lemma 3, it is obvious that $K_i(t,s) \le K_i(s)$ for each $t, s \in [0,1]$.

Now, we show that the form (ii) holds. In fact, from (16) and (9), we have

$$\begin{aligned}\min_{t\in[a,b]} K_i(t,s) &\ge \gamma_i K_{i1}(s) + \frac{a^{n_i-1}}{1-\beta_i}\int_0^1 n_i(t) K_{i1}(t,s)\,dt \\ &\ge \gamma_i^*\left[K_{i1}(s) + K_{i2}(1,s)\right] \\ &= \gamma_i^* K_i(s), \quad \forall s \in [0,1].\end{aligned} \tag{18}$$

Then, the proof of Lemma 6 is completed. □

Remark 7. From the definition of γ_i^*, it is obvious that $0 < \gamma_i^* < 1$.

It is easy to prove that $(u,v) \in C^{n_1}(0,1) \cap C[0,1] \times C^{n_2}(0,1) \cap C[0,1]$ is a positive solution of systems (1) if and only if $(u,v) \in C[0,1]\times C[0,1]$ is a positive solution of systems of integral equations

$$\begin{aligned}u(t) &= \int_0^1 K_1(t,s)\,a_1(s)\,f_1(s,u(s),v(s))\,ds, \\ v(t) &= \int_0^1 K_2(t,s)\,a_2(s)\,f_2(s,u(s))\,ds,\end{aligned} \tag{19}$$

where $K_i(t,s)$, $i = 1,2$, are Green's functions defined by (7).

It follows from (19) that we can obtain the integral equation:

$$u(t) = \int_0^1 K_1(t,s)\,a_1(s) \times f_1\left(s, u(s), \int_0^1 K_2(s,r)\,a_2(r)\,f_2(r,u(r))\,dr\right) ds. \tag{20}$$

In a real Banach space $C[0,1]$, the norm is defined by $\|u\| = \max_{t\in[0,1]}|u(t)|$. Set

$$P = \left\{u(t) \in C[0,1] \mid u(t) \ge 0, t \in [0,1], \min_{t\in[a,b]} u(t) \ge \gamma\|u\|\right\}, \tag{21}$$

where $\gamma = \min\{\gamma_1^*, \gamma_2^*\}$. Obviously, P is a positive cone in $C[0,1]$.

Define the operator $T : P \to E$ by

$$Tu(t) = \int_0^1 K_1(t,s)\,a_1(s) \times f_1\left(s, u(s), \int_0^1 K_2(s,r)\,a_2(r)\,f_2(r,u(r))\,dr\right) ds, \quad \forall t \in [0,1]. \tag{22}$$

Lemma 8. *Suppose that* (H_1)*–*(H_3) *are satisfied; then the operator* $T : P \to P$ *is completely continuous.*

Proof. Let $u \in P$; consider (22); from Lemma 3 and (21), we have

$$\begin{aligned}0 &\le Tu(t) \le \|Tu\| \\ &\le \int_0^1 K_1(s)\,a_1(s) \times f_1\left(s, u(s), \int_0^1 K_2(s,r)\,a_2(r)\,f_2(r,u(r))\,dr\right) ds \\ \min_{t\in[a,b]} Tu(t) &\ge \gamma\int_0^1 K_1(s)\,a_1(s) \times f_1\left(s, u(s), \int_0^1 K_2(s,r)\,a_2(r)\,f_2(r,u(r))\,dr\right) ds.\end{aligned} \tag{23}$$

It follows from (23) that we have $\min_{t\in[a,b]}Tu(t) \ge \gamma\|Tu\|$; therefore, operator $T : P \to P$. It is easy to prove that operator $T : P \to P$ is completely continuous since $K_1(t,s)$, $K_2(t,s)$, $f_1(t,u,v)$, $f_2(t,u)$, $a_1(t)$, and $a_2(t)$ are continuous. □

Lemma 9 (see [18]). *Suppose* E *is a real Banach space and* P *is cone in* E*, and let* Ω_1, Ω_2 *be bounded open sets in* E *such that* $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. *Let operator* $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ *be completely continuous. Suppose that one of two conditions holds*

(i) $\|Tu\| \le \|u\|$, *for all* $u \in P \cap \partial\Omega_1$; $\|Tu\| \ge \|u\|$, *for all* $u \in P \cap \partial\Omega_2$;

(ii) $\|Tu\| \ge \|u\|$, *for all* $u \in P \cap \partial\Omega_1$; $\|Tu\| \le \|u\|$, *for all* $u \in P \cap \partial\Omega_2$.

Then, operator T *has at least one fixed point in* $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 10 (see [18]). *Suppose* E *is a real Banach space and* P *is cone in* E*, and let* Ω_1, Ω_2*, and* Ω_3 *be bounded open sets in* E *such that* $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$*, and* $\overline{\Omega}_2 \subset \Omega_3$. *Let operator* $T : P \cap (\overline{\Omega}_3 \setminus \Omega_1) \to P$ *be completely continuous, such that*

(1) $\|Tu\| \ge \|u\|$, *for all* $u \in P \cap \partial\Omega_1$;

(2) $\|Tu\| \le \|u\|$, $Tu \ne u$, *for all* $u \in P \cap \partial\Omega_2$;

(3) $\|Tu\| \ge \|u\|$, *for all* $u \in P \cap \partial\Omega_3$.

Then, operator T *has at least two fixed points* u_1 *and* u_2 *in* $P \cap (\overline{\Omega}_3 \setminus \Omega_1)$ *with* $u_1 \in (\Omega_2 \setminus \Omega_1)$ *and* $u_2 \in (\overline{\Omega}_3 \setminus \overline{\Omega}_2)$.

3. Main Results

Theorem 11. *Suppose that assumptions* (H_1)*–*(H_5) *are satisfied; then systems* (1) *have at least one positive solution* (u,v) *satisfying* $u(t) > 0$, $v(t) > 0$.

Proof. At first, let $\rho_1 = M_1\gamma^{-1}$, and set $\Omega_1 = \{u \in C[0,1] : \|u\| < \rho_1\}$ and $u \in P \cap \partial\Omega_1$; then $\min_{t\in[a,b]} u(t) \geq \gamma\|u\| = M_1$. By Lemma 6 and the assumption (H_4), we have

$$\begin{aligned} v(t) &= \int_0^1 K_2(t,s)\,a_2(s)\,f_2(t,u(s))\,ds \\ &\geq C_1\int_0^1 K_2(t,s)\,a_2(s)\,u^{1/\alpha}(s)\,ds \\ &\geq \gamma C_1\int_a^b K_2(s)\,a_2(s)\,u^{1/\alpha}(s)\,ds \\ &\geq \gamma C_1\int_a^b K_2(s)\,a_2(s)\,ds(\gamma\|u\|)^{1/\alpha} \\ &= \gamma C_1\delta_2 M_1^{1/\alpha} \geq M_1, \quad t\in[a,b], \\ \min_{t\in[a,b]}(Tu)(t) &\geq \gamma\int_0^1 K_1(s)\,a_1(s)\,f_1(s,u(s),v(s))\,ds \\ &\geq \gamma\lambda_1\int_a^b K_1(s)\,a_1(s)\,v^{\alpha}(s)\,ds \\ &\geq \gamma\lambda_1\delta_1(\gamma C_1\delta_2)^{\alpha}(\gamma\|u\|) \geq \|u\|. \end{aligned} \tag{24}$$

Therefore, we have

$$\|Tu\| \geq \|u\|, \quad u\in P\cap\partial\Omega_1. \tag{25}$$

Further, set $\Omega_2 = \{u \in C[0,1] : \|u\| < \rho_2\}$, for $u \in P\cap\partial\Omega_2$; by the assumption ($H_5$), we have

$$\begin{aligned} v(t) &\leq C_2\int_0^1 K_2(s)\,a_2(s)\,u^{1/\beta}(s)\,ds \\ &\leq C_2\mu_2\|u\|^{1/\beta} \\ &\leq \rho_2^{1+(1/\beta)} \leq \rho_2, \quad t\in[0,1], \\ (Tu)(t) &\leq \int_0^1 K_1(s)\,a_1(s)\,f_1(s,u(s),v(s))\,ds \\ &\leq \mu_1\lambda_2\|v\|^{\beta} \leq \mu_1\lambda_2(C_2\mu_2)^{\beta}\|u\| \leq \|u\|. \end{aligned} \tag{26}$$

Therefore, we have

$$\|Tu\| \leq \|u\|, \quad u\in P\cap\partial\Omega_2. \tag{27}$$

Thus, from (25), (27), Lemma 8, and Lemma 9, operator T has a fixed point u in $P\cap(\overline{\Omega}_1\setminus\Omega_2)$. This means that systems (1) have at least one positive solution (u,v) satisfying $u(t) > 0$, $v(t) > 0$. □

Theorem 12. *Suppose that assumptions (H_1)–(H_3) and (H_6)-(H_7) are satisfied; then systems* (1) *have at least one positive solution* (u,v) *satisfying* $u(t) > 0$, $v(t) > 0$.

Proof. At first, it follows from the assumption (H_6) that we have

$$\begin{aligned} &(Tu)(t) \\ &= \int_0^1 K_1(t,s)\,a_1(s)\,f_1(t,u(s),v(s))\,ds \\ &\leq \int_0^1 K_1(s)\,a_1(s)\left(\lambda_3 v^p(s) + M_2\right)ds \\ &\leq \int_0^1 K_1(s)\,a_1(s)\left[\lambda_3\left(\int_0^1 K_2(s)\,a_2(s)\,f_2(s,u(s))\,ds\right)^p + M_2\right]ds \\ &\leq \mu_1\lambda_3\mu_2^p\left(C_3u^{1/p} + M_2\right)^p + \mu_1M_2 \\ &\leq \mu_1\lambda_3\mu_2^p\left(C_3\|u\|^{1/p} + M_2\right)^p + \mu_1M_2. \end{aligned} \tag{28}$$

By means of simple calculation, we have

$$\lim_{u\to+\infty}\frac{\left(\mu_1\lambda_3\mu_2^p\left(C_3\|u\|^{1/p}+M_2\right)^p+\mu_1M_2\right)}{\|u\|} = \frac{1}{2}. \tag{29}$$

Then, there exists a sufficiently large $M > 0$ such that

$$\mu_1\lambda_3\mu_2^p\left(C_3\|u\|^{1/p}+M_2\right)^p+\mu_1M_2 \leq \|u\|. \tag{30}$$

Set $\Omega_3 = \{u \in C[0,1] : \|u\| < M\}$. For $u \in P\cap\partial\Omega_3$, by (28),(30), we obtain that

$$\|Tu\| \leq \|u\|, \quad u\in P\cap\partial\Omega_3. \tag{31}$$

Further, since $f_2(t,0) \equiv 0$ and $f_2(t,u)$ is continuous in $[0,1]\times[0,+\infty)$, there exists $\rho\in(0,\varepsilon)$ such that

$$f_2(t,u) < \mu_2^{-1}\rho, \quad (t,u)\in[0,1]\times(0,\rho). \tag{32}$$

Set $\Omega_4 = \{u \in C[0,1] : \|u\| < \rho\}$. For $u\in P\cap\partial\Omega_4$, we have

$$\begin{aligned} v(t) &= \int_0^1 K_2(t,s)\,a_2(s)\,f_2(t,u(s))\,ds \\ &< \mu_2^{-1}\rho\int_0^1 K_2(s)\,a_2(s)\,ds = \rho. \end{aligned} \tag{33}$$

It follows from the assumption (H_7) and Lemma 6 that we have

$$\begin{aligned} &\min_{t\in[a,b]}(Tu)(t) \\ &\geq \gamma\int_0^1 K_1(s)\,a_1(s)\,f_1(s,u(s),v(s))\,ds \end{aligned}$$

$$\begin{aligned}
&\geq \gamma\lambda_4 \int_a^b K_1(s)\, a_1(s)\, ds \\
&\quad \times \left(\int_a^b K_2(s,r)\, a_2(r)\, f_2(r,u(r))\, dr \right)^q \\
&\geq \gamma\lambda_4\delta_1 \left(\gamma \int_a^b K_2(r)\, a_2(r)\, C_4 u^{1/q}(r)\, dr \right)^q \\
&\geq \gamma^{2+q}\lambda_4\delta_1 (C_4\delta_2)^q \|u\| \geq \|u\|.
\end{aligned} \tag{34}$$

Hence, we have

$$\|Tu\| \geq \|u\|, \quad u \in P \cap \partial\Omega_4. \tag{35}$$

Thus, from (31),(35), Lemmas 8 and 9, operator T has a fixed point u in $P \cap (\overline{\Omega}_3 \setminus \Omega_4)$. This means that systems (1) have at least one positive solution (u, v) satisfying $u(t) > 0$, $v(t) > 0$. □

Theorem 13. *Suppose that assumptions (H_1)–(H_4) and (H_7)-(H_8) hold. Then, systems* (1) *have at least two positive solutions* (u_1, v_1) *and* (u_2, v_2).

Proof. Set $\Omega_5 = \{u \in E : \|u\| < R\}$. For $u \in P \cap \partial\Omega_5$, from ($H_8$), we obtain that

$$\begin{aligned}
&(Tu)(t) \\
&\leq \int_0^1 K_1(s)\, a_1(s)\, f_1\left(s, R, \int_0^1 K_2(r)\, a_2(r)\, f_2(r,R)\, dr\right) ds \\
&< \mu_1^{-1} R \int_0^1 K_1(s)\, a_1(s)\, ds = R.
\end{aligned} \tag{36}$$

Thus, we have

$$\|Tu\| < \|u\|, \quad u \in P \cap \partial\Omega_5. \tag{37}$$

By (H_4) and (H_7), we can get

$$\begin{aligned}
&\|Tu\| \geq \|u\|, \quad u \in P \cap \partial\Omega_1, \\
&\|Tu\| \geq \|u\|, \quad u \in P \cap \partial\Omega_4.
\end{aligned} \tag{38}$$

So, we can choose ρ, R, and ρ_1 such that $\rho < R < \rho_1$ and satisfying the above three inequalities. By Lemma 8 and Lemma 10, we guarantee that operator T has two fixed points $u_1 \in P \cap (\overline{\Omega}_1 \setminus \Omega_5)$ and $u_2 \in P \cap (\overline{\Omega}_5 \setminus \Omega_4)$. This means that systems (1) have at least two positive solutions (u_1, v_1) and (u_2, v_2).

In order to illustrate that our assumptions (H_4)–(H_7) are suitable for more general functions, we give some examples. □

Example 14. In systems (1), let $n_1 = 3$, $n_2 = 4$, $a_1(t) = a_2(t) = 1$, $n_1(t) = n_2(t) = t$, $f_1(t,u,v) = (1 + t + e^{-u})v^{3/2}$, and $f_2(t,u) = u^{5/2}$, so the assumptions (H_1)–(H_3) are satisfied. Choose $\alpha = 1/2$, $\beta = 3/2$; then

$$\begin{aligned}
&\liminf_{u \to +\infty} \frac{f_2(t,u)}{u^{1/\alpha}} = +\infty, \\
&\liminf_{v \to +\infty} \frac{f_1(t,u,v)}{v^{\alpha}} > 0; \\
&\limsup_{u \to 0^+} \frac{f_2(t,u)}{u^{1/\beta}} = 0, \\
&\limsup_{v \to 0^+} \frac{f_1(t,u,v)}{v^{\beta}} < +\infty
\end{aligned} \tag{39}$$

uniformly with respect to $t \in [0,1]$ and $(t,u) \in [0,1] \times [0,+\infty)$. It is easy to verify that the assumptions (H_4)-(H_5) hold. By Theorem 11, systems (1) have at least one position solution.

Example 15. In systems (1), let $n_1 = 3$, $n_2 = 4$, $a_1(t) = a_2(t) = 1$, $n_1(t) = n_2(t) = t$, $f_1(t,u,v) = (1+t+e^{-u})v^{1/2}$, and $f_2(t,u) = u^{1/2}$, so the assumptions (H_1)–(H_3) are satisfied. Choose $p = q = 1/2$; then

$$\begin{aligned}
&\limsup_{u \to +\infty} \frac{f_2(t,u)}{u^{1/p}} = 0, \\
&\limsup_{v \to +\infty} \frac{f_1(t,u,v)}{v^{p}} < +\infty, \\
&\liminf_{u \to 0^+} \frac{f_2(t,u)}{u^{1/q}} = +\infty, \\
&\limsup_{v \to 0^+} \frac{f_1(t,u,v)}{v^{q}} > 0
\end{aligned} \tag{40}$$

uniformly with respect to $t \in [0,1]$ and $(t,u) \in [0,1] \times [0,+\infty)$. It is easy to verify that the assumptions (H_6)-(H_7) hold. By Theorem 12, systems (1) have at least one position solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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