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Research Article

Computing the Electric and Magnetic Matrix Green's Functions in a Rectangular Parallelepiped with a Perfect Conducting Boundary

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A method for the approximate computation of frequency-dependent magnetic and electric matrix Green's functions in a rectangular parallelepiped with a perfect conducting boundary is suggested in the paper. This method is based on approximation (regularization) of the Dirac delta function and its derivatives, which appear in the differential equations for magnetic and electric Green's functions, and the Fourier series expansion meta-approach for solving the elliptic boundary value problems. The elements of approximate Green's functions are found explicitly in the form of the Fourier series with a finite number of terms. The convergence analysis for finding the number of the terms is given. The computational experiments have confirmed the robustness of the method.

1. Introduction

In recent years a lot of attention has been devoted to derivation of the electric and magnetic fields inside bounded domains with perfect conducting boundaries [1–4]. The study of the distribution of electromagnetic fields inside a real indoor environment (cabinets, desks, etc.) has the great practical interest (see, e.g., [5]). The theory (existence, uniqueness, and stability estimate theorems) of time-dependent Maxwell's equations in bounded domains with perfect conducting boundary conditions and smooth current and charge densities has been described by Dautray and Lions in [6]. These theoretical results form the background for many computational methods such as finite element method, finite-difference method, finite-difference time domain method, modal expansion method, and Nitsche type method, which have been handled for computation of the electric and magnetic fields in bounded domains with a perfect conducting boundary [1–3, 6–8].

We need to note that the solution (both analytical and numerical) of the initial value and initial boundary value problems in electrodynamics is often facilitated by introduction of electric and magnetic Green's functions [9–17], but the derivations of electric and magnetic Green's functions have been made in a free space or unbounded domain. The computation of time-dependent electric and magnetic Green's functions for the bounded domain with perfect conducting conditions has not been achieved so far.

In our paper we suggest a new method for the approximate computation of frequency-dependent magnetic and electric matrix Green's functions in a rectangular parallelepiped with perfect conducting boundary. This method consists of the following. The equations for magnetic Green's function are written in the special form which does not contain elements of electric Green's function. These equations are elliptic partial differential equations and contain the components of the generalized vector functions (distributions) $\operatorname{curl}_x[\delta(x - x^0)\overrightarrow{\mathbf{e}^s}]$, $s = 1, 2, 3$ in their right hand sides. We approximate components of these generalized vector functions by the Fourier series with a finite number of terms and then applied standard Fourier series expansion meta-approach for solving the boundary value problem for the elliptic partial differential equations in the bounded region. After calculation of all columns of magnetic matrix Green's function the approximate computation of the columns of electric Green's function is given by explicit calculation of the differential operator curl_x from each column of magnetic matrix Green's function with respect to 3D space variable $x = (x_1, x_2, x_3)$.

2. Frequency-Dependent Magnetic and Electric Green's Functions

Let b_1, b_2, b_3 be given positive numbers and let

$$V = \{x \in \mathbf{R}^3 : 0 < x_1 < b_1, 0 < x_2 < b_2, 0 < x_3 < b_3\} \quad (1)$$

be a rectangular parallelepiped; Γ is the boundary of V; $\omega \in \mathbf{R}$ (frequency, $\omega > 0$), let $x^0 = (x_1^0, x_2^0, x_3^0) \in V$ be 1D and 3D parameters, respectively. The 3×3 matrices

$$\mathbf{G}^H(x,\omega,x^0) = \begin{pmatrix} H_1^1(x,\omega,x^0) & H_1^2(x,\omega,x^0) & H_1^3(x,\omega,x^0) \\ H_2^1(x,\omega,x^0) & H_2^2(x,\omega,x^0) & H_2^3(x,\omega,x^0) \\ H_3^1(x,\omega,x^0) & H_3^2(x,\omega,x^0) & H_3^3(x,\omega,x^0) \end{pmatrix},$$

$$\mathbf{G}^E(x,\omega,x^0) = \begin{pmatrix} E_1^1(x,\omega,x^0) & E_1^2(x,\omega,x^0) & E_1^3(x,\omega,x^0) \\ E_2^1(x,\omega,x^0) & E_2^2(x,\omega,x^0) & E_2^3(x,\omega,x^0) \\ E_3^1(x,\omega,x^0) & E_3^2(x,\omega,x^0) & E_3^3(x,\omega,x^0) \end{pmatrix}, \quad (2)$$

whose columns $\mathbf{H}^s(x,\omega,x^0)$, $\mathbf{E}^s(x,\omega,x^0)$, and $s = 1,2,3$ satisfy

$$\text{curl}_x \mathbf{H}^s(x,\omega,x^0) = -i\omega\epsilon\mathbf{E}^s(x,\omega,x^0) + \delta(x - x^0)\overrightarrow{\mathbf{e}^s}, \quad x \in V, \quad (3)$$

$$\text{curl}_x \mathbf{E}^s(x,\omega,x^0) = i\mu\omega\mathbf{H}^s(x,\omega,x^0), \quad (4)$$

$$(\mathbf{E}^s \times \vec{\mathbf{n}})|_\Gamma = 0, \quad (\mathbf{H}^s \cdot \vec{\mathbf{n}})|_\Gamma = 0, \quad s = 1,2,3, \quad (5)$$

are called frequency-dependent magnetic and electric Green's functions in V, respectively. Here i is the imaginary unit ($i^2 = -1$); $\epsilon > 0$ and $\mu > 0$ are given constants characterizing the permittivity and permeability of the electromagnetic medium, which is located inside V; $\overrightarrow{\mathbf{e}^s}$ are the basis vectors from Euclidean space $\mathbf{R}^3$ ($\overrightarrow{\mathbf{e}^1} = (1,0,0)^T, \overrightarrow{\mathbf{e}^2} = (0,1,0)^T, \overrightarrow{\mathbf{e}^3} = (0,0,1)^T$); $x^0 = (x_1^0, x_2^0, x_3^0) \in V$ is the 3D parameter; $\delta(x - x^0) = \delta(x_1 - x_1^0)\delta(x_2 - x_2^0)\delta(x_3 - x_3^0)$, and $\delta(x_j - x_j^0)$ are the Dirac delta distribution concentrated at $x_j = x_j^0$ and $j = 1,2,3$; $\vec{\mathbf{n}}$ is an outward unit normal vector on Γ. The relations in (5) are called the perfect conducting boundary conditions.

Remark 1. We note that $\text{div}_x \mathbf{H}^s(x,\omega,x^0) = 0$ for all $x \in V$ is a consequence of (4).

Applying curl_x to (3) and using (4), we find

$$\omega^2\epsilon\mu\mathbf{H}^s - \text{curl}_x\text{curl}_x\mathbf{H}^s = -\text{curl}_x\left[\delta(x - x^0)\overrightarrow{\mathbf{e}^s}\right], \quad x \in V. \quad (6)$$

Taking the cross product of both sides of (3) and $\vec{\mathbf{n}}$, and using (5) we get

$$(\text{curl}_x\mathbf{H}^s \times \vec{\mathbf{n}})|_\Gamma = 0, \qquad (\mathbf{H}^s \cdot \vec{\mathbf{n}})|_\Gamma = 0. \quad (7)$$

The following equation follows from (3)

$$\mathbf{E}(x,\omega,x^0) = \frac{i}{\omega\epsilon}\left[\text{curl}_x\mathbf{H}^s(x,\omega,x^0) - \delta(x - x^0)\overrightarrow{\mathbf{e}^s}\right]. \quad (8)$$

Remark 2. If $\mathbf{H}^s$ is a vector function satisfying (6) and (7) and $\mathbf{E}^s$ is a vector function satisfying (8) for a fixed value of $s = 1,2,3$, then $\mathbf{H}^s$ and $\mathbf{E}^s$ satisfy (3)–(5); that is, $\mathbf{H}^s$ and $\mathbf{E}^s$ are columns of frequency-dependent magnetic and electric Green's functions. Really, from (8), we find $\text{curl}_x\mathbf{H}^s(x,\omega,x^0) = -i\omega\epsilon\mathbf{E}^s(x,\omega,x^0) + \delta(x - x^0)\overrightarrow{\mathbf{e}^s}$. Substituting the expression for $\text{curl}_x\mathbf{H}^s$ into (6) we have $\text{curl}_x\mathbf{E}^s(x,\omega,x^0) = i\mu\omega\mathbf{H}^s(x,\omega,x^0)$. Moreover, taking the cross product of (8) and $\vec{\mathbf{n}}$, letting $x \in \Gamma$ and, using (7) we find that $\mathbf{E}^s$ satisfies the first condition of (5). Therefore, we obtain that if $\mathbf{H}^s$ and $\mathbf{E}^s$ satisfy (6), (7), and (8) then $\mathbf{H}^s$ and $\mathbf{E}^s$ satisfy (3)–(5) also; that is, $\mathbf{H}^s$ and $\mathbf{E}^s$ are s-columns of frequency-dependent electric and magnetic Green's functions.

Remark 3. Let $\mathbf{H}^s$ and $\mathbf{E}^s$ satisfy (6) and (8). Then $\text{div}_x\mathbf{H}^s = 0$, $x \in V$.

3. The Eigenvalue-Eigenfunction Problem for the Operator $\Delta_x\mathbf{I}_3$ in a Rectangular Parallelepiped

Let us consider the problem of finding all values of λ (eigenvalues) for which there exists a nonzero vector-valued function $\mathbf{U}(x) = (U_1(x), U_2(x), U_3(x))^T$ (vector-valued eigenfunction) satisfying

$$\Delta_x\mathbf{I}_3\mathbf{U}(x) + \lambda\mathbf{U}(x) = 0, \quad x \in V, \quad (9)$$

$$(\text{curl}_x\mathbf{U}(x) \times \vec{\mathbf{n}})|_\Gamma = 0, \qquad (\mathbf{U} \cdot \vec{\mathbf{n}})|_\Gamma = 0, \quad (10)$$

where $\mathbf{I}_3$ is the 3×3 identity matrix.

In this section we show that the eigenvalue-eigenfunction problem (9) and (10) can be decomposed into three eigenvalue-eigenfunction problems for the scalar-valued Laplace operator in the rectangular parallelepiped with the mixed boundary conditions. These subproblems can be solved by a standard technique [18].

Lemma 4. *Let the boundary* Γ *of the parallelepiped* V *be presented as* $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, *where* Γ_1, Γ_2, *and* Γ_3 *are the sets defined by*

$$\begin{aligned}\Gamma_1 &= \{x = (x_1, x_2, x_3) : x_1 = 0 \vee x_1 = b_1, \\ &\qquad 0 < x_2 < b_2, 0 < x_3 < b_3\}, \\ \Gamma_2 &= \{x = (x_1, x_2, x_3) : x_2 = 0 \vee x_2 = b_2, \\ &\qquad 0 < x_1 < b_1, 0 < x_3 < b_3\}, \\ \Gamma_3 &= \{x = (x_1, x_2, x_3) : x_3 = 0 \vee x_3 = b_3, \\ &\qquad 0 < x_1 < b_1, 0 < x_2 < b_2\}.\end{aligned} \tag{11}$$

The boundary condition in (10) *can be written as*

$$\left(\alpha_j(x)\frac{\partial U_j(x)}{\partial n} + \beta_j(x) U_j(x)\right)\Bigg|_{\Gamma} = 0, \tag{12}$$

where the functions $\alpha_j(x)$ *and* $\beta_j(x)$, $j = 1, 2, 3$ *are defined by*

$$\alpha_j(x) = \begin{cases} 0, & x \in \Gamma_j, \\ 1, & x \in \Gamma \setminus \Gamma_j; \end{cases} \qquad \beta_j(x) = \begin{cases} 1, & x \in \Gamma_j, \\ 0, & x \in \Gamma \setminus \Gamma_j. \end{cases} \tag{13}$$

Proof. Let us prove this for $j = 1$ and let us consider the part of the boundary where $x_1 = b_1$. Thus, the outward normal unit vector is $\overrightarrow{\mathbf{e}^1} = (1, 0, 0)^T$. The second condition of (10) reads

$$U_1(b_1, x_2, x_3) = 0, \quad \text{for } 0 < x_2 < b_2,\ 0 < x_3 < b_3. \tag{14}$$

The first condition of (10) takes the form

$$\begin{aligned}\frac{\partial U_2}{\partial x_1}(b_1, x_2, x_3) - \frac{\partial}{\partial x_2}[U_1(b_1, x_2, x_3)] &= 0, \\ \frac{\partial U_3}{\partial x_1}(b_1, x_2, x_3) - \frac{\partial}{\partial x_3}[U_1(b_1, x_2, x_3)] &= 0.\end{aligned} \tag{15}$$

Combining the last three equalities we can write the boundary condition (10) in the form

$$\left(\alpha_j(x)\frac{\partial U_j(x)}{\partial n} + \beta_j(x) U_j(x)\right)\Bigg|_{x_1 = b_1} = 0, \quad j = 1, 2, 3, \tag{16}$$

where $\alpha_1 = 0$, $\beta_1 = 1$; $\alpha_2 = 1$, and $\beta_2 = 0$; $\alpha_3 = 1$ and $\beta_3 = 0$. Similar arguments hold for the lower part of Γ_1 and for the parts of the boundaries Γ_2 and Γ_3 leading to (12). □

Let λ_{kmn} denote the eigenvalues and $\mathbf{U}^{kmn}(x)$ the corresponding vector-valued eigenfunctions of (9) and (10). The eigenfunctions $\mathbf{U}^{kmn}(x)$ can be written in the form

$$\mathbf{U}^{kmn}(x) = U_1^{kmn}(x)\overrightarrow{\mathbf{e}^1} + U_2^{kmn}(x)\overrightarrow{\mathbf{e}^2} + U_3^{kmn}(x)\overrightarrow{\mathbf{e}^3}, \tag{17}$$

and λ_{kmn}, $U_j^{kmn}(x)$, and $j = 1, 2, 3$ satisfy

$$\Delta_x U_j^{kmn}(x) + \lambda_{kmn} U_j^{kmn}(x) = 0, \quad x \in V; \tag{18}$$

$$\left(\alpha_j(x)\frac{\partial U_j^{kmn}(x)}{\partial n} + \beta_j(x) U_j^{kmn}(x)\right)\Bigg|_{x_1 = b_1} = 0. \tag{19}$$

Applying the standard technique of the spectral theory (see, e.g., [18]) we find all eigenvalues and corresponding eigenfunctions of the problem (18) and (19) for each j as follows.

For $j = 1$,

$$\begin{aligned}\lambda_{kmn} &= \left(\frac{k\pi}{b_1}\right)^2 + \left(\frac{m\pi}{b_2}\right)^2 + \left(\frac{n\pi}{b_3}\right)^2, \\ U_1^{kmn}(x) &= A_{kmn}\sin\left(\frac{k\pi}{b_1}x_1\right)\cos\left(\frac{m\pi}{b_2}x_2\right)\cos\left(\frac{n\pi}{b_3}x_3\right),\end{aligned} \tag{20}$$

$k = 1, 2, \ldots; n, m = 0, 1, \ldots$ are eigenvalues and corresponding eigenfunctions.

For $j = 2$,

$$\begin{aligned}\lambda_{kmn} &= \left(\frac{k\pi}{b_1}\right)^2 + \left(\frac{m\pi}{b_2}\right)^2 + \left(\frac{n\pi}{b_3}\right)^2, \\ U_2^{kmn}(x) &= B_{kmn}\cos\left(\frac{k\pi}{b_1}x_1\right)\sin\left(\frac{m\pi}{b_2}x_2\right)\cos\left(\frac{n\pi}{b_3}x_3\right),\end{aligned} \tag{21}$$

$m = 1, 2, \ldots; k, n = 0, 1, \ldots$ are eigenvalues and corresponding eigenfunctions.

For $j = 3$,

$$\begin{aligned}\lambda_{kmn} &= \left(\frac{k\pi}{b_1}\right)^2 + \left(\frac{m\pi}{b_2}\right)^2 + \left(\frac{n\pi}{b_3}\right)^2, \\ U_3^{kmn}(x) &= C_{kmn}\cos\left(\frac{k\pi}{b_1}x_1\right)\cos\left(\frac{m\pi}{b_2}x_2\right)\sin\left(\frac{n\pi}{b_3}x_3\right),\end{aligned} \tag{22}$$

$n = 1, 2, \ldots; k, m = 0, 1, \ldots$ are eigenvalues and corresponding eigenfunctions.

Here A_{kmn}, B_{kmn}, and C_{kmn} are constants, such that

$$\begin{aligned}&\int_0^{b_3}\int_0^{b_2}\int_0^{b_1} U_j^{kmn}(x)\cdot U_j^{k'm'n'}(x)\,dx \\ &\quad = \begin{cases} 0, & \text{if } (k, m, n) \neq (k', m', n') \\ 1, & \text{if } k = k',\ m = m',\ n = n'. \end{cases}\end{aligned} \tag{23}$$

$k, m, n = 0, 1, \ldots$; $j = 1, 2, 3$. That is, $A_{kmn} = B_{kmn} = C_{kmn} = \sqrt{8/b_1 b_2 b_3}$, $A_{k0n} = A_{km0} = B_{0mn} = B_{km0} = C_{0mn} = C_{k0n} = \sqrt{4/b_1 b_2 b_3} A_{k00} = B_{0m0} = C_{00n} = \sqrt{2/b_1 b_2 b_3}$, and $k = 1, 2, \ldots$; $m = 1, 2, \ldots$; $n = 1, 2, \ldots$.

Remark 5. Let $U_j^{kmn}(x)$, $j = 1, 2, 3$, be eigenfunctions of (18) and (19) defined above. By the direct computation we can show that $\mathbf{U}^{kmn}(x) = (U_1^{kmn}(x), U_2^{kmn}(x), U_3^{kmn}(x))^T$ satisfies the boundary conditions (10) for the rectangular parallelepiped V.

Remark 6. For each $j = 1, 2, 3$ the system of eigenfunctions $\{U_j^{kmn}(x)\}$ is orthogonal and complete in the space of the square integrable functions over V. That is, any scalar-valued function $h(x) \in C_0^\infty(V)$ can be written in the form of uniformly convergent series over $\overline{V}$ (see, e.g., [18])

$$h(x) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_j^{kmn} U_j^{kmn}(x), \tag{24}$$

where

$$h_j^{kmn} = \iiint_V h(x)\, U_j^{kmn}(x)\, dx. \tag{25}$$

The constants h_j^{kmn} defined in (25) are called the Fourier coefficients of $h(x)$ relative to the basis functions $\{U_j^{kmn}(x)\}$.

Lemma 7 (see [18]). *Let $h(x) \in C_0^\infty(V)$. Then*

$$\iiint_V |h(x)|^2 dx = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left[h_j^{kmn}\right]^2 < \infty \tag{26}$$

holds. Here h_j^{kmn} are the Fourier coefficients (25). *The equality* (26) *is known as Parseval's equality. The proof of this lemma can be found, for example, in [18].*

As a result, any vector function $\Psi(x) = (\psi_1(x), \psi_2(x), \psi_3(x))^T$ with components from $C_0^\infty(V)$ can be presented in the form

$$\Psi(x) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left(\psi_1^{kmn} U_1^{kmn}(x)\, \overrightarrow{\mathbf{e}^1} + \psi_2^{kmn} U_2^{kmn}(x)\, \overrightarrow{\mathbf{e}^2} + \psi_3^{kmn} U_3^{kmn}(x)\, \overrightarrow{\mathbf{e}^3}\right), \quad \forall x \in \overline{V}, \tag{27}$$

where

$$\psi_j^{kmn} = \iiint_V \psi_j(x)\, U_j^{kmn}(x)\, dx, \quad j = 1, 2, 3. \tag{28}$$

We note that if $\psi_j(x) \in C_0^\infty(V)$ then $\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \psi_j^{kmn} U_j^{kmn}(x)$; $j = 1, 2, 3$, is uniformly convergent series over $\overline{V}$.

4. Computation of Frequency-Dependent Magnetic Matrix Green's Function

4.1. The Regularization of $\operatorname{curl}_x[\delta(x-x^0)\overrightarrow{\mathbf{e}^s}]$. Let $s = 1$, $x = (x_1, x_2, x_3) \in \mathbf{R}^3$, $x^0 = (x_1^0, x_2^0, x_3^0) \in \mathbf{R}^3$, $\delta(x - x^0) = \delta(x_1 - x_1^0)\delta(x_2 - x_2^0)\delta(x_3 - x_3^0)$ be the Dirac delta distribution. The vector distribution $-\operatorname{curl}_x[\delta(x - x^0)\overrightarrow{\mathbf{e}^1}] = \operatorname{curl}_{x^0}[\delta(x - x^0)\overrightarrow{\mathbf{e}^1}]$ can be written as follows:

$$\begin{aligned} -\operatorname{curl}_x\left[\delta\left(x - x^0\right)\overrightarrow{\mathbf{e}^1}\right] &= \operatorname{curl}_{x^0}\left[\delta\left(x - x^0\right)\overrightarrow{\mathbf{e}^1}\right] \\ &= 0 \cdot \overrightarrow{\mathbf{e}^1} + \frac{\partial\delta\left(x - x^0\right)}{\partial x_3^0}\overrightarrow{\mathbf{e}^2} - \frac{\partial\delta\left(x - x^0\right)}{\partial x_2^0}\overrightarrow{\mathbf{e}^3}. \end{aligned} \tag{29}$$

Using formulas (26) and (28) we find the Fourier series expansion of $\operatorname{curl}_{x^0}[\delta(x - x^0)\overrightarrow{\mathbf{e}^1}]$ in the form of (27), where

$$\psi_1^{kmn} = 0, \qquad \psi_2^{kmn} = \frac{\partial U_2^{kmn}\left(x^0\right)}{\partial x_3^0}, \qquad \psi_3^{kmn} = -\frac{\partial U_3^{kmn}\left(x^0\right)}{\partial x_2^0}. \tag{30}$$

(See, e.g., [19–22].)

Lemma 8. *Let $x^0 \in V$ be fixed, and $h(x) \in C_0^\infty(V)$. Then the series*

$$\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{\partial U_2^{kmn}\left(x^0\right)}{\partial x_3^0} h_2^{kmn}, \tag{31}$$

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{\partial U_3^{kmn}\left(x^0\right)}{\partial x_2^0} h_3^{kmn} \tag{32}$$

are absolutely convergent. Here h_2^{kmn} and h_3^{kmn} are defined by (25) *for the given function $h(x)$.*

The proof of Lemma 8 can be done by standard arguments of calculus (see Appendix A).

Using Lemma 8 the distributions $\partial\delta(x - x^0)/\partial x_3^0$ and $\partial\delta(x - x^0)/\partial x_2^0$ can be defined by

$$\begin{aligned} \left\langle \frac{\partial\delta\left(x - x^0\right)}{\partial x_3^0}, h(x) \right\rangle &= \frac{\partial h\left(x^0\right)}{\partial x_3^0} = \sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{\partial U_2^{kmn}\left(x^0\right)}{\partial x_3^0} h_2^{kmn}, \\ \left\langle \frac{\partial\delta\left(x - x^0\right)}{\partial x_2^0}, h(x) \right\rangle &= \frac{\partial h\left(x^0\right)}{\partial x_2^0} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{\partial U_3^{kmn}\left(x^0\right)}{\partial x_2^0} h_3^{kmn}, \end{aligned} \tag{33}$$

for any function $h(x) \in C_0^\infty(V)$.

Let us consider the infinitely differentiable functions $\psi_2^{1,N}(x,x^0)$, $\psi_3^{1,N}(x,x^0)$ defined by

$$\psi_2^{1,N}\left(x,x^0\right)=\sum_{k=0}^{N}\sum_{m=1}^{N}\sum_{n=0}^{N}\frac{\partial U_2^{kmn}\left(x^0\right)}{\partial x_3^0}U_2^{kmn}\left(x\right), \quad (34)$$

$$\psi_3^{1,N}\left(x,x^0\right)=-\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=1}^{N}\frac{\partial U_3^{kmn}\left(x^0\right)}{\partial x_2^0}U_3^{kmn}\left(x\right). \quad (35)$$

Lemma 9. *Let $x^0\in V$ be fixed, and let functions $\psi_2^{1,N}(x,x^0)$ and $\psi_3^{1,N}(x,x^0)$ be defined by (34) and (35). Then for any $h(x)\in C_0^\infty(V)$*

$$\lim_{N\to\infty}\iiint_V\psi_2^{1,N}\left(x,x^0\right)h\left(x\right)dx=\left\langle\frac{\partial\delta\left(x-x^0\right)}{\partial x_3^0},h\left(x\right)\right\rangle, \quad (36)$$

$$\lim_{N\to\infty}\iiint_V\psi_3^{1,N}\left(x,x^0\right)h\left(x\right)dx=-\left\langle\frac{\partial\delta\left(x-x^0\right)}{\partial x_2^0},h\left(x\right)\right\rangle. \quad (37)$$

The proof of Lemma 7 follows from Lemma 4 and standard arguments of the theory of distributions (see, e.g., [19, 20]).

Remark 10. Equalities (36) and (37) mean that

$$\lim_{N\to\infty}\psi_2^{1,N}\left(x,x^0\right)=\frac{\partial\delta\left(x-x^0\right)}{\partial x_3^0},$$
$$\lim_{N\to\infty}\psi_3^{1,N}\left(x,x^0\right)=-\frac{\partial\delta\left(x-x^0\right)}{\partial x_2^0} \quad (38)$$

in the space of distributions $\mathscr{D}'(V)$ (see, e.g., [19, 20]).

Remark 11. The functions $\psi_2^{1,N}(x,x^0)$ and $\psi_3^{1,N}(x,x^0)$ can be taken as regularizations (approximations) of the distributions $\partial\delta(x-x^0)/\partial x_3^0$ and $-\partial\delta(x-x^0)/\partial x_2^0$, respectively. Here N is a parameter of the regularization (see, e.g., [19, 20]).

Remark 12. Let

$$\begin{aligned}\Psi^{1,N}\left(x,x^0\right)&=\left(0,\psi_2^{1,N},\psi_3^{1,N}\right)^T\\&=\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\left(\frac{\partial U_2^{kmn}\left(x^0\right)}{\partial x_3^0}U_2^{kmn}\left(x\right)\overrightarrow{\mathbf{e}^2}\right.\\&\qquad\left.-\frac{\partial U_3^{kmn}\left(x^0\right)}{\partial x_2^0}U_3^{kmn}\left(x\right)\overrightarrow{\mathbf{e}^3}\right).\end{aligned} \quad (39)$$

Using Lemma 9 and Remarks 10 and 11 we find that

$$\lim_{N\to\infty}\Psi^{1,N}\left(x,x^0\right)=\operatorname{curl}_{x^0}\left[\delta\left(x-x^0\right)\overrightarrow{\mathbf{e}^1}\right] \quad (40)$$

in the space of distributions $\mathscr{D}'(V)$ and therefore the vector function $\Psi^{1,N}(x,x^0)$, with infinitely differentiable components, can be taken as a regularization of the generalized vector function $\operatorname{curl}_{x^0}[\delta(x-x^0)\overrightarrow{\mathbf{e}^1}]$.

Similarly, we find that the vector functions $\Psi^{2,N}(x,x^0)$ and $\Psi^{3,N}(x,x^0)$, defined by

$$\begin{aligned}\Psi^{2,N}\left(x,x^0\right)&=\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\left(-\frac{\partial U_1^{kmn}\left(x^0\right)}{\partial x_3^0}U_1^{kmn}\left(x\right)\overrightarrow{\mathbf{e}^1}\right.\\&\qquad\left.+\frac{\partial U_3^{kmn}\left(x^0\right)}{\partial x_1^0}U_3^{kmn}\left(x\right)\overrightarrow{\mathbf{e}^3}\right),\end{aligned} \quad (41)$$

$$\begin{aligned}\Psi^{3,N}\left(x,x^0\right)&=\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\left(\frac{\partial U_1^{kmn}\left(x^0\right)}{\partial x_2^0}U_1^{kmn}\left(x\right)\overrightarrow{\mathbf{e}^1}\right.\\&\qquad\left.-\frac{\partial U_2^{kmn}\left(x^0\right)}{\partial x_1^0}U_2^{kmn}\left(x\right)\overrightarrow{\mathbf{e}^2}\right),\end{aligned} \quad (42)$$

can be taken as the regularizations of $\operatorname{curl}_{x^0}[\delta(x-x^0)\overrightarrow{\mathbf{e}^2}]$ and $\operatorname{curl}_{x^0}[\delta(x-x^0)\overrightarrow{\mathbf{e}^3}]$.

4.2. Solving (6) *and* (7) *with the Regularized Inhomogeneous Term.* Let us consider (6) and (7) for frequency-dependent magnetic Green's function. We replace $\operatorname{curl}_x[\delta(x-x^0)\overrightarrow{\mathbf{e}^s}]$ by their regularizations defined by (39), (41), and (42) for $s=1,2,3$, respectively. Moreover, (6) and (7) are replaced by the following equations:

$$\Delta_x\mathbf{I_3}\mathbf{H}^{s,N}+\omega^2\epsilon\mu\mathbf{H}^{s,N}=\Psi^{s,N}\left(x,x^0\right), \quad (43)$$

$$\left(\operatorname{curl}_x\mathbf{H}^{s,N}\times\vec{\mathbf{n}}\right)\Big|_\Gamma=0,\qquad\left(\mathbf{H}^{s,N}\cdot\vec{\mathbf{n}}\right)\Big|_\Gamma=0. \quad (44)$$

We will find solutions of the problem (43) and (44) in the form

$$\begin{aligned}\mathbf{H}^{s,N}\left(x,\omega,x^0\right)&=\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\left(Y_1^{s,kmn}\left(\omega,x^0\right)U_1^{kmn}\left(x\right)\overrightarrow{\mathbf{e}^1}\right.\\&\qquad+Y_2^{s,kmn}\left(\omega,x^0\right)U_2^{kmn}\left(x\right)\overrightarrow{\mathbf{e}^2}\\&\qquad\left.+Y_3^{s,kmn}\left(\omega,x^0\right)U_3^{kmn}\left(x\right)\overrightarrow{\mathbf{e}^3}\right),\end{aligned} \quad (45)$$

where $x\in V$; $U_j^{kmn}(x)$ are eigenfunctions of the eigenvalue-eigenfunction problem (18) and (19); $Y_j^{s,kmn}(\omega,x^0)$ and $j=1,2,3$, are the unknown Fourier coefficients.

Substituting (45) into (43) and using orthogonality of $U_j^{kmn}(x)$, we find

$$\left(\omega^2\epsilon\mu - \lambda_{kmn}\right) Y_j^{s,kmn}\left(\omega, x^0\right) = \psi_j^{s,kmn}\left(x^0\right), \quad (46)$$

for $j = 1,2,3$; $s = 1,2,3$; $k = 0,1,\ldots$, $m = 0,1,\ldots$, and $n = 0,1,\ldots$.

Let $\omega^2 \neq \lambda_{kmn}/\epsilon\mu$; then $Y_j^{s,kmn}(\omega, x^0)$ is uniquely determined by

$$Y_j^{s,kmn}\left(\omega, x^0\right) = \frac{\psi_j^{s,kmn}\left(x^0\right)}{\omega^2\epsilon\mu - \lambda_{kmn}}, \quad (47)$$

where

$$\psi_j^{s,kmn}\left(x^0\right) = \iiint_V \Psi_j^s\left(x, x^0\right) U_j^{kmn}(x)\, dx, \quad (48)$$

$k, m, n = 0, 1, \ldots$. As a result we obtain the following explicit formula:

$$\begin{aligned}\mathbf{H}^{1,N}\left(x,\omega,x^0\right) &= \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}} \\ &\times\left(\frac{\partial U_2^{kmn}\left(x^0\right)}{\partial x_3^0}U_2^{kmn}(x)\,\overrightarrow{\mathbf{e}^2} - \frac{\partial U_3^{kmn}\left(x^0\right)}{\partial x_2^0}U_3^{kmn}(x)\,\overrightarrow{\mathbf{e}^3}\right),\end{aligned} \quad (49)$$

for all $x \in V$, $x^0 \in V$, and all parameter ω satisfying $\omega^2\epsilon\mu \neq \lambda_{kmn}$ ($k = 0,1,\ldots$; $m = 0,1,\ldots$; $n = 0,1,\ldots$).

Similarly, the solutions of (43) and (44) for $s = 2$ and $s = 3$ are obtained in the form

$$\begin{aligned}\mathbf{H}^{2,N}\left(x,\omega,x^0\right) &= \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}} \\ &\times\left(\frac{\partial U_1^{kmn}\left(x^0\right)}{\partial x_3^0}U_1^{kmn}(x)\,\overrightarrow{\mathbf{e}^1} - \frac{\partial U_3^{kmn}\left(x^0\right)}{\partial x_1^0}U_3^{kmn}(x)\,\overrightarrow{\mathbf{e}^3}\right),\end{aligned} \quad (50)$$

$$\begin{aligned}\mathbf{H}^{3,N}\left(x,\omega,x^0\right) &= \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}} \\ &\times\left(\frac{\partial U_1^{kmn}\left(x^0\right)}{\partial x_2^0}U_1^{kmn}(x)\,\overrightarrow{\mathbf{e}^1} - \frac{\partial U_2^{kmn}\left(x^0\right)}{\partial x_1^0}U_2^{kmn}(x)\,\overrightarrow{\mathbf{e}^2}\right).\end{aligned} \quad (51)$$

Using the explicit formulas for $\mathbf{H}^{s,N}(x,\omega,x^0)$ and $U_j^{kmn}(x)$ we can check directly that $\mathbf{H}^{s,N}(x,\omega,x^0)$ satisfy the boundary condition (44).

Remark 13. By direct computation we can show that the vector functions $\mathbf{H}^{s,N}(x,\omega,x^0)$, defined by, (49)–(51), satisfy

$$\operatorname{div}_x \mathbf{H}^{s,N}\left(x,\omega,x^0\right) = 0, \qquad s = 1,2,3. \quad (52)$$

4.3. Frequency-Dependent Magnetic Green's Function and Its Regularization. In this section we show that $\lim_{N\to\infty} H_j^{s,N}(x,\omega,x^0)$; ($s = 1,2,3$; $j = 1,2,3$) exists in the space of distributions $\mathscr{D}'(V)$, and the distributions $H_j^s = \lim_{N\to\infty} H_j^{s,N}$ are the entries of matrix-valued Green's function $\mathbf{G}^H(x,\omega,x^0)$.

Let us consider for example, $s = 1$, $j = 2$ and an arbitrary function $h(x) \in C_0^\infty(V)$ of the form (24) and (25). The integrable over V function $H_j^{s,N}(x,t,x^0)$ defines a regular functional (distribution) by

$$\begin{aligned}&\left\langle H_2^{1,N}\left(x,\omega,x^0\right), h(x)\right\rangle \\ &= \sum_{k=0}^{N}\sum_{m=1}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}}\frac{\partial U_2^{kmn}\left(x^0\right)}{\partial x_3^0}h_2^{kmn}.\end{aligned} \quad (53)$$

Using Lemma 8 we find that $\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(1/(\omega^2\epsilon\mu - \lambda_{kmn}))(\partial U_2^{kmn}(x^0)/\partial x_3^0)h_2^{kmn}$ is absolutely convergent for any $h(x) \in C_0^\infty(V)$. This means that $\lim_{N\to\infty}\langle H_2^{1,N}(x, \omega, x^0), h(x)\rangle$ exists for any $h(x) \in C_0^\infty(V)$ and defines the linear continuous functional $H_2^{1,N}(x,\omega,x^0)$ over $C_0^\infty(V)$.

Similarly, we can find $H_j^s(x,\omega,x^0)$ as $\lim_{N\to\infty} H_j^{s,N}(x,\omega,x^0)$ in the complete space $\mathscr{D}'(V)$ (see, e.g., [19, 20]) for $s = 1,2,3$ and $j = 1,2,3$.

Moreover, since $\mathbf{H}^{s,N}(x,\omega,x^0) = (H_1^{s,N}(x,\omega,x^0), H_2^{s,N}(x, \omega,x^0), H_3^{s,N}(x,t\omega,x^0))^T$ satisfies equalities (43) and (44) then $\mathbf{H}^s(x,\omega,x^0) = (H_1^s(x,\omega,x^0), H_2^s(x,\omega,x^0), H_3^s(x,\omega,x^0))^T$ satisfies the following equalities:

$$\begin{gathered}\omega^2\epsilon\mu\mathbf{H}^s - \operatorname{curl}_x\operatorname{curl}_x\mathbf{H}^s = -\operatorname{curl}_x\left[\delta\left(x - x^0\right)\overrightarrow{\mathbf{e}^s}\right], \\ \left(\operatorname{curl}_x\mathbf{H}^s \times \vec{\mathbf{n}}\right)\big|_\Gamma = 0, \qquad \left(\mathbf{H}^s\cdot\vec{\mathbf{n}}\right)\big|_\Gamma = 0,\end{gathered} \quad (54)$$

that is, equalities (6) and (7). This means that $\mathbf{H}^s(x,\omega,x^0)$ is the s-column of frequency-dependent magnetic matrix Green's function.

5. Approximate Solution of Electric Green's Function

Let columns of magnetic Green's function be derived approximately; that is, let solutions $\mathbf{H}^{s,N}$, $s = 1,2,3$ of the problem (43) and (44) be computed by formulae (49), (50), and (51). For the approximate derivation of $\mathbf{E}^s(x,\omega,x^0)$, satisfying (8), we replace $\operatorname{curl}_x\mathbf{H}^s(x,\omega,x^0)$ by $\operatorname{curl}_x\mathbf{H}^{s,N}$

(x, ω, x^0) and $\delta(x - x^0)\overrightarrow{\mathbf{e}^s}$ by its regularization $\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} U_s^{kmn}(x^0)U_s^{kmn}(x)\overrightarrow{\mathbf{e}^s}$ in (8) for $s = 1, 2, 3$, respectively, and find an explicit formula for $\mathbf{E}^{s,N}(x, \omega, x^0)$ as follows:

$$\mathbf{E}^{s,N}\left(x, \omega, x^0\right) = \frac{i}{\omega\epsilon}\left[\operatorname{curl}_x \mathbf{H}^{s,N}\left(x, \omega, x^0\right) - \sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} U_s^{kmn}\left(x^0\right) U_s^{kmn}(x)\,\overrightarrow{\mathbf{e}^s}\right], \tag{55}$$

for all $x \in V$, $x^0 \in V$, and all parameter ω satisfying $\omega^2\epsilon\mu \neq \lambda_{kmn}$.

5.1. The Regularization of $\mathbf{E}^s(x, \omega, x^0)$. Let us consider the differentiable function $E_2^{2,N}(x, \omega, x^0)$, given by (55). This function defines a linear continuous regular functional over $C_0^\infty(V)$ by the following relation (see, e.g., [19, 20]):

$$\begin{aligned}
\left\langle E_2^{2,N}, h(x)\right\rangle
&= -\frac{i}{\omega\epsilon}\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu - \lambda_{kmn}} \\
&\quad\times\left[\frac{\partial U_1^{kmn}\left(x^0\right)}{\partial x_3^0}\times\iiint_V U_1^{kmn}(x)\,\frac{\partial h(x)}{\partial x_3}dx + \frac{\partial U_3^{kmn}\left(x^0\right)}{\partial x_1^0}\times\iiint_V U_3^{kmn}(x)\,\frac{\partial h(x)}{\partial x_1}dx\right] \\
&\quad+\frac{i}{\omega\epsilon}\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N} U_2^{kmn}\left(x^0\right) h_2^{kmn},
\end{aligned} \tag{56}$$

for an arbitrary $h(x) \in C_0^\infty(V)$. The series in the right-hand side of the last equality is absolutely convergent for any $h(x) \in C_0^\infty(V)$ (see, Appendix B). This means that $\lim_{N\to\infty}\langle E_2^{2,N}(x, \omega, x^0), h(x)\rangle$ exists for any $h(x) \in C_0^\infty(V)$ and defines a linear continuous regular functional over $C_0^\infty(V)$. We denote this functional as $E_2^2(x, \omega, x^0)$; that is, $E_2^2(x, \omega, x^0) = \lim_{N\to\infty} E_2^{2,N}(x, \omega, x^0)$ in the complete space $\mathscr{D}'(V)$. Similarly, we can find $E_j^s(x, \omega, x^0)$ as $\lim_{N\to\infty} E_j^{s,N}(x, \omega, x^0)$ in the space $\mathscr{D}'(V)$ for $s = 1, 2, 3$ and $j = 1, 2, 3$.

The s-column of electric Green's function is

$$\mathbf{E}^s\left(x, \omega, x^0\right) = \frac{i}{\omega\epsilon}\left[\operatorname{curl}_x \mathbf{H}^s\left(x, \omega, x^0\right) - \delta\left(x - x^0\right)\overrightarrow{\mathbf{e}^s}\right]. \tag{57}$$

Let now $\mathbf{E}^s(x, \omega, x^0) = (E_1^s(x, \omega, x^0), E_2^s(x, \omega, x^0), E_3^s(x, \omega, x^0))^T$ be the generalized vector function, whose components are distributions defined as $E_j^s = \lim_{N\to\infty} E_j^{s,N}$, $j = 1, 2, 3$. Since $\mathbf{E}^{s,N}(x, \omega, x^0) = (E_1^{s,N}(x, \omega, x^0), E_2^{s,N}(x, \omega, x^0), E_3^{s,N}(x, \omega, x^0))^T$ satisfies (55), then the generalized vector function $\mathbf{E}^s(x, \omega, x^0)$ satisfies (4).

Remark 14. We note that the generalized vector functions $\mathbf{H}^s(x, \omega, x^0)$, $\mathbf{E}^s(x, \omega, x^0)$, defined in Sections 4.3 and 5.1, satisfy (6), (7), and (8). It follows that $\mathbf{H}^s(x, \omega, x^0)$ and $\mathbf{E}^s(x, \omega, x^0)$ satisfy (3)–(5), and therefore, $\mathbf{H}^s(x, \omega, x^0)$ and $\mathbf{E}^s(x, \omega, x^0)$ are s-columns of frequency-dependent electric and magnetic Green's functions $\mathbf{G}^E(x, \omega, x^0)$ and $\mathbf{G}^H(x, \omega, x^0)$, respectively.

6. Computational Experiments

In this section, we present computational experiments to show applicability of the approximate computation of frequency-dependent magnetic and electric matrix Green's functions in the rectangular parallelepiped. Here we consider the parallelepiped

$$V = \left\{x \in \mathbf{R}^3 : 0 < x_1 < 3, 0 < x_2 < 4, 0 < x_3 < 2.5\right\} \tag{58}$$

which contains a dielectric material characterized by the electric permittivity ϵ and the magnetic permeability μ. We choose ϵ and μ as $c = 1/\sqrt{\epsilon\mu}$ ($c \approx 3 \cdot 10^8$ m/s is the speed of light) and ω is equal to the value of c. The boundary of V, denoted as Γ, is a perfect conducting boundary on which the tangential components of the electric field component and normal component of the magnetic field component vanish. There is an electric current $\mathbf{J}(x, \omega)$ inside V which produces the electric and magnetic fields. In our experiments we take $\mathbf{J}^s(x, \omega)$, $s = 1, 2, 3$, as

$$\mathbf{J}^s(x, \omega) = \delta\left(x - x^0\right)\overrightarrow{\mathbf{e}^s}, \tag{59}$$

where $\overrightarrow{\mathbf{e}^s}$ are the basis vectors from $\mathbf{R}^3$; $x^0 = (x_1^0, x_2^0, x_3^0)$ is the 3D parameter from V and chosen as $x_1^0 = 2$, $x_2^0 = 3$, $x_3^0 = 1.5$; $\delta(x - x^0) = \delta(x_1 - x_1^0)\delta(x_2 - x_2^0)\delta(x_3 - x_3^0)$, $\delta(x_j - x_j^0)$ is the Dirac delta distribution at $x_j = x_j^0$, $j = 1, 2, 3$.

Two types of computational experiments will be presented. The first one is related to convergence and error analysis of our method; the second one is the computation of the elements of matrix magnetic and electric Green's functions in the rectangular parallelepiped V and the visualization of the computed magnetic and electric fields components by MATLAB graphic tools.

6.1. Convergence Analysis. Green's function of the magnetic field is 3×3 matrix whose elements are distributions. Generally speaking, distributions do not have values at fixed points [19, 20]. The elements of regularized (approximate) matrix Green's function, which we have computed by our method, are classical differentiable and integrable functions over V having the values at fixed points. These classical functions define the regular distributions by the standard rule (see, [19, 20]). The comparison of the distributions can be made by the comparison of their integral characteristics. Some integral characteristics of the elements of matrix Green's function of the magnetic field can be found explicitly. For example, let $H_2^1(x,\omega,x^0)$ be a solution of (6) and let (7) for $s=1$ and $j=2$ and let $h_2^1(x_3,\omega,x^0)$ be defined by

$$h_2^1\left(x_3,\omega,x^0\right) = \int_0^{b_1}\int_0^{b_2} H_2^1\left(x,\omega,x^0\right)\cos\left(\frac{\pi x_1}{b_1}\right)\sin\left(\frac{\pi x_2}{b_2}\right)dx_2\,dx_1. \tag{60}$$

We can show that $h_2^1 = (x_3,\omega,x^0)$ satisfies

$$\frac{\partial^2 h_2^1}{\partial x_3^2} + k^2 h_2^1 = -\frac{\partial}{\partial x_3}\delta\left(x_3 - x_3^0\right)\cos\left(\frac{\pi x_1^0}{b_1}\right)\sin\left(\frac{\pi x_2^0}{b_2}\right), \quad 0 < x_3 < b_3, \tag{61}$$

$$\left.\frac{dh_2^1}{dx_3}\right|_{x_3=0} = 0, \qquad \left.\frac{dh_2^1}{dx_3}\right|_{x_3=b_3} = 0, \tag{62}$$

where $k^2 = \omega^2\epsilon\mu - (\pi/b_1)^2 - (\pi/b_2)^2$. A solution of (61) and (62) can be written in the form

$$h_2^1\left(x_3,\omega,x^0\right) = \begin{cases} \dfrac{\sin\left(k\left(b_3 - x_3^0\right)\right)}{\sin\left(kb_3\right)}\cos\left(kx_3\right) \times\cos\left(\dfrac{\pi x_1^0}{b_1}\right)\sin\left(\dfrac{\pi x_2^0}{b_2}\right), & 0 < x_3 < x_3^0, \\ -\dfrac{\sin\left(kx_3^0\right)}{\sin\left(kb_3\right)}\cos\left(k\left(b_3 - x_3\right)\right) \times\cos\left(\dfrac{\pi x_1^0}{b_1}\right)\sin\left(\dfrac{\pi x_2^0}{b_2}\right), & x_3^0 < x_3 < b_3. \end{cases} \tag{63}$$

We can compare the values of $h_2^1(x_3,\omega,x^0)$, as the values of the integral characteristics of H_2^1, with the values of the integral characteristic

$$\begin{aligned} h_2^{1,N}\left(x_3,\omega,x^0\right) &= \int_0^{b_1}\int_0^{b_2} H_2^{1,N}\left(x,\omega,x^0\right)\cos\left(\frac{\pi x_1}{b_1}\right)\sin\left(\frac{\pi x_2}{b_2}\right)dx_2\,dx_1 \\ &= \sum_{n=1}^{N}\frac{-1}{k^2 - (n\pi/b_3)^2}\frac{2n\pi}{(b_3)^2} \\ &\quad\times\cos\left(\frac{\pi x_1^0}{b_1}\right)\sin\left(\frac{\pi x_2^0}{b_2}\right)\sin\left(\frac{n\pi x_3^0}{b_3}\right)\cos\left(\frac{n\pi x_3}{b_3}\right) \end{aligned} \tag{64}$$

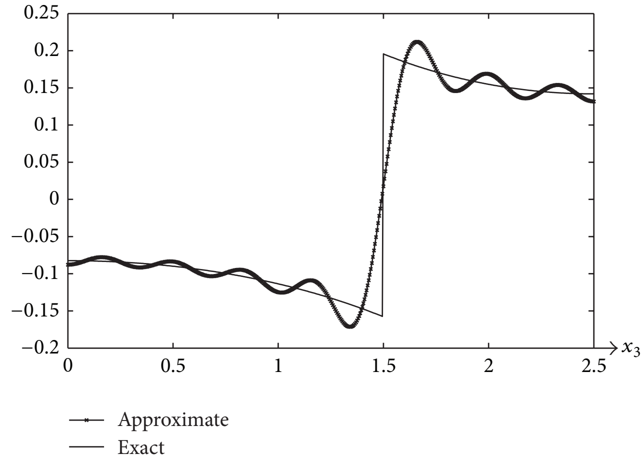


Figure 1: The graphs of h_2^1 and $h_2^{1,N}$ for $N = 15$, $x_1^0 = 2$, $x_2^0 = 3$, and $x_3^0 = 1.5$.

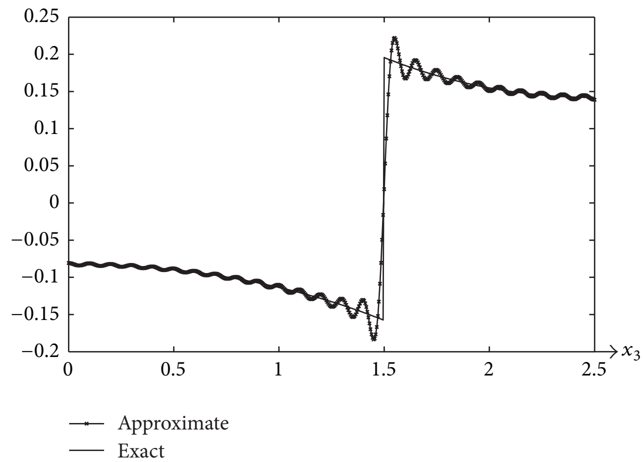


Figure 2: The graphs of h_2^1 and $h_2^{1,N}$ for $N = 50$, $x_1^0 = 2$, $x_2^0 = 3$, and $x_3^0 = 1.5$.

of $H_2^{1,N}$. Here $H_2^{1,N}$ is the element of approximate matrix Green's function computed by our method and N is the parameter of the approximation corresponding to the number of the terms in each series of the right hand side of (64).

In Figures 1, 2, 3, and 4 and Table 1 the values of $h_2^{1,N}$ are compared with the values of h_2^1 for fixed $x_1^0 = 2$, $x_2^0 = 3$, $x_3^0 = 1.5$, and ϵ and μ such that $c = 1/\sqrt{\epsilon\mu}$ and $\omega = c$ and different numbers N. The maximum of relative error $r_N(x_3) = |h_2^1 - h_2^{1,N}|/|h_2^1|$ at the point $x_3 = 1.3$ of the interval $[0, 2.5]$ for $N = 15, 50, 100, 150$ is $r_{15} = 0.1806$, $r_{50} = 0.0625$, $r_{100} = 0.0316$, and $r_{150} = 0.0211$, respectively. Making the error analysis we have concluded that the biggest N corresponds to the smallest error between values of $h_2^{1,N}$ and h_2^1. We have taken $N = 100$ as a reasonable value between accuracy and the time for our computation in MATLAB.

6.2. Approximate Computation and Visualization of the Elements of Matrix Magnetic and Electric Green's Functions. The method described in Sections 4 and 5 has been applied

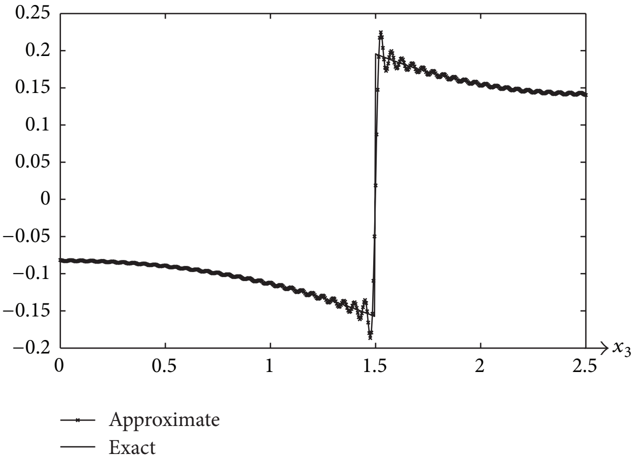


Figure 3: The graphs of h_2^1 and $h_2^{1,N}$ for $N = 100$, $x_1^0 = 2$, $x_2^0 = 3$, and $x_3^0 = 1.5$.

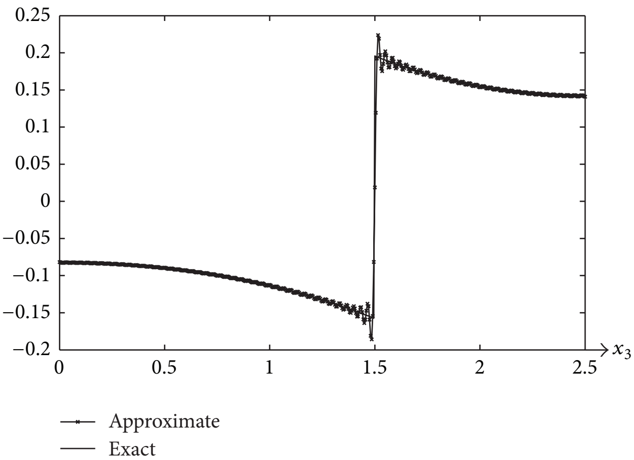


Figure 4: The graphs of h_2^1 and $h_2^{1,N}$ for $N = 150$, $x_1^0 = 2$, $x_2^0 = 3$, and $x_3^0 = 1.5$.

Table 1: The accuracy of the calculation of the magnetic field for $x_1^0 = 2$; $x_2^0 = 3$; $x_3^0 = 1.5$; $r_N = |h_2^1 - h_2^{1,N}|/|h_2^1|$; $N = 15, 50, 100$, and 150.

x_3	r_{15}	r_{50}	r_{100}	r_{150}
0.5	0.0705	0.0213	0.0107	0.0071
0.7	0.0597	0.0232	0.0116	0.0078
0.9	0.0208	0.0271	0.0136	0.0090
1.1	0.0492	0.0356	0.0179	0.0119
1.3	0.1806	0.0625	0.0316	0.0211
1.7	0.1525	0.0531	0.0268	0.0179
1.9	0.0417	0.0317	0.0159	0.0106
2.1	0.0213	0.0251	0.0126	0.0084
2.2	0.0645	0.0235	0.0118	0.0078
2.4	0.0233	0.0220	0.0110	0.0073

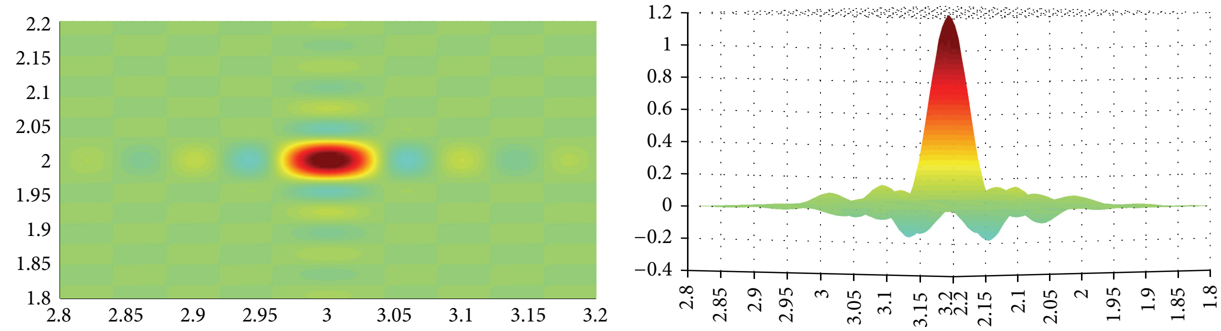
for the approximate computation of frequency-dependent magnetic and electric Green's matrices in the rectangular parallelepiped $V = \{x \in \mathbf{R}^3 : 0 < x_1 < b_1, 0 < x_2 < b_2, 0 < x_3 < b_3\}$, respectively. We note that each column of magnetic and electric frequency-dependent Green's matrices is the magnetic and electric fields arising from the pulse dipole polarized in the direction of the basis vector of $\mathbf{R}^3$. For computation we take $b_1 = 3$, $b_2 = 4$, and $b_3 = 2.5$ and ε and μ such that $c = 1/\sqrt{\varepsilon\mu}$, $\omega = c$ (c is the value of the light velocity), $x_1^0 = 2$, $x_2^0 = 3$, and $x_3^0 = 1.5$. Computation of $H_j^{s,N}$ and $E_j^{s,N}$ has been implemented in MATLAB and their graphs are presented. Figures 5(a), 5(b), 6(a), and 6(b) present the second and third components of the first column of the regularized magnetic matrix, respectively, $H_2^{1,100}(x,\omega,x^0)$ and $H_3^{1,100}(x,\omega,x^0)$. Figures 7(a)-7(b), 8(a)-8(b), and 9(a)-9(b) present each component of the first column of the regularized electric matrix, respectively, $E_1^{1,100}(x,\omega,x^0)$, $E_2^{1,100}(x,\omega,x^0)$, and $E_3^{1,100}(x,\omega,x^0)$. The views of the surfaces $z = H_2^{1,N}$, $z = H_3^{1,N}$, $z = E_1^{1,N}$, $z = E_2^{1,N}$, and $z = E_3^{1,N}$ (N=100) from the top of z-axis are presented in Figures 5(a), 6(a), 7(a), 8(a), and 9(a), where x_1-axis is vertical and x_2-axis is a horizontal one. Figures 5(b), 6(b), 7(b), 8(b), and 9(b) are 3D images of the surfaces $H_2^{1,100}(x,\omega,x^0)$, $H_3^{1,100}(x,\omega,x^0)$, $E_1^{1,100}(x,\omega,x^0)$, $E_2^{1,100}(x,\omega,x^0)$, and $E_3^{1,100}(x,\omega,x^0)$ for $x_3 = 1.5$, respectively. Here, x_1 and x_2 are horizontal axes and z-axis is the magnitude.

7. Conclusion

The new method for the approximate computation of frequency-dependent magnetic and electric Green's matrices in the rectangular parallelepiped with a perfect conducting boundary has been developed. This method has the following steps.

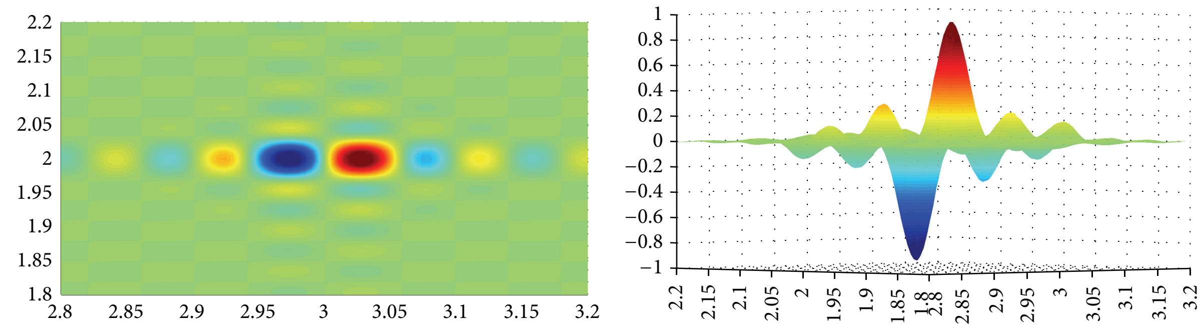
The first step is determination of the eigenvalues and corresponding eigenfunctions of the vector Laplace differential operator of the form $\Delta_x \mathbf{I}_3$ (where Δ is the Laplace operator and $\mathbf{I}_3$ is the 3×3 identity matrix) in a parallelepiped subject to some boundary conditions. The components of these vector eigenfunctions form an orthogonal set which is complete in the space of the square integrable functions over the considered parallelepiped.

The second step is an approximation (regularization) of the generalized vector functions (distributions) $\operatorname{curl}_x[\delta(x - x^0)\overrightarrow{\mathbf{e}^s}]$; $s = 1,2,3$, appear in the differential equations for the columns of magnetic Green's matrix as the free terms. Here $\overrightarrow{\mathbf{e}^s}$ are the basis vectors of $\mathbf{R}^3$($\overrightarrow{\mathbf{e}^1} = (1,0,0)$, $\overrightarrow{\mathbf{e}^2} = (0,1,0)$, $\overrightarrow{\mathbf{e}^3} = (0,0,1)$); $x^0 = (x_1^0, x_2^0, x_3^0)$ is a 3D parameter from the considered parallelepiped, $\delta(x - x^0) = \delta(x_1 - x_1^0)\delta(x_2 - x_2^0)\delta(x_3 - x_3^0)$, and $\delta(x_j - x_j^0)$ is the Dirac delta function considered at $x_j = x_j^0$ and $j = 1,2,3$. We find the Fourier coefficients of these free terms relative to the obtained set of eigenfunctions and approximate them by the Fourier series with a finite number of terms. We note that the Dirac delta function is very often used for modeling the point source in physics and engineering (see, [19, 20, 23]). The Dirac delta function does not have point-wise values



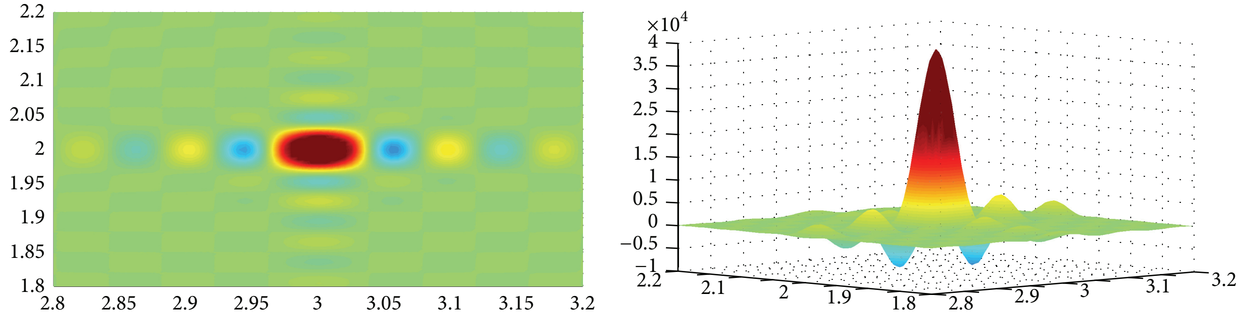
(a) $H_2^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$ (b) $H_2^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$

Figure 5: The second component of the first column of regularized magnetic Green's function.



(a) $H_3^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$ (b) $H_3^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$

Figure 6: The third component of the first column of regularized magnetic Green's function.



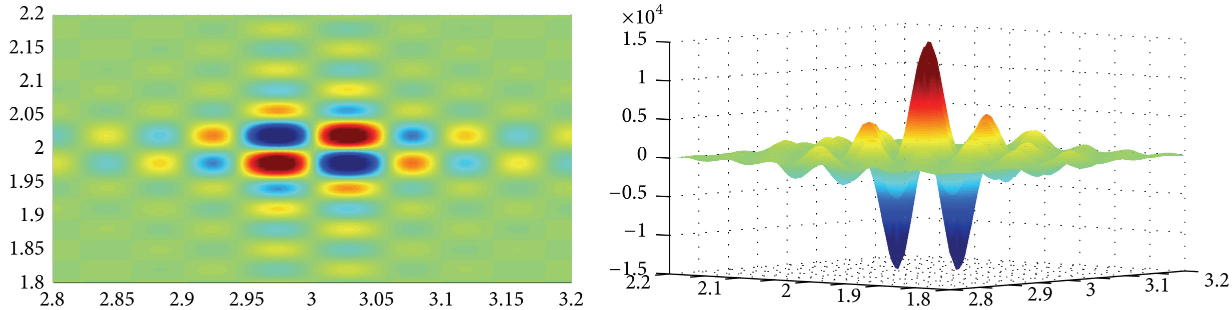
(a) $E_1^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$ (b) $E_1^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$

Figure 7: The first component of the first column of regularized electric Green's function.

and can not be drawn. For this reason, a regularization (approximation) of the Dirac delta function in the form of the classical function which has point-wise values is usually used for drawing and computation. For the Dirac delta function, as a singular generalized function, there are different types of approximation (see, e.g., [19, 20, 23]). Unfortunately, the known approximations of the Dirac delta function are not convenient for our approach. For this reason we have suggested a new approximation which is very useful for the computation of considered Green's function.

The third step is an approximate computation of elements of Green's matrix in the form of the Fourier series with a finite number of terms relative to the considered set of eigenfunctions. All terms of this series contain known functions which are given explicitly. The simple implementation of our method for computing magnetic Green's function in a rectangular parallelepiped is based on the obtained presentation and does not contain any type of discretization.

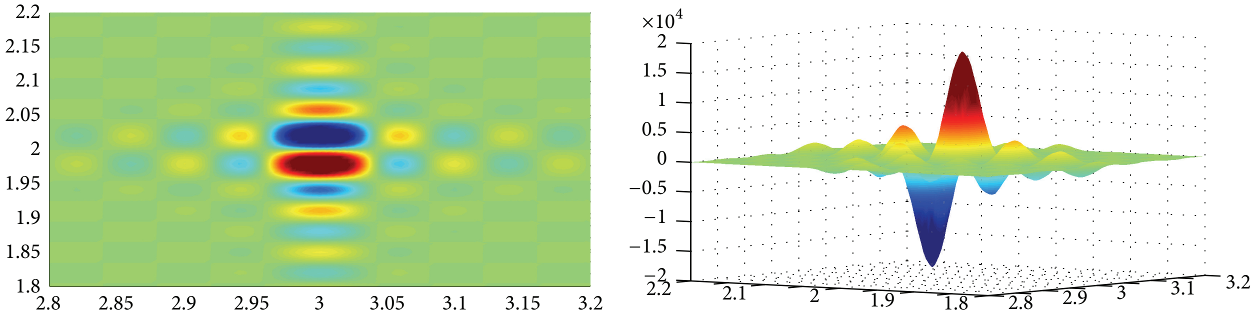
Using explicit formulas for the approximate elements of magnetic Green's matrix we derive explicitly the approximate elements of electric Green's matrix. Finally, we have obtained the columns of magnetic and electric matrix Green's function in the space of generalized functions (distributions)



(a) $E_2^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$

(b) $E_2^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$

Figure 8: The second component of the first column of regularized electric Green's function.



(a) $E_3^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$

(b) $E_3^{1,N}(x_1, x_2, 1.5, \omega, x^0)$, $N = 100$

Figure 9: The third component of the first column of regularized electric Green's function.

and their regularizations (approximations). The method has been implemented in MATLAB. The computational experiments have confirmed its robustness. We can note also that application, for example, finite elements or finite difference approaches can be done if we approximate the generalized vector functions $\operatorname{curl}_x[\delta(x - x^0)\overrightarrow{\mathbf{e}^s}]$ by vector functions with components which are classical functions. We note here that numerical computations of the values of magnetic Green's function by equations, where $\operatorname{curl}_x[\delta(x - x^0)\overrightarrow{\mathbf{e}^s}]$ are replaced by their approximation, will contain the error depending on the grid of discretization. For this reason our method of the computation of the magnetic Green function has the accuracy which is significantly better than other methods based on any type of discretization.

Appendices

A. Proof of Lemma 8

Let $D^\alpha = \partial^{\alpha_1+\alpha_2+\alpha_3}/\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ be the differential operator, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha_1, \alpha_2, \alpha_3$ are nonnegative integers. We have

$$\iiint_V [D^\alpha h(x)]^2 dx < \infty, \tag{A.1}$$

for an arbitrary $h(x) \in C_0^\infty(V)$.

Using Parseval's equality (26) we find

$$\iiint_V [D^\alpha h(x)]^2 dx = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[h_{j,\alpha}^{kmn}\right]^2 < \infty. \tag{A.2}$$

Here $h_{j,\alpha}^{kmn}$ denotes the Fourier coefficients of the function $D^\alpha h(x)$ relative to basis functions $\{U_j^{kmn}(x)\}$, $k, m, n = 0, 1, \ldots$; $j = 1, 2, 3$.

For even positive integers $\alpha_1, \alpha_2, \alpha_3$ we have

$$h_{j,\alpha}^{kmn} = (-1)^{|\alpha|/2}\left(\frac{k\pi}{b_1}\right)^{\alpha_1}\left(\frac{m\pi}{b_2}\right)^{\alpha_2}\left(\frac{n\pi}{b_3}\right)^{\alpha_3} h_j^{kmn}, \tag{A.3}$$

where $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

Using (21) and (A.3) we have

$$\begin{aligned}\left|\frac{\partial U_2^{kmn}(x^0)}{\partial x_3^0} h_2^{kmn}(x)\right| &\le \left|\frac{n\pi}{b_3} B_{kmn} h_2^{kmn}\right| \\ &\le 2c_1\left|\frac{1}{k^{\alpha_1} m^{\alpha_2} n^{\alpha_3-1}} h_{2,\alpha}^{kmn}\right| \\ &\le c_1\left[\frac{1}{k^{2\alpha_1} m^{2\alpha_2} n^{2\alpha_3-2}} + \left|h_{2,\alpha}^{kmn}\right|^2\right],\end{aligned} \tag{A.4}$$

where $c_1 = \pi^{1-|\alpha|}\sqrt{2/(b_1 b_2 b_3)}\, b_1^{\alpha_1} b_2^{\alpha_2} b_3^{\alpha_3-1}$.

The convergence of $\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|h_{2,\alpha}^{kmn}|^2$ follows from (A.2). The series

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^{2\alpha_1}m^{2\alpha_2}n^{2\alpha_3-2}} \tag{A.5}$$

converges for $2\alpha_1 > 1$, $2\alpha_2 > 1$, and $2(\alpha_3 - 1) > 1$. Therefore, taking even positive integers $\alpha_1, \alpha_2, \alpha_3$ and using equality (A.4), we find that the series (31) is convergent for any fixed $x^0 \in V$ and $h(x) \in C_0^{\infty}(V)$. The convergence of (32) can be established similarly.

Lemma 8 is proved.

B. Convergence of Series (56)

Let us consider the convergence of the following series for any fixed $x^0 \in V$ and any function $h(x) \in C_0^{\infty}(V)$:

$$-\frac{i}{\omega\epsilon}\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}}\frac{\partial U_1^{kmn}\left(x^0\right)}{\partial x_3^0} \times \iiint_V U_1^{kmn}(x)\frac{\partial h(x)}{\partial x_3}dx, \tag{B.1}$$

$$-\frac{i}{\omega\epsilon}\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}}\frac{\partial U_3^{kmn}\left(x^0\right)}{\partial x_1^0} \times \iiint_V U_3^{kmn}(x)\frac{\partial h(x)}{\partial x_1}dx, \tag{B.2}$$

$$\frac{i}{\omega\epsilon}\sum_{k=0}^{N}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\omega^2\epsilon\mu-\lambda_{kmn}}U_2^{kmn}\left(x^0\right)h_2^{kmn}. \tag{B.3}$$

Let $h(x)$ be an arbitrary function from $C_0^{\infty}(V)$ and $\phi(x) = \partial h(x)/\partial x_3$; $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, where α_j, $j = 1, 2, 3$ are natural numbers, such that $2\alpha_1 > 1$, $2\alpha_2 > 1$, and $2(\alpha_3 - 1) > 1$. Let $\phi_1^{kmn}$and$\phi_{1,\alpha}^{kmn}$ be the Fourier coefficients of $\phi(x)$ and $D^{\alpha}\phi(x)$ relative to the basis $\{U_1^{kmn}(x)\}$, $k = 1, 2, \ldots$; $m, n = 0, 1, 2, 3, \ldots$. Then we have

$$\begin{aligned}&\left|\frac{1}{\omega\epsilon\left(\omega^2\epsilon\mu-\lambda_{kmn}\right)}\frac{\partial U_1^{kmn}\left(x^0\right)}{\partial x_3^0}\right.\\&\left.\times\iiint_V U_1^{kmn}(x)\,\phi(x)\,dx\right|\\&\leq\left(\frac{A_{kmn}}{\omega\epsilon\left(\lambda_{kmn}-\omega^2\epsilon\mu\right)}\right)\frac{n\pi}{b_3}\left|\phi_1^{kmn}\right|\\&\leq\frac{A_{kmn}}{\omega\epsilon}\frac{\pi n}{b_3}\left|\phi_1^{kmn}\right|\\&\leq c_2\left[\frac{1}{k^{2\alpha_1}m^{2\alpha_2}n^{2(\alpha_3-1)}}+\left|\phi_{1,\alpha}^{kmn}\right|^2\right],\end{aligned} \tag{B.4}$$

where $c_2 = (\pi^{1-|\alpha|}/\omega\epsilon)\sqrt{2/(b_1b_2b_3)}b_1^{\alpha_1}b_2^{\alpha_2}b_3^{(\alpha_3-1)}$. Using the technique similar to Appendix A we find that $\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c_2[1/k^{2\alpha_1}m^{2\alpha_2}n^{2(\alpha_3-1)}+|\phi_{1,\alpha}^{kmn}|^2]$ converges and, as a result of it, series (B.1) converges absolutely for any fixed $x^0 \in V$ and any function $h(x) \in C_0^{\infty}(V)$. The convergence of series (B.2) and (B.3) for any fixed $x^0 \in V$ and an arbitrary function $h(x) \in C_0^{\infty}(V)$ can be established in a similar way. Convergence of series (B.1), (B.2), and (B.3) implies the absolute convergence of the series in (56) for any fixed $x^0 \in V$ and arbitrary $h(x) \in C_0^{\infty}(V)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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